6 Partial and General Equilibrium

In this chapter we study equilibrium allocations and prices in the market of a single good (partial equilibrium) or several goods (general equilibrium). In section 6.1 on partial equilibrium, we implicitly assume that (1) such a good represents a small proportion of the economy, which guarantees that changes in the price of that good do not significantly affect equilibrium conditions in markets of other goods, and (2) the budget share that individuals spend on the good we analyze is relatively minor, and thus its wealth effects are negligible (which allows us to use the change in consumer surplus as a relatively accurate measure of welfare change). In this context, we are particularly interested in identifying prices that guarantee that consumption and production decisions are compatible, so that the market clears, meaning no excess supply or excess demand exists in equilibrium. Last, we evaluate how equilibrium prices and quantities are affected by small changes in some parameters (section 6.2), and apply our results to the analysis of how the introduction of a sales tax impacts equilibrium prices and welfare (section 6.3).

In section 6.4, we study the markets of several goods simultaneously, by first examining, for simplicity, economies without production and later extending our results to economies with production. In the first type of economies (barter economies), individuals are endowed with a set of goods that they can exchange with one another until reaching a satisfactory allocation for all parties. Mathematically, these simultaneous decisions problems by different consumers are equivalent to solving several utility maximization problems (UMPs), one for each consumer, guaranteeing that their Walrasian demands are compatible. In economies with production, however, individuals must first determine how to allocate inputs in the production of different goods, and then market prices help consumption and production decisions clear, so no excess supply or demand arises in any market. In section 6.5, we explore comparative statics results under a general equilibrium setting, where either the price of one good changes or the initial endowment of one of the inputs changes. Section 6.6 examines the effect of sale taxes and of taxes on inputs in a general equilibrium setting. Two appendixes describe large economies, and how equilibrium results naturally arise in this context.

6.1 Partial Equilibrium Analysis

In a competitive equilibrium allocation, all agents must select an optimal allocation given their resources: that is, firms choose their production plans to maximize profits given their technologies, and consumers choose bundles that maximize their utility levels given their budget constraints. A competitive equilibrium allocation will emerge at a price that makes consumers' purchasing plans (as captured by the aggregate demand function) to coincide with the firms' production decisions (as represented by the aggregate supply function). Let us analyze each of these agents starting with the firm.

6.1.1 Firms

For a given price $p^* \in \mathbb{R}_+$, every firm j's equilibrium output level q_j^* must solve the PMP:

$$\max_{q_j \geq 0} \; p^* q_j - c_j(q_j),$$

which yields the necessary and sufficient condition

$$p^* \leq c'_j(q_j^*) \quad \text{with equality if} \quad q_j^* > 0 .$$

In the case of interior solutions, this result states that every firm j operating in a perfectly competitive market increases output until the point at which the marginal cost of producing such output equals market prices, as described in previous chapters.

6.1.2 Consumers

For simplicity, we consider that every consumer in the economy has a quasi-linear utility function $u_i(m_i, x_i) = m_i + v_i(x_i)$, where $m_i > 0$ denotes the numeraire and $v_i(x_i)$ represents the utility from x_i units of the good. Additionally $v_i'(x_i) > 0$, but $v_i''(x_i) < 0$ for all $x_i > 0$, that is, each consumer obtains a positive but diminishing marginal utility from an additional unit of good x_i. Examples of this utility function include $u_i(m, x_i) = m_i + \sqrt{x_i}$ and $u_i(m_i, x_i) = m_i + \ln x_i$.

