Exercise #1 - Externalities and car accidents

1. Consider an economy with two individuals $i = \{1, 2\}$ with the following quasi-linear utility function

$$u_i(s^i, q^i) = v^i(s^i) + \alpha w^i$$

where s^i denotes the speed at which individual i drives his car, w^i is his wealth, and $\alpha > 0$. The utility that individual i obtains from driving fast is $v^i(s^i)$, which is increasing but concave in speed, whereby $\frac{\partial v^i(s^i)}{\partial s^i} > 0$ and $\frac{\partial^2 v^i(s^i)}{(\partial s^i)^2} < 0$. Driving fast, however, increases the probability of suffering a car accident, represented by $\gamma(s^i, s^j)$. This probability is increasing both in the speed at which individual i drives, s^i, and the speed at which other individuals drive, s^j, where $j \neq i$. Hence, the speed of other individuals imposes a negative externality on driver i, since it increases his risk of suffering a car accident. If individual i suffers an accident, he bears a cost of $c^i > 0$, which intuitively embodies the cost of fixing his car, health-care expenses, etc.

(a) *Unregulated equilibrium.* Set up individual i's expected utility maximization problem. Take first-order conditions with respect to s^i, and denote the (implicit) solution to this first-order condition as $\widehat{s}^i$.

- With probability $\gamma(s^i, s^j)$, the individual suffers a car accident, and thus his utility is $v^i(s^i) + \alpha w^i - c^i$, and with probability $1 - \gamma(s^i, s^j)$ he does not suffer the accident, leaving his utility level at $v^i(s^i) + \alpha w^i$.
- Hence, his expected utility is

$$\gamma(s^i, s^j)[v^i(s^i) + \alpha w^i - c^i] + (1 - \gamma(s^i, s^j))[v^i(s^i) - \alpha w^i],$$

which reduces to $v^i(s^i) + \alpha w^i - \gamma(s^i, s^j)c^i$. Hence, every individual i maximizes his expected utility by choosing an speed level s^i that solves

$$\max_{s^i} \; v^i(s^i) + \alpha w^i - \gamma(s^i, s^j) \times c^i$$

Taking first-order conditions with respect to s^i we obtain

$$\frac{\partial v^i(s^i)}{\partial s^i} - \frac{\partial \gamma}{\partial s^i} c^i = 0 \tag{4}$$

Hence, driver i independently selects the speed, $\widehat{s}^i$, that solves $\frac{\partial v^i(s^i)}{\partial s^i} = \frac{\partial \gamma}{\partial s^i} c^i$.

- Intuitively, driver i increases his speed s^i until the point where the additional utility from marginally increasing s^i, $\frac{\partial v^i(s^i)}{\partial s^i}$, coincides with its associated expected individual cost from speed, i.e., a higher probability of suffering a car accident times its associated cost, as measured by $\frac{\partial \gamma}{\partial s^i} c^i$.

- *Parametric example.* Consider, for instance, a utility from driving fast of $v(s^i) = \sqrt{s^i}$ (which is increasing and concave in s^i, as required), and that the probability of suffering a car accident is $\gamma(s^i, s^j) = \beta_i s^i + \beta_j s^j$, where $\beta_i > \beta_j$ (indicating that my own speed increases the probability that I suffer a car accident more than other drivers' speeds). First order condition (1) in this context becomes

$$\frac{1}{2\sqrt{s^i}} = \beta_i c^i,$$

and solving for s^i, we obtain an equilibrium speed of $\widehat{s}^i = \frac{1}{4(\beta_i c^i)^2}$ for every individual driver $i = \{1, 2\}$.

(b) *Social optimum.* Set up the social planner's expected welfare maximization problem. Take first-order conditions with respect to s^1 and s^2. Denote the (implicit) solution to this first-order condition as $\overline{s}^i$.

- The social planner solves the expected welfare maximization problem

$$\max_{s^1, s^2} \; v^1(s^1) + \alpha w^1 + v^2(s^2) + \alpha w^2 - \gamma(s^1, s^2) \times \left(c^1 + c^2\right)$$

Taking first-order conditions with respect to s^1, we obtain that $\bar{s}^1$ solves

$$\frac{\partial v^1(s^1)}{\partial s^1} = \frac{\partial \gamma}{\partial s^1} \left(c^1 + c^2\right) \tag{5}$$

and similarly with respect to s^2, we obtain that $\bar{s}^2$ solves

$$\frac{\partial v^2(s^2)}{\partial s^2} = \frac{\partial \gamma}{\partial s^2} \left(c^1 + c^2\right) \tag{6}$$

Intuitively, at the social optimum every driver i increases his speed s^i until the point where the additional utility from marginally increasing s^i coincides with its associated expected social cost from speed, measured by not only the higher probability of him suffering a car accident but also by the higher probability that the other individual $j \neq i$ suffers a car accident because of the speed s^i of individual i.

