# EconS 501 - Micro Theory I[1]
# Recitation #11 - Public Goods

## Exercise 1

[**Public goods with different degrees of publicness**] Consider two consumers (1, 2), each with income $M$ to allocate between two goods. Good 1 provides 1 unit of consumption to its purchaser and $\alpha$, $0 \leq \alpha \leq 1$, units of consumption to the other consumer. Each consumer $i$, $i = 1, 2$, has the utility function $U^i = \log(x_1^i) + x_2^i$, where $x_1^i$ is the consumption of good 1 and $x_2^i$ is the consumption of good 2.

(a) Provide an interpretation of $\alpha$.

(b) Assume that good 2 is a private good. Find the Nash equilibrium levels of consumption when both goods have a price of 1.

(c) By maximizing the sum of utilities, show that the equilibrium is Pareto-efficient if $\alpha = 0$ but inefficient for all other values of $\alpha$.

(d) Now assume that good 2 also provides 1 unit of consumption to its purchaser and $\alpha$, $0 \leq \alpha \leq 1$, units of consumption to the other consumer. For the same preferences, find the Nash equilibrium and show that it is efficient for all values of $\alpha$.

(e) Explain the conclusion in part d.

**Solution:**

(a) The parameter $\alpha$ measures the degree of publicness of the good.

(b) $U^1 = \log(y_1^1 + \alpha y_1^2) + x_2^1$ where $y_1^i$ is the purchase of good 1 by $i$. Using the budget constraint (and assuming both goods have unit price) obtains

$$U^1 = \log\left(y_1^1 + \alpha y_1^2\right) + M - y_1^1.$$

the choice of $y_1^1$ satisfies:

$$\frac{1}{y_1^1 + \alpha y_1^2} - 1 = 0$$

The game is symmetric. So the solution is $y_1^1 = y_1^2 = y_1 = \frac{1}{1+\alpha}$. Hence the consumption level in equilibrium is:

$$x_1^1 = x_1^2 = x_1 = [1 + \alpha]y_1 = 1.$$

(c) The social welfare function is:

$$W = \log(y_1^1 + \alpha y_1^2) + M - y_1^1 + \log(y_1^2 + \alpha y_1^2) + M - y_1^2$$

Applying symmetry yields:

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$$W = 2\log((1+\alpha)y_1) + 2[M - y_1]$$

Hence, taking first order conditions with respect to $y_1$,

$$\frac{\partial W}{\partial y_1} = \frac{2}{y_1} - 2 = 0.$$

Thus, $y_1 = 1$ which entails $x_1 = 1+\alpha$. The Nash equilibrium from part (a), $y_1 = \frac{1}{1+\alpha}$, and the social optimum we just found, $y_1 = 1$, coincide only if $\alpha = 0$. In contrast, when $\alpha \neq 0$, the Nash equilibrium outcome would be diffierent from Pareto-efficient outcome. In particular, when $\alpha \neq 0$, the individual contributions to the public good in the Nash equilibrium are lower than in the social optimum.

(d) Utility now becomes

$$U^1 = \log\left(y_1^1 + \alpha y_1^2\right) + M - y_1^1 + \alpha(M - y_1^2).$$

Taking first-order conditions with respect to $y_1^1$, we obtain individual 1's best response function, $y_1^1 = 1 - \alpha y_1^2$. Similarly operating for individual 2, we find his best response function $y_1^2 = 1 - \alpha y_1^1$. Simultaneously solving for $y_1^1$ and $y_1^2$, we obtain the Nash equilibrium $y_1^1 = y_1^2 = y_1 = \frac{1}{1+\alpha}$.
We can now find the social optimal allocation. With symmetry the social welfare function is:

$$W = 2\log((1+\alpha)y_1) + 2(1+\alpha)[M - y_1],$$

Taking first-order conditions with respect to $y_1$ and solving for $y_1$ yields $y_1 = \frac{1}{1+\alpha}$. The Nash equilibrium and social optimum thus are inow dentical for alll values of $\alpha$.

(e) In part b there is one private good and one public good when $\alpha \neq 0$. So free riding takes place when $\alpha \neq 0$. With $\alpha = 0$, there are two private goods, so the outcome is efficient. In part d both goods have an identical degree of publicness so the consumption externalities are balanced.

## Exercise 2

**(Based on M.W.G. 11.D.4)** Reconsider the nondepletable externality example discussed in section 11.D, but now assume that the externalities produced by the $J$ firms are not homogeneous. In particular, suppose that if $h_1, h_2, ..., h_J$ are the firms' externality levels, then consumer i's derived utility is given by $\phi_i(h_1, h_2, ..., h_J) + w_i$ for each $i = 1, ..., I$. Compare the equilibrium and efficient levels of $h_1, h_2, ..., h_J$. What tax/subsidy scheme can restore efficiency? Under what condition should each firm face the same tax/subsidy rate?