In this scenario, individual i has an initial endowment of $w_i \geq 0$, and that he owns a share θ_{ij} of firm j, where $\theta_{ij} \in [0,1]$, and for every firm j, $\sum_{i=1}^{I} \theta_{ij} = .$ Hence the total amount of resources that individual i can use to purchase goods is $w_i + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j(q_j^*) \right)$ from his endowment and his participation in the profits of the J firms. Therefore consumer i's UMP is

$$\max_{m_i \in \mathbb{R}_+, x_i \in \mathbb{R}_+} \quad m_i + v_i(x_i)$$

$$\text{subject to} \quad m_i + p^* x_i \leq w_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot q_j^* - c_j(q_j^*)) .$$

Since the budget constraint must hold with equality (by Walras's law), this consumer's UMP can be rewritten as

$$m_i = -p^* x_i + \left[w_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot q_j^* - c_j(q_j^*)) \right],$$

and after we substitute the budget constraint into the objective function, the UMP can be simplified to the following unconstrained maximization problem:

$$\max_{x_i \in \mathbb{R}_+} \quad v_i(x_i) - p^* x_i + \left[w_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot q_j^* - c_j(q_j^*)) \right],$$

where now the only choice variable for consumer i is the amount of good x_i. Taking first-order conditions with respect to x_i yields

$$v'_i(x_i) \leq p^* \text{ with equality if } x_i^* > 0 ,$$

which intuitively states that the consumer increases the amount of good x_i he buys until the point at which the marginal utility he obtains from the last unit exactly coincides with its market price.

Summarizing, an allocation $(x_1^*, x_2^*, ..., x_I^*, q_1^*, q_2^*, ..., q_J^*)$ and a price p* constitute a competitive equilibrium (CE) if

$$p^* \leq c'_j(q_j^*) \text{ with equality if } q_j^* > 0,$$

$$v'_i(x_i) \leq p^* \text{ with equality if } x_i^* > 0, \quad \text{and}$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^* .$$

Note that the previous conditions do not depend on the consumer's initial endowment.[1] We next provide a graphical illustration of the conditions above. Figure 6.1

1. This result arises from quasi-linearity, whereby an increase in the initial endowment raises consumer i's initial wealth. A larger wealth helps him increase the amount consumed of all other goods but leaves his demand of good x_i unaffected. In other words, there are no wealth effects for good x_i.

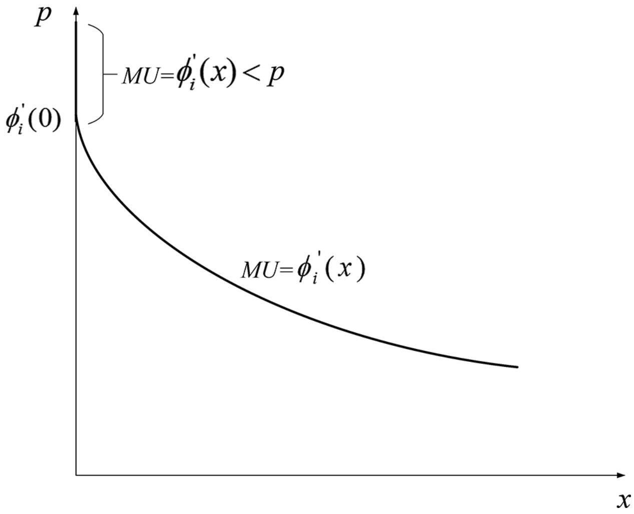

Figure 6.1
Consumer i's demand

represents consumer i's demand for good x_i. For prices above $v_i'(0)$, the consumer's marginal utility from purchasing the first unit of the good is lower than its market price p, leading him to buy zero units of the good. For prices below this cutoff, the consumer purchases a positive amount of the good, increasing x_i until the point in which the utility from buying the last unit coincides with the current market price.[2]

We can now horizontally sum individual demands in order to obtain the aggregate demand for this good originating from individual 1's and 2's demand, as figure 6.2 illustrates. Interestingly, we can identify three segments in this aggregate demand curve. First, when market prices are above $\max_i \ \phi_i'(0)$, no consumer demands a positive amount of the good, implying that aggregate demand is also zero. Intuitively, in this range of (high) market prices the marginal utility that all consumers obtain from buying the first unit of good is still lower than the current market price, and hence no positive units are demanded. For intermediate prices, however, individual 2 in the figure obtains a positive marginal utility from buying positive amounts while individual 1 does not. As a result aggregate demand coincides with individual 2's demand for this range of prices. Last, when market prices are sufficiently low, aggregate demand reflects the horizontal sum of all individuals' demand curves.