(c) *Comparison.* Show that drivers have individual incentives to drive too fast, relative to the socially optimal speed, i.e., show that $\widehat{s}^i > \overline{s}^i$.

- Comparing expressions (1) and (2), yields

$$\frac{\partial v^1(\hat{s}^1)}{\partial s^1} < \frac{\partial v^1(\bar{s}^1)}{\partial s^1}$$

Since $\frac{\partial^2 v^i(s^i)}{(\partial s^i)^2} < 0$ by definition, $\frac{\partial v^i(s^i)}{\partial s^i}$ is a decreasing function. Therefore, the speed that individual 1 independently selects, $\widehat{s}^1$, is excessive from a social point of view, i.e., $\widehat{s}^1 > \overline{s}^1$. Similarly, comparing (1) and (3), we have that $\widehat{s}^2 > \overline{s}^2$. Intuitively, every driver does not internalize the negative externality that his speed imposes on other drivers (in the form of a higher probability of suffering a car accident) when he independently selects his driving speed.

- Figure 9.1 represents the marginal utility, $\frac{\partial v^i(s^i)}{\partial s^i}$, and marginal expected costs, individual marginal costs, $\frac{\partial \gamma}{\partial s^i} c^i$, and social marginal costs, $\frac{\partial \gamma}{\partial s^i}(c^i + c^j)$, to support the above explanation. Since the social marginal cost curve is higher for any speed level s^i than the individual marginal cost curve, the former crosses the marginal utility curve at a lower speed level, i.e., $\overline{s}^i < \widehat{s}^i$. Intuitively, the social planner internalizes the externality that additional speed imposes on other drivers (who could suffer a car accident due to the speed of driver i), and thus reduces the speed of both drivers.

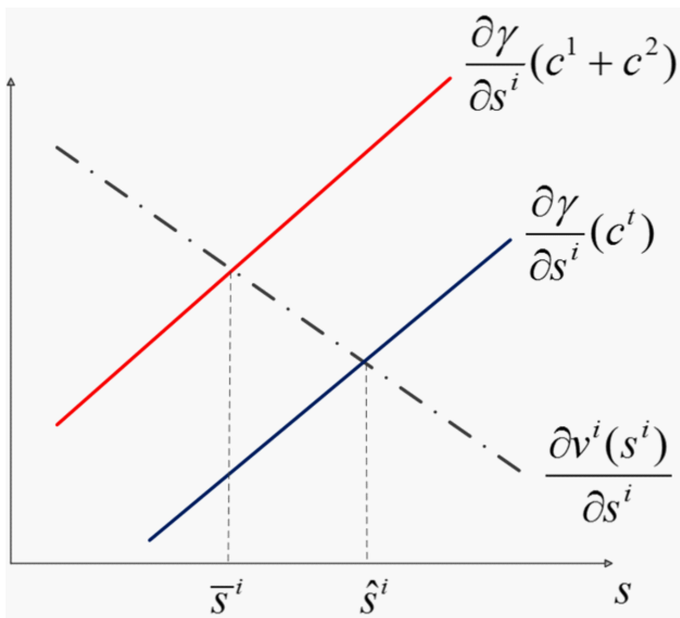

Figure 9.1. Efficient and socially optimal speed.

  - Note that for simplicity, we consider that the marginal utility decreases in s^i at a constant rate, i.e., $\frac{\partial^2 v^i(s^i)}{\partial s^{i2}}$ is constant in s^i or, alternatively, $\frac{\partial^3 v^i(s^i)}{(\partial s^i)^3} = 0$; implying that the marginal utility curve is a straight line. In addition, we also assume that further increases in speed s^i imply a constant increase in the probability of an accident, i.e., $\frac{\partial^2 \gamma}{(\partial s^i)^2} > 0$ but constant or, alternatively, that $\frac{\partial^3 \gamma}{(\partial s^i)^3} = 0$. This property entails the marginal cost curve is also a straight line.

– *Parametric example.* Continuing with the previous example in which $v^i(s^i) = \sqrt{s^i}$ and $\gamma(s^i, s^j) = \beta_i s^i + \beta_j s^j$, the socially optimal speed that the social planner would select, $\bar{s}^i$, is that satisfying

$$\frac{1}{2\sqrt{\bar{s}^i}} = \beta_i(c^i + c^j)$$

and solving for $\bar{s}^i$ yields $\bar{s}^i = \frac{1}{4[\beta_i(c^i+c^j)]^2}$, which clearly falls below the speed level independently selected by every driver $\widehat{s}^i = \frac{1}{4(\beta_i c^i)^2}$.

(d) *Restoring the social optimum.* Let us now evaluate the effect of speeding tickets (fines) to individuals driving too fast, i.e., to those drivers with a speed $\widehat{s}^i$ satisfying, $\widehat{s}^i > \bar{s}^i$. What is the dollar amount of the fine m^i that induces every individual i to fully internalize the externality he imposes onto others?