**Solution:**
For the Pareto optimal outcome we solve:

$$\max_{\{h_i\}} \sum_{i=1}^{I} \phi_i(h_1, h_2, ..., h_J) + \sum_{j=1}^{J} \pi_j(h_j)$$

which yields the F.O.C.s

$$\pi_j'(h_j^o) \leq -\sum_{i=1}^{I} \left( \frac{\partial \phi_i(h_1^o, h_2^o, ..., h_J^o)}{\partial h_j} \right) \text{ with equality if } h_j^o > 0 \text{ for all } j = 1, ..., J.$$

On the other hand, in a competitive equilibrium each firm maximizes profits individually, and we get the FOC:

$$\pi_j(h_j^*) \leq \pi_j'(h_j^*) \leq 0, \text{ with equality if } h*_j > 0.$$

To restore Pareto-optimal outcome in a competitive equilibrium, we must set an individual tax for each $j$ of

$$t_j = -\sum_{i=1}^{I} \left( \frac{\partial \phi_i(h_1^o, h_2^o, ..., h_J^o)}{\partial h_j} \right)$$

Each firm will face the same tax rate if and only if we have $\sum_{i=1}^{I} \left( \frac{\partial \phi_i(h_1^o, h_2^o, ..., h_J^o)}{\partial h_j} \right) = \sum_{i=1}^{I} \left( \frac{\partial \phi_i(h_1^o, h_2^o, ..., h_J^o)}{\partial h_k} \right)$ for all $j, k$.

## Exercise 3

**(Based on M.W.G. 11.D.7)** A continuoum of individuals can build their houses in one of two neighborhoods, A or B. It costs $c_A$to build a house in neighborhood A and $c_b < c_A$ to build in neighborhood B. Individuals care about the prestige of the people living in their neighborhood. Individuals have varying levels of prestige, denoted by the parameter $\theta$. Prestige varies between 0 and 1 and is uniformly distributed across the population. The prestige of neighborhood $k$ ($k = A, B$) is a function of the average value of $\theta$ in that neighborhood, denoted by $\overline{\theta}_k$. If individual $i$ has prestige parameter $\theta$ and builds her house in neighborhood $k$, her derived utility net of building costs is $(1+\theta)(1+\overline{\theta}_k)-c_k$. Thus, individuals with more prestige value a prestigious neighborhood more. Assume that $c_A$ and $c_B$ are less than 1 and that $(c_A - c_B) \in (\frac{1}{2}, 1)$.

(a) Show that in any building-choice equilibrium (technically, the Nash equilibrium of the simultaneous-move game in which individuals simultaneously choose where to build their house) both neighborhoods must be occupied.

(b) Show that in any equilibrium in which the prestige levels of the two neighborhoods differ, every resident of neighborhood A must have at least as high a prestige level as every resident of neighborhood B; that is, there is a cutoff level of $\theta$, say $\hat{\theta}$, such that all types $\theta \geq \hat{\theta}$ build in neighborhood A and all $\theta < \hat{\theta}$ build in neighborhood B. Characterize this cutoff level.

(c) Show that in any equilibrium of the type identified in (b), a Pareto improvement can be achieved by altering the cutoff value of $\theta$ slightly and allowing transfers between individuals.

**Solution:**
(a) Assume in negation that only one neighborhood is occupied. First assume it is B, and consider the most prestigious individual with $\theta = 1$. Since $\overline{\theta}_B = \frac{1}{2}$, then this individual's utility from staying in neighborhood B is $(1+1)(1+\frac{1}{2}) - c_B = 3 - c_B \leq 3$. If he would move to neighborhood A his utility would be $(1+1)(1+1) - c_A = 4 - c_A > 3$, so all individuals in neighborhood B cannot be an equilibrium. Now assume that only A is occupied and again consider the most prestigious individual with $\theta = 1$. His utility from staying in the neighborhood A is $(1+1)(1+\frac{1}{2}) - c_A = 3 - c_A$, and his utility from moving to neighborhood B is $(1+1)(1+1) - c_B = 4 - c_B > 3 - c_A$ so all individuals in neighborhood A cannot be an equilibrium - contradiction.

(b) Let an equilibrium be a pair $(\Theta_A, \Theta_B)$, where

$$\Theta_B \equiv \{\theta : \text{ type}\, \theta \,\text{locates in neighborhood}\, i\}$$

, and let $\overline{\Theta}_A$, $\overline{\Theta}_B$ be the average prestige levels associated with such an equilibrium.