2. By inverting the marginal utility function $v_i'(x_i)$, we can obtain this consumer's Walrasian demand $x_i(p)$.

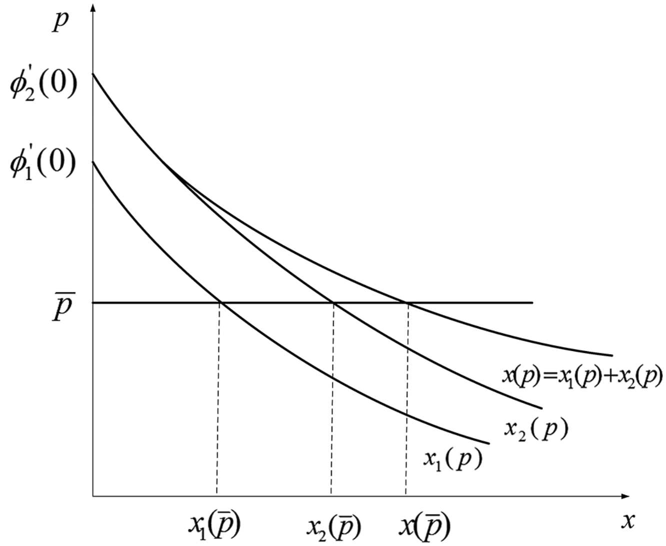

Figure 6.2
Aggregate demand

Let us now examine the firm's supply curve. Figure 6.3 represents the supply curve for an individual firm j. Note that when market prices are sufficiently low, that is, $p < c_j'(0)$, firm j's marginal cost of producing the first unit is higher than current market prices, leading the firm to supply zero units of the good. However, when market prices are above that cutoff, the firm increases production until the point in which the marginal cost of such level of output exactly coincides with the market price the firm obtains from selling those units in the market, that is, $p = c_j'(q_j)$, as described in previous chapters.

Aggregate supply can be obtained by horizontally summing individual supply curves. As in our discussion of individual demand, we can now solve for q_j in $p = c_j'(q_j)$ in order to obtain firm j's supply curve, $q_j(p)$. As in the case of aggregate demand, we can identify three regions in the aggregate demand curve $q(p)$, as figure 6.4 shows. First, when market prices are below the marginal cost of producing the first unit for the most efficient firm (the firm with the lowest marginal cost of production, i.e., firm 2 in our figure), no firm supplies positive units to the market, and aggregate supply is zero. More formally, $\min_j \; c_j'(0) = c_2'(0)$, and hence for all $p < c_2'(0)$, aggregate supply is zero, $q(p) = 0$, in the vertical spike coinciding with the vertical axes in the figure. When market prices are intermediate, only the most efficient firm finds profitable to supply positive units, and the aggregate supply curve coincides with the individual supply for