- Comparing (1) and (2) for driver 1, we must impose a fine of $m^1 = c^2$ in order to guarantee that (1) coincides with (2). Intuitively, this fine induces driver 1 to internalize the negative externality (higher chances of suffering a car accident and, in this case, an associated monetary cost of repairs) that he imposes on driver 2. Similarly comparing (1) and (3) for driver 2, we must impose a fine of $m^2 = c^1$ in order to guarantee that (1) coincides with (3).

(e) Let us now consider that individuals obtain a utility from driving fast, $v^i(s^i)$, only in the case that no accident occurs. Repeat steps (a)-(c), finding the optimal fine m^i that induces individuals to fully internalize the externality.

- *Equilibrium speed.* In this section of the exercise, driver i only obtains utility from driving fast, $v^i(s^i)$, when no accident occurs. Given that the probability that an accident does *not* occur is $1 - \gamma(s^1, s^2)$, the utility of driver i is

$$\underbrace{\left[1-\gamma(s^1,s^2)\right]\left(v^i(s^i)+\alpha w^i\right)}_{\text{No accident}} + \underbrace{\gamma(s^i,s^j)(\alpha w^i - c^i)}_{\text{Accident}}$$

which can be rearranged as

$$v^i(s^i) + \alpha w^i - \gamma(s^i, s^j)\left[c^i + v^i(s^i)\right]$$

Taking first order conditions with respect to s^i, we obtain that the individual driver i independently selects the speed $\bar{s}^i$ that solves

$$\frac{\partial v^i(s^i)}{\partial s^i}\left[1-\gamma(s^i,s^j)\right] = \frac{\partial \gamma}{\partial s^i}(c^i + v^i(s^i)) \tag{4}$$

where conveniently separates the marginal utility of driving faster in the left-hand side, which only arises if driver i does not suffer a car accident, an event with probability $1-\gamma(s^i,s^j)$; and its associated marginal cost in the right-hand side, which captures the higher probability of suffering a car accident, $\frac{\partial\gamma}{\partial s^i}$, and its two costs: one explicit, c^i, and one implicit, namely, the utility from driving that driver i would have to give up (since he can only benefit from driving when he does not suffer a car accident).

– *Parametric example.* Following with the on-going parametric example, the above first order condition (4) becomes

$$\frac{1}{2\sqrt{\overline{s}^i}}\left[1-(\beta_i\overline{s}^i+\beta_j\overline{s}^j)\right]=\beta_i(c^i+\sqrt{\overline{s}^i}),$$

and similarly for driver j. Before solving for $\overline{s}^i$ in order to driver i's best response function, let us assume (in order to keep our parametric example compact) that $\beta_i=\beta_j=\frac{1}{2}$ and $c^i=c^j=\frac{2}{3}$. In this context, solving for $\overline{s}^i$ we obtain

$$\overline{s}^i(\overline{s}^j)=2-3\overline{s}^j-\frac{4}{3}\sqrt{\overline{s}^j}$$

Since both drivers are symmetric, $\overline{s}^i=\overline{s}^j$, we can solve for $\overline{s}^i$ yielding a symmetric equilibrium speed level of $\overline{s}^i=0.313$.

- *Socially optimal speed.* The social planner's maximization problem in this case becomes

$$\max_{s^1,s^2}\ v^1(s^1)+\alpha w^1-\gamma(s^1,s^2)\left[c^1+v^1(s^1)\right]+v^2(s^2)+\alpha w^2-\gamma(s^1,s^2)\left[c^2+v^2(s^2)\right]$$

Taking first order conditions with respect to s^i, we obtain that the socially optimal speed, $\widehat{s}^i$, solves

$$\frac{\partial v^i(s^i)}{\partial s^i}\left[1-\gamma(s^i,s^j)\right]=\frac{\partial\gamma}{\partial s^i}\left[c^i+v^i(s^i)\right]+\frac{\partial\gamma}{\partial s^i}\left[c^j+v^j(s^j)\right] \tag{5}$$

- *Comparison.* Comparing expressions (4) and (5), we obtain that the fine m^i that induces every individual i to internalize the externality that his driving imposes on others is

$$m^i=c^j+v^j(s^j)$$

Intuitively, now an increase in the speed of driver i not only increases the probability that driver j suffers a car accident, and thus needs to incur a cost of c^j, it also reduces the utility from driving that driver j can only experience if he is not involved in a car accident.

– *Parametric example.* Following with the on-going parametric example, the above first order condition (4) becomes

$$\frac{1}{2\sqrt{\widehat{s}^{ii}}}\left[1-(\beta_i\widehat{s}^i+\beta_j\widehat{s}^j)\right]=\beta_i\left[(c^i+\sqrt{\widehat{s}^i})+(c^j+\sqrt{\widehat{s}^j})\right],$$

Before solving for $\widehat{s}^i$ in order to driver i's best response function, let us assume (in order to keep our parametric example compact) that $\beta_i = \beta_j = \frac{1}{2}$ and $c^i = c^j = \frac{2}{3}$. In this context, we can simultaneously solve for s

when drivers independently choose their own driving speed, $\overline{s}^i = 0.313$.

