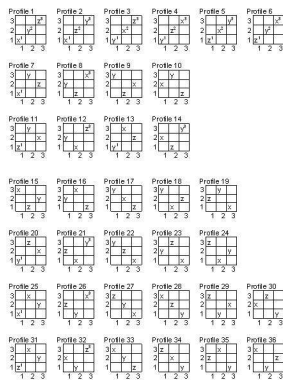


# Another Graphical Proof of Arrow's Impossibility Theorem

Paul Hansen's article

- Two individuals A and B
- Three alternatives x,y,z
- Six possible rankings for each individual: xyz, xzy, yxz, yzx, zxy, zyx
- Six rankings for A and six rankings for B mean 36 pairs of ranking (6x6) comprising of all possible combinations of two individual's ranking of x,y,z



## Profiles

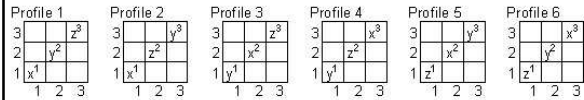
		Profile 1			Profile 7		
B's	3			$z^3$		$y$	
	2		$y^2$				$z$
	1	$x^1$			$x^1$		
		1	2	3	1	2	3
		A's ranking			A's ranking		

- Appreciating the direction of the preference conflicts between individuals is important.
- Conflict  $y_z$  (A prefers y to z and B z to y) is not the same as its inverse  $z_y$  (A prefers z to y and B y to z)
- There are 6 types of preference conflicts across the 36 profiles:  $y_z, y_x, x_z, z_x, z_y, y_x$
- Resolving these conflicts is the essence of welfare economics in general and social-choice rules in particular.

- The central objective...of welfare economics is to provide a framework which permits meaningful statements to be made about whether some alternatives are socially preferable to others" Boadway and Bruce
- We need to aggregate these 36 individual preference profiles into a social ranking. This is achieved by a social-choice rule.

- The essence of Arrow's theorem is that given the assumptions of Pareto, Unrestricted Domain, Independence of Irrelevant Alternatives, a Dictatorship by either A or B is the *only* social choice rule capable of socially ordering (Transitivity) the alternatives of *all* 36 preference profiles
- Emphasis on *all* (unrestricted domain) requires consistency that the rule be universally applicable so that it applies not just to a particular profile that may exist at a point in time but to the full set of profiles that are theoretically possible.

Figure 1



- To completely socially order the remaining profiles (7 to 36) stronger value judgments than the Pareto principle are required because at least one preference conflict is involved in each of these profiles.
- The IIA requires that any value judgment applied to a *particular* conflict in a particular profile must hold for *all* profiles with the *same* type of conflict.
- IIA and the other assumptions ensure that the resulting social orderings of all 36 profiles are identical to the rankings of *one* of the individuals.

## The proof

- Objective to find a social ranking for all 36 profiles: superscript all
- Profiles 1-6 already ranked
- Thus must rank profiles 7-36 one-at-a-time
- Choice of 7 is arbitrary (but whatever we choose a similar proof would follow and a dictatorship would ensue)

- SCR must impose the value judgment that either A gets her way (yz) or B gets her his (zy)
- Suppose arbitrarily that A gets her way and y is *socially* ranked ahead of z (social ordering xyz)
- Key issue henceforth is application of IIA

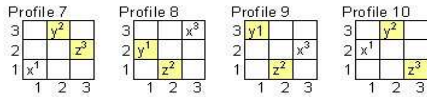
		Profile 7		
B's	3		y	
	2			z
	1	x <sup>1</sup>		
		1	2	3
		A's ranking		

## IIA

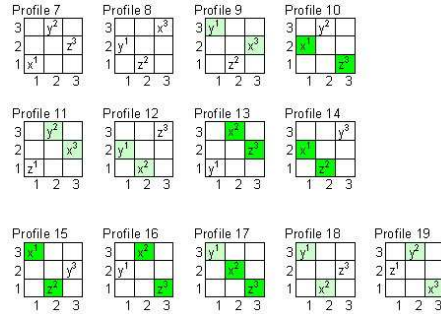
- According to IIA any value judgment imposed by the scr to resolve a particular conflict must hold for *all* other profiles with the *same* type of conflict.
- As noted earlier scr has 6 generic conflicts to resolve
- Hence the social ordering imposed for profile 7 must apply to the 8 other profiles with conflict  $y_z$
- Profiles 8,9,10,16,17,18,22, and 33
- Moreover this social ordering is independent of the ranking of the third alternative x . Whether ranked first in profile 7, second in 17 or 3<sup>rd</sup> in 8. x is irrelevant to the social order of y versus z

### Figure 2

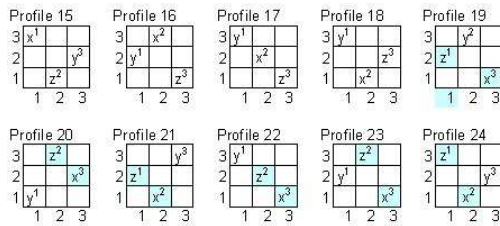
- By IIA social ordering imposed on profile 7 must also be imposed on 8,9,10,16,17,18,22,33
- For 8,9,10 below using IIA, Pareto and Transitivity we arrive a social ordering



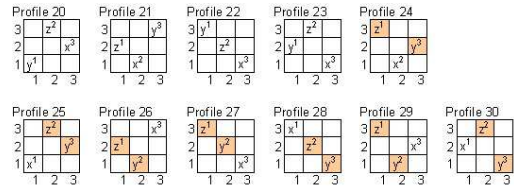
### Figure 3



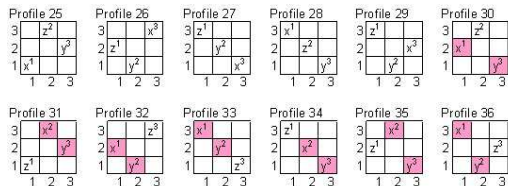
### Figure 4



### Figure 5



### Figure 6



### Generalizing the Dictatorship Result

- Once judgment was made on Profile 7, the force of IIA meant that A's preferences were destined to dictate the social orderings for all 36 profiles.
- Had B's preference been used for Profile 7 then IIA would have led to B being the dictator
- Result is not sufficient to prove AIT. We have proved that *one* set of social orderings corresponds to a dictatorship; it remains to show that no other set of orderings is possible.