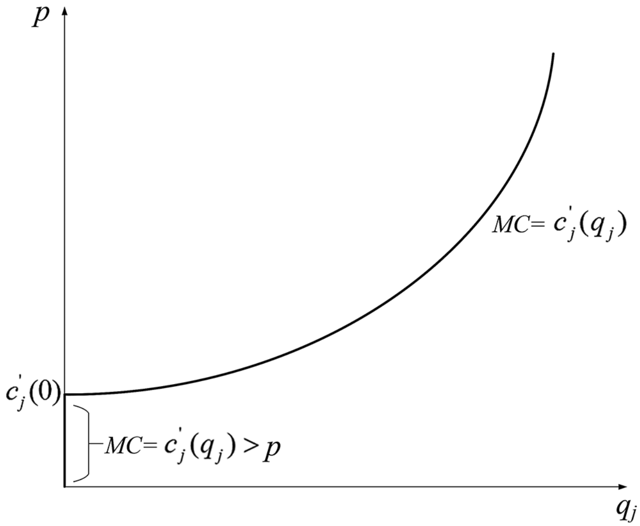

Figure 6.3
Firm j's supply

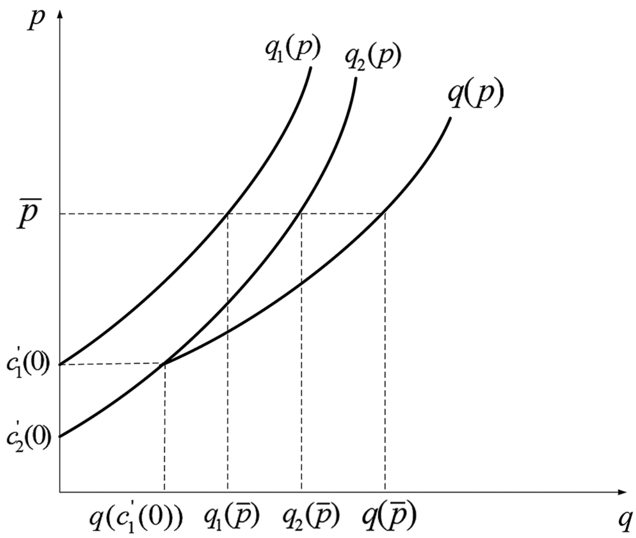

Figure 6.4
Aggregate supply

the most efficient firm (firm 2 in the figure). Note that this occurs for prices above $c_2'(0)$ and below $c_1'(0)$. Finally, when market prices are sufficiently high, that is, $p > c_1'(0)$, both firms supply positive units and, as a consequence, aggregate supply consists of the individual supply of firms 1 and 2.

We can now superimpose aggregate demand and aggregate supply in a single figure in order to obtain the competitive equilibrium allocation of good x. First, note that in order to guarantee that a competitive equilibrium exists (i.e., aggregate demand crosses aggregate supply in figure 6.5), we need to confirm that the equilibrium price p^* satisfies

$$\max_i \; v_i'(0) \geq p^* \geq \min_j \; c_j'(0).$$

Graphically, this condition states that the vertical intercept of the aggregate demand curve lies above that of the aggregate supply curve. Intuitively, this assumption simply implies that, for the goods to be exchanged, the consumer with the highest willingness to pay must assign a value to the first unit, $\max_i v_i'(0)$, that exceeds the marginal cost of this unit for the most efficient firm, $\min_j \; c_j'(0)$. If this condition holds, a competitive equilibrium price p^* exists, entailing that $x(p^*)=q(p^*)$ units of the product are exchanged.

Note that if, instead, $\max_i \; v_i'(0) < \min_j \; c_j'(0)$ holds, we cannot guarantee that there is a positive production or consumption of good x, as figure 6.6 illustrates. Intuitively,