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Exercise #3 - Positive and Negative Externalities

3. Consider an economy with two firms which produce a homogeneous good. Firm 1 produces q_1 units of the good, and its cost function is $c_1(q_1, q_2) = 2q_1^2 + 5q_1 + q_2$, while firm 2 produces q_2 units of the same good and its cost function is $c_2(q_2, q_1) = q_2^2 + 3q_2 - 4q_1$. Note that every firm i's costs depends on its rival's output, q_j, where $j \neq i$. Finally, inverse market demand is given by $p(Q) = 34 - Q$, where $Q = q_1 + q_2$ denotes aggregate output.

   (a) *Unregulated equilibrium.* Considering that every firm independently and simultaneously selects its production level, determine equilibrium output q_1 and q_2. What are the associated profits for each firm? Measure consumer surplus, profits and social welfare.

   - Since $\frac{\partial c_1(q_1,q_2)}{\partial q_2} = 1 > 0$ and $\frac{\partial c_2(q_2,q_1)}{\partial q_1} = -4 < 0$, firm 2 generates a negative externality on firm 1 (i.e., q_2 increases firm 1's costs), while firm 1 produces a positive externality on firm 2. In order to determine the equilibrium level of q_1 and q_2, we need to separately consider each firm's profit-maximization problem. First, firm 1 chooses the level of q_1 that solves

$$\max_{q_1 \geq 0} \; (34 - q_1 - q_2)\, q_1 - \left(2q_1^2 + 5q_1 + q_2\right)$$

   Taking first order condition with respect to q_1, we obtain $29 - 6q_1 - q_2 = 0$. Solving for q_1 we find firm 1's best response function, $q_1(q_2) = \frac{29 - q_2}{6}$. Similarly, firm 2 solves

$$\max_{q_2 \geq 0} \ (34 - q_1 - q_2)\, q_2 - \left(q_2^2 + 3q_2 - 4q_1\right)$$

and taking first order condition with respect to q_2, we have $31 - q_1 - 4q_2 = 0$. Thus, solving for q_2 we obtain firm 2's best response function, $q_2(q_1) = \frac{31-q_1}{4}$. Plugging $q_2(q_1)$ into $q_1(q_2)$, we find the equilibrium output levels $q_1^* = \frac{85}{23} \simeq 3.69$ and $q_2^* = \frac{157}{23} \simeq 6.82$. Therefore, the aggregate supply is $Q^S(p) = q_1^* + q_2^* = \frac{242}{23} \simeq 10.52$, with an equilibrium price of $p = 34 - \frac{242}{23} = \frac{540}{23} \simeq \$23.47$.

- Equilibrium profits are therefore $\pi_1 = 34.14$ and $\pi_2 = 107.97$ for firm 1 and 2, respectively, and aggregate profits are $\pi = 142.11$.
- Consumer surplus is, hence, given by the area of the triangle below the inverse demand curve and above the equilibrium price of \$23.47.

$$CS = \frac{1}{2}(34 - 23.47) \cdot 10.52 = 55.39.$$

Thus, social welfare is $W = CS + \pi = 197.5$.

- Finally, notice that this output allocation is inefficient: firm 1 (the agent who generates the positive externality) produces too little, whereas firm 2 (the agent who causes the negative externality) produces too much. We formally show this result in the next question, where firms are allowed to merge and thus internalize the positive and negative externalities of their production decisions.

(b) *Merger.* Assume that the government is aware of these mutual externalities between firm 1 and 2, but does not want to directly regulate their production by the imposition of quotas or fees. Instead, the regulator allows both firms to merge. Determine the equilibrium level of q_1 and q_2 that the newly merged firm will choose, and check if firm 1 and 2 have incentives to merge.

- This merge is equivalent to a horizontal integration, whereby firms choose the level of q_1 and q_2 in order to maximize their joint profits, as follows

$$\max_{q_1 \geq 0, q_2 \geq 0} \ (34 - q_1 - q_2)\,(q_1 + q_2) - \left(2q_1^2 + 5q_1 + q_2\right) - \left(q_2^2 + 3q_2 - 4q_1\right)$$

Taking first order conditions with respect to q_1 and q_2, we obtain

$$\begin{aligned} 33 - 6q_1 - 2q_2 &= 0, \quad \text{and} \\ 30 - 2q_1 - 4q_2 &= 0 \end{aligned}$$

where we can simultaneously solve for q_1 and q_2 to find $q_1 = \frac{18}{5} = 3.6$ and

$q_2 = \frac{57}{10} = 5.7.$

- Thus, the production of firm 2 (which generates a negative externality on firm 1) is significantly reduced, from 6.28 to 5.7 units. Aggregate supply is hence $Q^S(p) = \frac{93}{10} = 9.3$ units, with an equilibrium price of $p = 34 - 9.3 = \$24.7$. Aggregate output thus decreases and the equilibrium price increases as a consequence, from \$23.47 to \$24.7.
- Equilibrium profits are therefore $\pi_1 = 39.3$ and $\pi_2 = 105.6$ for firm 1 and 2, respectively, and aggregate profits are $\pi = 144.9$. Aggregate profits increase as a result of the merger, and hence firms have incentives to merge.(While aggregate profits increase, the individual profits of firm 2 decrease, suggesting that firm 2 will only be attracted to merge if firm 1 compensates it.)

