Elegant proof of Arrow's Impossibility Theorem

Impossibility Theorem

- Impossible to have a SWF with *universal domain,* satisfying *independence,* the *Pareto principle,* and *nondictatorship*
- Proof in three simple steps (Sen R&F, p.267):
- 1. Field-expansion lemma: *If a group is decisive over any pair of states, it is decisive*
- *2.* Group-contraction lemma: *If a group (of more than one person) is decisive, then so is some smaller group contained in it.*

Impossibility Theorem

- 3. By Pareto-principle, the group of all individuals is decisive. Since it is finite by successive partitioning (and each time picking the decisive set) we arrive at a decisive individual, who must, thus, be a dictator.

Field Expansion Lemma Proof

- take two pairs of alternative states (x,y) and (a,b) (all distinct – proof similar when not all distinct)
- Group G is decisive over (x,y); we have to show that it is decisive over (a,b) as well.
- By unrestricted domain, let everyone in G prefer a to x to y to b, while all others prefer a to x and y to b, but rank other pairs in any way whatever

Field Expansion Lemma Proof

- By decisiveness of G over (x,y), x is socially preferred to y.
- By Pareto principle, a is socially preferred to x, and y to b.
- By transitivity a is socially preferred to b.

Field Expansion Lemma Proof

- If this result is influenced by individual preferences over any pair other than (a,b), then the condition of IIA would be violated.
- Thus, a must be ranked above b simply by virtue of everyone in G preferring a to b (since others can have any preference whatever over this pair).
- So G is indeed decisive over (a,b).

Group-Contraction Lemma Proof

- Take a decisive group G and partition it into G1 and G2.
- Let everyone in G1 prefer x to y and x to z, with any possible ranking of (y,z), and let
- Let everyone in G2 prefer x to y and z to y, with any possible ranking of (x,z).

Group-Contraction Lemma Proof

- It does not matter what those not in G prefer.
- If, now, x is socially preferred to z then the members of group G1 would be decisive over the pair, since they alone definitely prefer x to z (the others rank the pair in any way).
- If G1 is not to be decisive, we must have z at least as good as x for some individual preferences over (x,z) of nonmembers of G1.

Group-Contraction Lemma Proof

- Take that case, and combine this social ranking (that z is at least as good as x) with the social preference for x over y (a consequence of the decisiveness of G and the fact that everyone in G prefers x to y). By transitivity, z is socially preferred to y. But only G2 members definitely prefer z to y.

Group-Contraction Lemma Proof

- Thus G2 is decisive over this pair (z,y).
- Thus from the Field-Expansion Lemma, G2 is decisive. So either G1 or G2 must be decisive – proving the lemma.