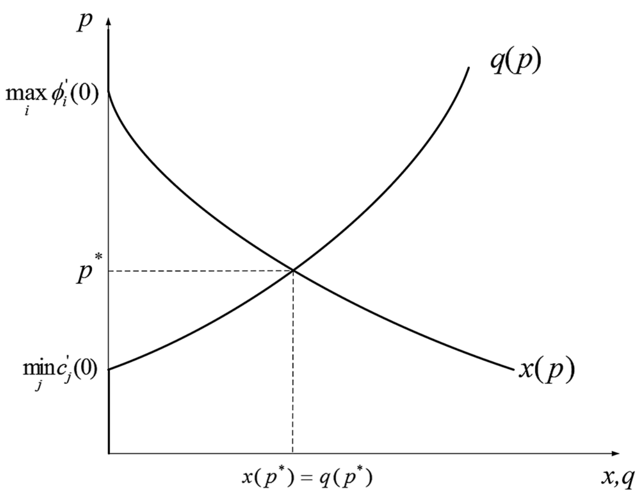

Figure 6.5
Both aggregate demand and aggregate supply combined

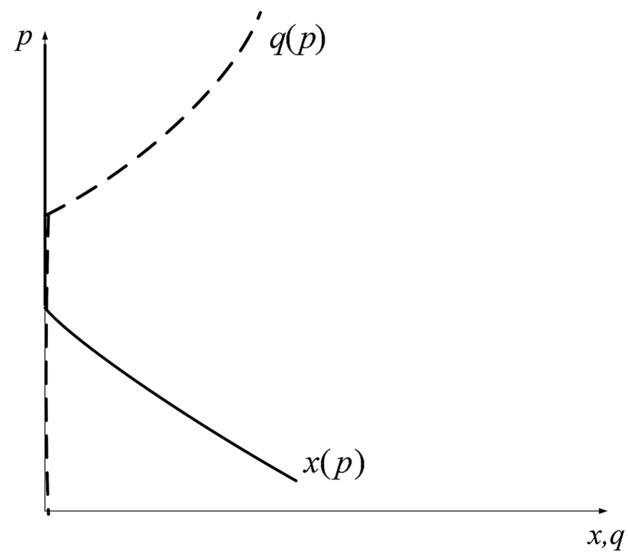

Figure 6.6
No positive production and consumption

this condition indicates that the willingness to pay of the consumer most interested in the good is still lower than the marginal cost of producing this unit for the most efficient firm. As a consequence there is no room for a profitable exchange, and no units of the good are produced or consumed.

Additionally, since the marginal utility $v'_i(x_i)$ is downward sloping for every consumer, $v''_i(x_i)<0$ for all i, and the marginal cost $c'_j(q_j)$ is upward sloping in output for every firm j, $c''_j(q_j)>0$ for all j, aggregate demand and supply cross at a unique point, implying that the CE allocation is unique.

Example 6.1: ***Finding equilibrium conditions*** Suppose that a perfectly competitive industry consists of two types of firms: 100 firms of type A and 30 firms of type B. Each type A firm has a short-run supply curve $s_A(p)=2p$. Each type B firm has a short-run supply curve $s_B(p)=10p$. The Walrasian market demand curve is $x(p)=5000-500p$. Assuming that no more firms enter the industry, we can obtain the short-run equilibrium price as follows:

First, we sum the individual supply curves of the 100 type-A firms and the 30 type-B firms, to obtain an aggregate supply curve of $S(P)=100(2p)+30(10p)=500p$. The short-run equilibrium occurs at the price at which quantity supplied equals quantity demanded,

$$5000 - 500p = 500p, \quad \text{or} \quad p = 5.$$

At this price, each type-A firm supplies $s_A(p) = 2p = 2 \times 5 = 10$ units, and each type-B firm supplies $s_B(p) = 10p = 2 \times 5 = 50$ units. ■

6.1.3 Experiments in Partial and General Equilibria

In the last decades, the sharp theoretical predictions of perfectly competitive markets (mainly, that a precise equilibrium price and quantity are given by the crossing point between demand and supply) were tested and confirmed in many controlled experiments in different countries and subject pools. See, for instance, Smith (1991, 156) and his famous quote "I am still recovering from the shock of the experimental results. The outcome was unbelievably consistent with competitive price theory." The results alluded to were in reference to a "double auction" in which the experimenter assigns a reservation value to every buyer and a reservation price to every seller, and then every seller is allowed to announce the price at which he is willing to sell the good, and every buyer announces the price at which he is willing to buy. In this setting, the experimenter then aggregates the reservation values (prices) for all buyers (sellers, respectively) in order to construct the market demand curve (supply curve) and find the point at which demand and supply cross each other. Such competitive equilibrium price and quantity (the theoretical prediction in this market) were then compared with the experimental results in the lab. Interestingly, while every seller (buyer) in this market only observed his reservation price (value), all sellers converged relatively fast to the equilibrium outcomes. Subsequently Gode and Sunder (1993) experimentally showed that behavior approaches the theoretical prediction even when some subjects are "dumb." The literature has also examined whether individuals in controlled experiments behave as predicted by general equilibrium theory. While the implementation of these markets is more involved than perfectly competitive markets of a single commodity, the results are generally positive as well. For references, see the seminal work of Goodfellow and Plott (1990), the role of credit constraints in Bosch-Domènech and Silvestre (1997), and the effect of money in Lian and Plott (1998) and Hey and Di Cagno (1998).