<u>*Claim*</u>: $\overline{\Theta}_A$ must take the on the form $[\hat{\theta}, 1]$ for some $\hat{\theta}$.

*Proof*: Assume type $\theta'$ prefers A to B:

$$(1+\theta')(1+\overline{\Theta}_A) - c_A > (1+\theta')(1+\overline{\Theta}_B) - c_B$$

Rearranging gives us: $(1+\theta') \geq \frac{c_A - c_B}{\overline{\Theta}_A - \overline{\Theta}_B}$, which implies that all types locates in which neighborhood, and it is calculated by solving:

$$(1+\hat{\theta})\left(1+\frac{1+\hat{\theta}}{2}\right) - c_A = (1+\hat{\theta})\left(1+\frac{\hat{\theta}}{2}\right) - c_B$$

which yields, $\hat{\theta} = 2(c_A - c_B) - 1$

(c) Starting at the equilibrium with $\hat{\theta}$ as given above, if a small group of individuals from the lower end of neighborhood A move to neighborhood B, then the average prestige in both neighborhoods will rise. In particular, if for some $\varepsilon > 0$ the segment $[\hat{\theta}, \hat{\theta}+\varepsilon]$ moved from A to B, the average prestige in both neighborhoods would rise by $\frac{\varepsilon}{2}$. So, in both neighborhoods, an individual of type $\theta$ who did not move will have a positive change in utility of $(1+\theta)\frac{\varepsilon}{2}$. For a type $\theta$ individual who moved from A to B, there will be a negative change in utility equal to $(1+\theta)\left(1+\frac{\hat{\theta}}{2}+\frac{\varepsilon}{2}\right)-c_B-\left[(1+\theta)\left(1+\frac{1}{2}+\frac{\hat{\theta}}{2}\right)-c_A\right]=(1+\theta)\left(\frac{\varepsilon-1}{2}\right)+(c_A+c_B)$. We denote the total benefit from such a change as $B$, and the total cost as $C$, so that we have:

$$B(\varepsilon)=\int_0^{\hat{\theta}}(1+\theta)\left(\frac{\varepsilon}{2}\right)d\theta+\int_{\hat{\theta}+\varepsilon}^{1}(1+\theta)\left(\frac{\varepsilon}{2}\right)d\theta, \text{ and}$$

$$C(\varepsilon)=\int_0^{\hat{\theta}+\varepsilon}\left[(1+\theta)\left(\frac{\varepsilon-1}{2}\right)(c_A-c_B)\right]d\theta,$$

and we can evaluate the effect of such a change when $\varepsilon=0$:

$$\frac{dB(\varepsilon)}{d\varepsilon}\mid_{\varepsilon=0}=\int_0^{\hat{\theta}}(1+\theta)\left(\frac{1}{2}\right)d\theta+\int_{\hat{\theta}+\varepsilon}^{1}(1+\theta)\left(\frac{1}{2}\right)d\theta-(1+\hat{\theta}+\varepsilon)\left(\frac{\varepsilon}{2}\right)$$

$$\frac{dB(\varepsilon)}{d\varepsilon}\mid_{\varepsilon=0}=\frac{\hat{\theta}}{2}+\frac{\hat{\theta}^2}{2}+\frac{1}{2}+\frac{1}{4}-\frac{\hat{\theta}+\varepsilon}{2}-\frac{\left(\hat{\theta}+\varepsilon\right)^2}{2}-(1+\hat{\theta}+\varepsilon)\left(\frac{\varepsilon}{2}\right)=\frac{3}{4}$$

and

$$\frac{dC(\varepsilon)}{d\varepsilon}\mid_{\varepsilon=0}=\int_0^{\hat{\theta}+\varepsilon}(1+\theta)\left(\frac{1}{2}\right)d\theta+\left[(1+\theta+\varepsilon)\left(\frac{\varepsilon}{2}\right)+c_A-c_B\right]$$

$$\frac{dC(\varepsilon)}{d\varepsilon}\mid_{\varepsilon=0}=\frac{\hat{\theta}+\varepsilon}{2}-\frac{\hat{\theta}}{2}+\frac{\left(\hat{\theta}+\varepsilon\right)^2}{4}-\frac{\hat{\theta}^2}{4}+\left[2(c_A-c_B)+\varepsilon\right]\left(\frac{\varepsilon-1}{2}\right)+c_A-c_B=0$$

Note that the last equality is true since from the conclusion of part (b). In particular, since $\frac{dB(\varepsilon)}{d\varepsilon}\mid_{\varepsilon=0}=\frac{3}{4}\geq 0=\frac{dC(\varepsilon)}{d\varepsilon}\mid_{\varepsilon=0}$, a Pareto improvement can be achieved by altering the cutoff value of $\theta$ slightly and allowing transfers between individuals.