(c) *Comparisons.* Compare consumer surplus, profits and welfare after the merger (as you found in part b) and before the merger (as found in part a). Does the merger ameliorate the negative externality that the production of firm 2 generates? Does social welfare increase as a result of the merger?

- After the merge, consumer surplus is

$$CS = \frac{1}{2}(34 - 24.7) \cdot 9.3 = 43.24$$

  Thus, social welfare is $W = CS + \pi = 188.14$.
- Comparing our results in parts (a) and (b), we can summarize that, as a result of the merger firms are better off, they not only maximize joint profits, as in a standard cartel exercise, but, in addition, they solve the mutual externality problem they face in part (a). However, the merger leads firms to reduce aggregate production, which increases market prices, ultimately reducing consumer surplus (and aggregate welfare). Hence, while the mutual externalities are internalized by the merger, their monopolistic effects yield a net welfare loss.

Exercise #5 - Regulating externalities under incomplete information

5. Consider a polluting firm with profit function $\pi(q) = 10q - q^2$ where q denotes units of the externality-generating activity (for instance, q can represent units of output if each unit generates one unit of pollution). Pollution damage to consumers is given by the convex damage function $d(q) = 3q^2$. Let us analyze a context in which the regulator does not observe the firm's profit function, but observes the damage which additional pollution causes on consumers. In particular, the regulator estimates that marginal profits are

$$\frac{\partial \pi(q,a)}{\partial q} = 10 - 2aq,$$

where the random parameter a takes two equally likely values, $a = 1$ or $a = \frac{1}{2}$. (Note that in our above description we assume that the firm privately observes that the realization of parameter a is $a = 1$, thus yielding a marginal profit function of $10 - 2q$.) We will first determine which is the best quota and emission fee that the regulator can design given that he operates under incomplete information. Afterwards, we will evaluate the welfare that arises under each of these policy instruments, to determine which is better from a social point of view.

   (a) *Unregulated equilibrium.* Find the equilibrium amount of pollution, q^E, if the firm is unregulated and no bargaining occurs between the affected consumers and the firm.

   - In this setting, the firm maximizes its profits by solving

$$\max_q \pi(q)$$

   Taking first-order conditions with respect to q, yields $\frac{\partial \pi(q^E)}{\partial q} \leq 0$. Since $\frac{\partial \pi(q)}{\partial q} = 10 - 2q$ by definition, then

$$\frac{\partial \pi(q^E, \alpha)}{\partial q} = 10 - 2q^E \leq 0, \text{ with equality for } q^E > 0$$

   Solving for q^E we obtain an equilibrium amount of pollution (in interior solutions) of $q^E = 5$ units.

   (b) *Setting a quota.* In this incomplete information setting, determine which is the best quota x_q that a social planner can select in order to maximize the expected value of aggregate surplus.

- The firm must produce an output level exactly equal to the quota. The social planner determines the optimal quantity $\widehat{q}$ by choosing the value of q that maximizes the expected value of aggregate surplus (since the social planner does not know the precise realization of parameter a),

$$\max_{q} \; E_a[\pi(q,a)] - d(q)$$

And taking first order condition with respect to q, we obtain

$$E_a\left[\frac{\partial \pi(\widehat{q},a)}{\partial q}\right] - \frac{\partial d(\widehat{q})}{\partial q} \leq 0$$

We can now substitute the functional forms for the marginal damage for consumers, $\frac{\partial d(q)}{\partial q}$, and the expected marginal profits for the firm, $\frac{\partial \pi(q,a)}{\partial q}$, yielding

$$\frac{1}{2}\left(10 - 2\cdot\widehat{q}\right) + \frac{1}{2}\left(10 - 2\cdot\frac{1}{2}\cdot\widehat{q}\right) - 6\widehat{q} \leq 0.$$

which reduces to

$$5 - \widehat{q} + 5 - \frac{1}{2}\cdot\widehat{q} - 6\widehat{q} \leq 0, \text{ or } \; \widehat{q} \geq \frac{4}{3}.$$

(c) *Setting an emission fee.* Find the best tax t^* that this social planner can set under the context of incomplete information described above.

