Solutions for Problem Set 1

1)

	t=3000			t=1800			t=1200			t=800		
	PRO	ES/UL	UG	PRO	ES/UL	UG	PRO	ES/UL	UG	PRO	ES/UL	UG
A	1400	1075	750	840	775	700	560	625	450	373.33	0	250
B	1000	875	750	600	575	500	400	425	450	266.67	0	250
C	400	575	750	240	275	300	160	125	200	106.67	300	200
D	200	475	750	120	175	300	80	25	100	53.33	500	100

2)

(i) The total amount of claims before the merge is

$$\sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n = x_N.$$

After the merge, we have $n-1$ agents and the total claims are x_{N-1}. We calculate the shares of agents i and j before the merge:

$$y_i = \frac{x_i}{x_N} t \text{ and } y_j = \frac{x_j}{x_N} t.$$

We also calculate the share of the new agent $i+j$:

$$y_{i+j} = \frac{x_i + x_j}{x_{N-1}} t.$$

Now, we want to compare the sum of the shares of agents i and j with the share of agent $i+j$.

$$y_i + y_j = \frac{x_i}{x_N} t + \frac{x_j}{x_N} t = \frac{x_i + x_j}{x_N} t$$

while the share of agent $i+j$ is

$$y_{i+j} = \frac{x_i + x_j}{x_{N-1}} t.$$

However we observe that the total amount of claims has not changed. This is because $x_{N-1} = x_1 + x_2 + \cdots + x_{i+j} + \cdots + x_n = x_1 + x_2 + \cdots + x_i + x_j + \cdots + x_n = x_N$.

Therefore the two quantities are equal, showing that agents are indifferent.

(ii) We calculate the sum of the shares of i and j before the merge:

$$y_i + y_j = x_i + \frac{1}{n}(t - x_N) + x_j + \frac{1}{n}(t - x_N) = x_i + x_j + \frac{2}{n}(t - x_N)$$

And we calculate the share of the "new" agent, $i+j$:

$y_{i+j} = x_i + x_j + \frac{1}{n-1}(t - x_{N-1}) = x_i + x_j + \frac{1}{n-1}(t - x_N)$ because the total amount of claims does not change.

Comparing the two quantities we derive that $y_i + y_j \geq y_{i+j}$ for $n \geq 2$, which makes sense. Therefore we showed that agents prefer not to merge under the equal surplus solution.

(iii) is very demanding so there is no need to put emphasis on it.

2) i) we have that: $x_A + x_B - y_A - y_B = 1000$, so $y_A + y_B = 24229$. Also, $y_A = u_A^2$ and $y_B = u_B^2$, so $u_A^2 + u_B^2 = 24229$, thus $u_A = \sqrt{24229 - u_B^2}$.

ii) Equal sacrifice means that $x_A^{1/2} - y_A^{1/2} = x_B^{1/2} - y_B^{1/2}$, so $y_A^{1/2} - y_B^{1/2} = 27$. We also have that $y_A + y_B = 24229$.

4)

a) Classical Utilitarian:

The optimization problem that we want to solve is:

$$\max_{a,b} \sum_{i \in \{A,B\}} U_i$$
$$s.t. \, a + b = 1$$

So the Lagrangean is:

$$\mathcal{L} = u(a) + \lambda_A u(b) + u(b) + \lambda_B u(a) - \lambda(a + b - 1)$$
$$= (1 + \lambda_B)u(a) + (1 + \lambda_A)u(b) - \lambda(a + b - 1)$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial a} = 0 \Longrightarrow (1 + \lambda_B)u'(a) = \lambda$$
$$\frac{\partial \mathcal{L}}{\partial b} = 0 \Longrightarrow (1 + \lambda_A)u'(b) = \lambda$$
$$a + b = 1$$

If we divide the first two relations we get:

$$\frac{u'(b)}{u'(a)} = \frac{1 + \lambda_B}{1 + \lambda_A} < 1$$

We thus observe that the ratio is smaller than unity because $\lambda_A > \lambda_B$. So the marginal utility of Ann is greater than the marginal utility of Bob, which means that, given that their utility functions have the same functional form and they are concave, Bob's share is greater than Ann's.

b) Egalitarian:

In this case, we want the utilities of the two agents to be equal, therefore:

$$u(a) + \lambda_A u(b) = u(b) + \lambda_B u(a)$$

By manipulations we get that:

$$\frac{u(a)}{u(b)} = \frac{1 - \lambda_A}{1 - \lambda_B} < 1$$

We can conclude that Bob's share is greater than Ann's, as in the previous case.

c) Nash:

In the Nash collective welfare function we want to maximize the product of the utilities of the two agents:

$$\max_{a,b} \prod_{i \in \{A,B\}} U_i$$

$$s.t. a + b = 1$$

The proper Lagrangean is:

$$\mathcal{L} = [u(a) + \lambda_A u(b)][u(b) + \lambda_B u(a)] - \lambda(a + b = 1)$$
$$= u(a)u(b) + \lambda_B u^2(a) + \lambda_A u^2(b) + \lambda_A \lambda_B u(a)u(b)$$

From the FOCs we get:

$$\frac{\partial \mathcal{L}}{\partial a} = 0 \Rightarrow u(b)u'(a) + 2\lambda_B u(a)u'(a) + \lambda_A \lambda_B u(b)u'(a) = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow u(a)u'(b) + 2\lambda_A u(b)u'(b) + \lambda_A \lambda_B u(a)u'(b) = \lambda$$

$$a + b = 1$$

From the first two equations we have that:

$$\frac{u'(a)}{u'(b)} = \frac{u(a) + 2\lambda_A u(b) + \lambda_A \lambda_B u(a)}{u(b) + 2\lambda_B u(a) + \lambda_A \lambda_B u(b)}$$

We want to find out which one is greater, a or b. Let's assume that $a = b$. This means that $u'(a) = u'(b)$ and $u(a) = u(b)$. Then we have that:

$$1 + 2\lambda_A + \lambda_A \lambda_B = 1 + 2\lambda_B + \lambda_A \lambda_B$$

Which cannot be the case since we know by assumption that $\lambda_A > \lambda_B$.

Now let's assume that $a > b$. Then we have that:

$$a > b \Leftrightarrow u(a) > u(b) \Leftrightarrow u'(a) < u'(b)$$

So from the previous ratio, we have that:

$$u(b) + 2\lambda_B u(a) + \lambda_A \lambda_B u(b) > u(a) + 2\lambda_A u(b) + \lambda_A \lambda_B u(a) \Rightarrow$$

$$u(b)(1 + \lambda_A \lambda_B - 2\lambda_A) > u(a)(1 + \lambda_A \lambda_B - 2\lambda_B)$$

We observe that $(1 + \lambda_A \lambda_B - 2\lambda_A) < (1 + \lambda_A \lambda_B - 2\lambda_B)$ since $\lambda_A > \lambda_B$. So, in order for the inequality to hold, we want $u(b) > u(a)$. However, this cannot be the case, since we have $u(a) > u(b)$ from our assumption. Therefore, our hypothesis that $a > b$ is rejected. We can only conclude that $b > a$