6.2 Comparative Statics

6.2.1 Competitive Equilibrium Prices

In this section we examine how equilibrium prices are affected by changes in the parameters of the model. Specifically, we will assume that consumers' preferences are affected by a vector of parameters $\alpha \in \mathbb{R}^M$, where $M \leq L$.[3] Hence, consumer i's utility from good x becomes $v_i(x_i, \alpha)$. Similarly firms' technology is affected by a vector of parameters $\beta \in \mathbb{R}^S$, where $S \leq L$, implying that firm j's cost function becomes $c_j(q_j, \beta)$.[4] When bearing a tax, we will use $\hat{p}_i(p,t)$ to denote the effective price paid by consumer i and $\hat{p}_j(p,t)$ to represent the effective price received by firm j.[5] If consumption and production are strictly positive in the CE, then the following conditions must hold:

$$v_i^{'}(x_i^*, \alpha) = \hat{p}_i(p^*, t) \quad \text{for every consumer } i,$$
$$c_j^{'}(q_j^*, \beta) = \hat{p}_j(p^*, t) \quad \text{for every firm } j, \text{and}$$
$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*.$$

We consequently have $I+J+1$ equations that depend on parameter values α, β, and t. In order to understand how optimal consumption bundles x_i^* and profit-maximizing production plans q_j^* depend on parameters α and β, we will use the implicit function theorem as long as the functions above are differentiable. (See the mathematical appendix, section A.14, for a description of the implicit function theorem using examples from consumer theory.)

Example 6.2: *Sales tax* The expression of the aggregate demand now becomes $x(p+t)$, since the effective price that the consumer pays is actually $p+t$, which is to say, the sales tax is equivalent to an increase in the price paid by consumers. In equilibrium, the market price after imposing the tax, $p^*(t)$, must hence satisfy

$$x(p^*(t)+t) = q(p^*(t)).$$

Thus, if the sales tax is marginally increased, and functions are differentiable at $p=p^*(t)$, we obtain

$$x'(p^*(t)+t) \cdot [p^{*\prime}(t)+1] = q'(p^*(t)) \cdot p^{*\prime}(t).$$

3. This implies that there are fewer parameters than goods. This normally facilitates identification issues when the results of the model are empirically tested.

4. This also assumes that there are fewer parameters affecting the firm's production decision than goods.

5. Hence, in order to denote a per unit tax (charged on every unit sold), we use $\hat{p}_i(p,t) = p+t$, where the consumer's total expenditure when buying q units of that good thus becomes $pq+tq=(p+t)q$. In contrast, to denote an ad valorem tax (i.e., a sales tax), we use $\hat{p}_i(p,t) = p+pt = p(1+t)$, where the consumer's total expenditure on that good now becomes $pq+tpq=(1+t)pq$.

After rearranging, we have

$$p^{*\prime}(t)\cdot\left[x'(p^*(t)+t)-q'(p^*(t))\right]=-x'(p^*(t)+t).$$

Hence

$$p^{*\prime}(t)=-\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'(p^*(t))}.$$

Since the aggregate demand function $x(p)$ is decreasing in prices, $x'(p^*(t)+t<0$, and the aggregate supply function $q(p)$ is increasing in prices, $q'(p^*(t))>0$, then $x'(p^*(t)+t)<0<q'(p^*(t))$, and we can determine the sign of the ratio above:

$$p^{*\prime}(t)=-\frac{x'(p^*(t)+t)}{\underbrace{x'(p^*(t)+t)}_{-}-\underbrace{q'(p^*(t))}_{+}}=-\frac{(-)}{(-)}=(-).$$