- Given a tax t^*, the government predicts firm's expected best response function by maximizing its expected profits.

$$\max_{t} \quad E_a[\pi(q,a)] - tq$$

Taking first order condition with respect to t, yields

$$E_a\left[\frac{\partial \pi(q,a)}{\partial q}\right] - t = 0$$

and plugging our functional forms we obtain

$$5 - q + 5 - \frac{1}{2}\cdot q = t$$

which yields an output function $q\,(t) = \frac{20-2t}{3}$. Provided this expected output function, we can now find the optimal tax that the social planner imposes, anticipating the firm's expected best response function, as follows

$$\max_t \; E_a[\pi(q(t),a)] - d\left(q\left(t\right)\right)$$

Taking first order conditions with respect to t, and applying the chain rule, yields

$$E\left[\frac{\partial \pi(q(t),a)}{\partial q} \cdot \frac{\partial q(t)}{\partial t}\right] = \frac{\partial d\left(q\left(t\right)\right)}{\partial q} \cdot \frac{\partial q(t)}{\partial t}$$

where we use the chain rule. Intuitively, the regulator equals the marginal disutility of additional pollution to consumers (which he can perfectly assess), as represented in the right-hand side of the equality; and the expected marginal profits from additional pollution for the firm (which he cannot observe), represented in the left-hand side of the above expression.

- Since $q\left(t\right) = \frac{20-2t}{3}$ then the derivative $\frac{\partial q(t)}{\partial t} = -\frac{2}{3}$ is a constant, that can be taken out of the expectation operator. That is,

$$\frac{\partial q(t)}{\partial t} E\left[\frac{\partial \pi(q(t),a)}{\partial q}\right] = \frac{\partial d\left(q\left(t\right)\right)}{\partial q} \cdot \frac{\partial q(t)}{\partial t}$$

Therefore, we can cancel out the $\frac{\partial q(t)}{\partial t}$ term on both sides of the equality, which yields

$$E\left[\frac{\partial \pi(q(t),a)}{\partial q}\right] = \frac{\partial d\left(q\left(t\right)\right)}{\partial q}.$$

Substituting the functional form of our marginal benefit and marginal profit functions, the above first-order condition becomes

$$\frac{1}{2}\left(10 - 2 \cdot q\left(t\right)\right) + \frac{1}{2}\left(10 - 2 \cdot \frac{1}{2} \cdot q\left(t\right)\right) = 6q\left(t\right)$$

$$q\left(t\right) = \frac{4}{3}$$

Substituting $q\left(t\right) = \frac{20-2t}{3}$, we can finally find the optimal tax t^* that solves

$$\frac{20-2t^*}{3} = \frac{4}{3}, \quad \text{or} \quad t^* = 8.$$

(d) *Policy comparison.* Compare the emission fee and the quota in terms of their associated deadweight loss. Under which conditions an uninformed regulator prefers to choose the emission fee?

- We need to compare the expected difference in losses in order to determine when a tax or a quota instrument is better. Figure 9.2 illustrates the welfare loss associated to tax t^*, which induces an externality level of $q(t^*)$.

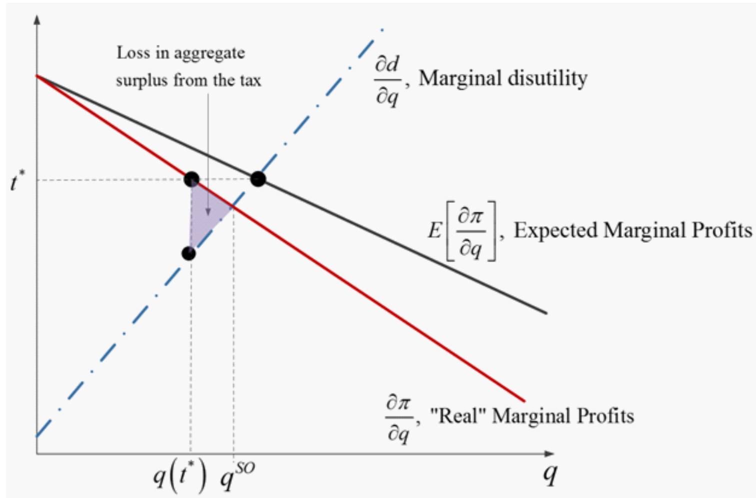

Figure 9.2. Welfare loss from a tax.

The figure considers that the regulator sets a tax based on the certain marginal disutility from the externality and the expected marginal profit. However, the realization of parameter a implies that the real and expected marginal profits do not coincide, thus giving rise to a welfare loss associated to a suboptimal tax due to the regulator's imprecise information.

- If the regulator, instead, imposes a quota, $\hat{q}$, figure 9.3 illustrates the associated welfare loss.

