Problem Set 1 for Social Choice

1. Four researchers A, B, C and D wish to get a grant from their university in order to run an experiment. Researcher A claims 700€, researcher B claims 500€, researcher C claims 200€ and researcher C claims 100€. The university can only justify a single expense of t €, awarded jointly to both researchers, who will have to agree on a division by themselves. What would their shares be if they decided to distribute the amount according to:
   a) The proportional solution
   b) The equal surplus solution (or uniform losses in case of a deficit)
   c) The uniform gains solution

   Calculate for: i) t=3000, ii) t=1800, iii) t=1200 and iv) t=800.

2. We want to distribute t units of a certain good among n individuals. The individuals' claims on the good are $x_1, x_2, \dots, x_n$.

   (i) Show that under the proportional solution with excess ($\frac{\sum_{i=1}^{n} x_i}{n} \leq t$), an agent is indifferent between merging with another agent and pursuing on his own claim (Hint: Formally, we model the merging of agents i, j with the formation of a "new" agent, $i + j$, whose claim is $x_i + x_j$. The number of agents after the merge is $n - 1$).

   (ii) Show that under the equal surplus solution agents have no incentive to merge, i.e. they are better off on their own.

   (iii) (Harder) Show that under the uniform gains solution with a surplus, agents have no incentive to merge and are weakly better off on their own.

3. Suppose that agent A's income is 15625€, while agent B's income is 9604€ and their utility functions are $u_1(x) = u_2(x) = x^{\frac{1}{2}}$. The local authorities want to raise an amount of 1000€ from taxes.
   i) Find the Utility Possibility Curve for the after-tax incomes of the agents (Hint: you need to find a relation between the utilities of the two agents.)
   ii) What would be the after-tax incomes of agent A and B if the authorities use the equal sacrifice principle? (You don't need to calculate the amounts exactly, it suffices to set up the system of equations)
   iii) Show your results on a graph along with the proportional and uniform losses solution (it should look like Figure 2.5 in Moulin). Once again you need not have the exact result from (ii) but explain approximately

where it should lie based on the functional form of utility.

4) (from Moulin p. 97) We assume two agents, Ann and Bob who want to divide a cake. Suppose that we can decompose their utility functions into two parts: the selfish and the altruistic part. We can write this as:
$U_A(a,b) = u(a) + \lambda_A u(b)$
$U_B(a,b) = u(b) + \lambda_B u(a)$

So, one's utility depends on the direct selfish effect $u(x)$ from the consumption of the cake, as well as on a portion of the other agent's utility, $\lambda_i u(x)$. Therefore, we assume that $\lambda_{A,B} \in [0,1]$. We also assume that Ann is more altruistic than Bob, i.e. $\lambda_A > \lambda_B$.

For the general cases of (a) the Classical Utilitarian, (b) the Egalitarian and (c) the Nash collective utility functions, derive a relation about Ann and Bob's share of the cake and try to figure out who gets the bigger share.