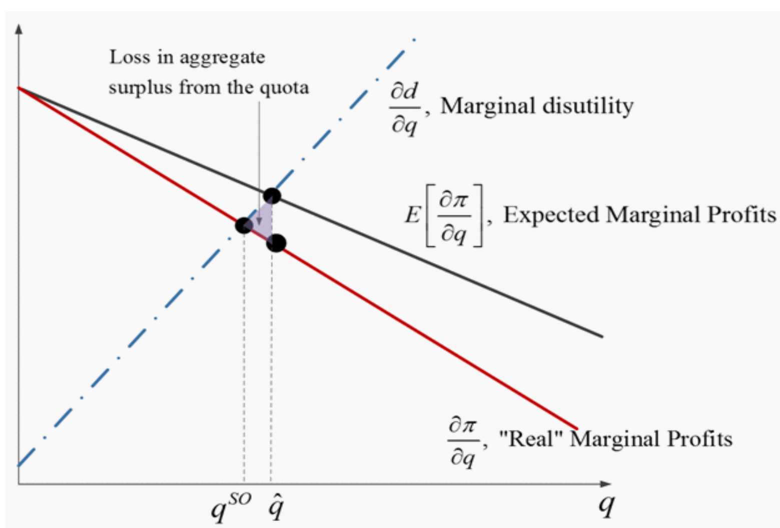

Figure 9.3. Welfare loss from a quota.

- *Welfare loss from the fee.* In order to compute the welfare loss from the tax,

WL_t, we first need to find the socially optimal level of externality, q^{SO}, given the true $a = 1$. In particular, q^{SO} solves

$$\begin{aligned}
\frac{\partial d(q^{SO})}{\partial q} &= \frac{\partial \pi(q^{SO})}{\partial q} \\
6q^{SO} &= 10 - 2q^{SO} \\
q^{SO} &= \frac{5}{4}
\end{aligned}$$

Moreover, given an emission fee, the firm maximizes profits. That is, it chooses the level of q that maximizes its profits (net of tax payments), as follows

$$\max_{q} \pi(q) - q \cdot t$$

The firm, hence, takes first order condition with repect to q, yielding

$$\frac{\partial \pi(q)}{\partial q} - t = 0$$

Since the firm knows its true marginal profit $\frac{\partial \pi(q)}{\partial q} = 10 - 2q$, the above expression becomes $10 - 2q - t = 0$, which yields an output function $q(t) = 5 - \frac{1}{2}t$. Given that $t^* = 8$, such fee induces an externality level of

$$q(t^*) = 5 - \frac{t^*}{2} = 1$$

Finally, we need to evaluate the marginal disutility function $\frac{\partial d(q)}{\partial q} = 6q$ at $q(t^*) = 1$, which yields

$$\frac{\partial d(q)}{\partial q} = 6q(t^*) = 6$$

Hence, the WL_t is given by the area of the shaded triangle in figure 9.2,

$$\begin{aligned}
WL_t &= \frac{1}{2}\left[q^{SO} - q(t^*)\right] \cdot \left[t^* - 6q(t^*)\right] \\
&= \frac{1}{2}\left[\frac{5}{4} - 1\right] \cdot [8 - 6] \\
&= \frac{1}{4}
\end{aligned}$$

- *Welfare loss from the quota.* If, in contrast, the regulator uses a quota of $\widehat{q} = \frac{4}{3}$, then we first need to evaluate the real marginal profits of the quota, that is

$$10 - 2q = 10 - 2 \times \frac{4}{3} = \frac{30}{3} - \frac{8}{3} = \frac{22}{3}$$

Second, we need to evaluate the expected marginal profit, $\frac{1}{2}(10 - 2 \cdot q) + \frac{1}{2}\left(10 - 2 \cdot \frac{1}{2} \cdot q\right)$, at the quota $\widehat{q} = \frac{4}{3}$, i.e.,

$$\frac{1}{2}\left(10 - 2 \cdot \frac{4}{3}\right) + \frac{1}{2}\left(10 - 2 \cdot \frac{1}{2} \cdot \frac{4}{3}\right) = 8.$$

Therefore, the welfare loss from the quota is the area of the shaded triangle in figure 9.3. That is,

$$\begin{aligned} WL_q &= \frac{1}{2}(\hat{q} - q^{SO})\left[8 - \frac{22}{3}\right] \\ &= \frac{1}{2}(\frac{4}{3} - \frac{5}{4})\left[\frac{24}{3} - \frac{22}{3}\right] \\ &= \frac{1}{36} \end{aligned}$$

- *Comparing welfare losses.* Comparing WL_t and WL_q, we obtain that

$$WL_t = \frac{1}{4} > \frac{1}{36} = WL_q$$

Hence, setting a quota is better than imposing an emission fee in this case.[33]

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[33] For more details about the welfare properties of emission fees and quotas under contexts in which the regulator is imperfectly informed, see Weitzman (1974).

Exercise #11 - Externalities in consumption

11. Consider two consumers with utility functions over two goods, x_1 and x_2, given by

$$\begin{aligned} u_A &= \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) \quad \text{for consumer } A\text{, and} \\ u_B &= \log(x_1^B) + x_2^B - \frac{1}{2}\log(x_1^A) \quad \text{for consumer } B. \end{aligned}$$

where the consumption of good 1 by individual $i = \{A, B\}$ creates a negative externality on individual $j \neq i$ (see the third term, which enters negatively on each individual's utility function). For simplicity, consider that both individuals have the same wealth, m, and that the price for both goods is 1.

(a) *Unregulated equilibrium.* Set up consumer A's utility maximization problem, and determine his demand for goods 1 and 2, as x_1^A and x_2^A. Then operate similarly to find consumer B's demand for good 1 and 2, as x_1^B and x_2^B.

- Consumer A chooses x_1^A and x_2^A to solve

$$\max_{(x_1^A, x_2^A)} \quad \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to} \quad x_1^A + x_2^A = M$$

The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M - x_1^A - x_2^A),$$

which yields first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - x_1^A - x_2^A = 0$$

Solving for x_1^A, we obtain $\frac{1}{x_1^A} = 1$, i.e., $x_1^A = 1$, which implies $M - 1 - x_2^A = 0$, or $x_2^A = M - 1$. Hence, consumer A's optimal consumption is

$$x_1^A = 1 \quad \text{and} \quad x_2^A = M - 1$$

A similar argument applies to consumer B,

$$x_1^B = 1 \quad \text{and} \quad x_2^B = M - 1$$

(b) *Social optimum.* Calculate the socially optimal amounts of x_1^A, x_2^A, x_1^B and x_2^B, considering that the social planner maximizes a utilitarian social welfare function, namely, $W = U_A + U_B$.

- The socially optimal consumption in this case solves

$$\max_{(x_1^A, x_2^A)} U^A + U^B \quad \text{subject to } x_1^A + x_2^A = M \text{ and } x_1^B + x_2^B = M$$

The Lagrangian for this social planner's problem is

$$\mathcal{L} = \frac{1}{2}\log(x_1^A) + \frac{1}{2}\log(x_1^B) + x_2^A + x_2^B + \lambda^A(M - x_1^A - x_2^A) + \lambda^B(M - x_1^B - x_2^B)$$

Taking first-order conditions, we find the socially optimal consumption profile:

$$x_1^A = \frac{1}{2} \text{ and } x_2^A = M - \frac{1}{2}$$

$$x_1^B = \frac{1}{2} \text{ and } x_2^B = M - \frac{1}{2}$$

Intuitively, the social planner recommends a lower consumption of good 1 (the good that generates the negative externality), and an increase in the consumption of good 2, for both individuals.

(c) *Restoring efficiency.* Show that the social optimum you found in part (b) can be induced by a tax on good 1 (so the after-tax price becomes $1+t$) with the revenue returned equally to both consumers in a lump-sum transfer.[36]

- With tax t^A placed on good 1 and with lump-sum transfer T^A, consumer A solves

$$\max_{(x_1^A, x_2^A)} \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$
$$\text{subject to } (1+t^A)x_1^A + x_2^A = M + T^A$$

where note that the price of good 1 increased from 1 to $(1+t^A)$, but this consumer also sees his wealth increase by the lump sum T^A. The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M + T^A - (1+t^A)x_1^A - x_2^A)$$

Taking first-order conditions, we obtain

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A(1+t^A) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M + T^A - (1+t^A)x_1^A - x_2^A = 0$$

Simultaneously solving for x_1^A and x_2^A, we find that consumer A's consumption

[36] Similarly as in the exercises about a polluting monopoly or oligopoly subject to emission fees, we assume that tax revenue is entirely returned to the agents being taxed as a lump-sum transfer. This assumption guarantees that the tax is revenue neutral, yet it helps modify agents' incentives ultimately correcting the externality, i.e., inducing the social optimum.

bundles after introducing the tax become

$$x_1^A = \frac{1}{1+t^A} \quad \text{and} \quad x_2^A = M + T^A - 1$$

Similarly we find the optimal consumption of consumer B who pays tax t^Bon good 1 and receives T^Bas a lump-sum transfer:

$$x_1^B = \frac{1}{1+t^B} \quad \text{and} \quad x_2^B = M + T^B - 1$$

- *Comparison.* Comparing the optimal consumption levels found in part (b) with the equilibrium outcomes found in part (c), the tax imposed on any individual $i = A, B$ must hence satisfy

$$\frac{1}{2} = \frac{1}{1+t^i},$$

which would guarantee that equilibrium and socially optimal amounts coincide. Solving for the tax t^i yields $t^i = \$1$. Hence, by setting a tax of $t^i = \$1$ on the consumption of good 1, and returning the tax revenue to this individual in a lump-sum transfer, efficiency is restored, yielding a consumption

$$x_1^i = \frac{1}{1+1} = \frac{1}{2} \text{ of good 1,}$$

and

$$\begin{aligned} x_2^i &= M + T^i - 1 \\ &= M + \frac{1}{2} - 1 = M - \frac{1}{2} \text{ of good 2,} \end{aligned}$$

as described in the socially optimal amounts found in part (b).