Hence $p^{*\prime}t<0$. However, the ratio above is larger than -1, which implies that $p^{*\prime}(t)$ lies in the interval $(-1,0]$. Therefore we can conclude that the equilibrium price $p^*(t)$ decreases in t, which means that the price received by producers falls in the tax but less than proportionally. In other words, a 1 percent increase in the tax produces a reduction in $p^*(t)$ of *less* than 1 percent. Additionally, since $p^*(t)+t$ is the price paid by consumers, then $p^{*\prime}(t)+1$ is the marginal increase in the price paid by consumers when the tax marginally increases. Since $p^{*\prime}(t)\in(-1,0)$, then $p^{*\prime}(t)+1<1$, and the consumers' cost of the product also raises less than proportionally with taxes.

Figure 6.7 summarizes the effect that the imposition of a tax produces on the competitive equilibrium price and quantity. Before the introduction of the tax, CE occurs at $p^*(0)$ and $x^*(p(0))$, where the aggregate demand $x(p)$ and aggregate supply $q(p)$ cross each other. The imposition of the tax produces a downward shift in the aggregate demand curve from $x(p)$ to $x(p+t)$, without affecting the supply curve, $q(p)$. (Note that the vertical distance between these two curves is equal to the tax, t, at any output level q.) This implies that the new CE, after the introduction of the tax, occurs at a lower output level, decreasing output from $x^*(p(0))$ to .egarding prices, note that consumers pay $p^*(t)+t$ after the imposition of the tax, rather than $p^*(0)$ before the tax was introduced, while producers receive a price $p^*(t)$ for the $x^*(t)$ units they sell after the tax is introduced rather than the price $p^*(0)$ they received before the tax was implemented.

6.2.2 Extreme Cases

We can examine the effect of the tax when the supply curve is very responsive to price changes, which is when the derivative $q'(p^*(t))$ is large. In such a case the change in the equilibrium price after introducing the tax becomes

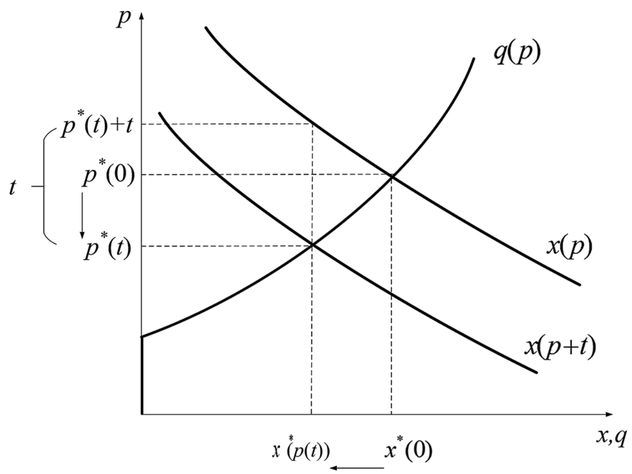

Figure 6.7
Effect of tax

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'\left(p^*(t)\right)} \to 0,$$

since the denominator becomes a large negative number. Therefore $p^{*\prime}(t) \to 0$, and the price received by producers before the tax, $p^*(0)$, does not fall after the introduction of the tax, $p^*(t)$, as depicted in figure 6.8, which describes a perfectly elastic supply curve whose $q'(p^*(t))$ is very large. However, consumers still have to pay $p^*(t)+t$. A marginal increase in taxes therefore provides an increase in the consumer's price of $q^{*\prime}(t)+1=1+0=1$. That is, the tax is solely borne by consumers. Moreover, as figure 6.8 illustrates, the price paid by consumers increases by exactly the amount of the tax.

If, in contrast, the supply curve is not responsive to price changes, meaning $q'(p^*(t))$ is close to zero, then the change in the equilibrium price as a result of the tax is

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - \underbrace{q'\left(p^*(t)\right)}_{0}} = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)} = -1.$$

Therefore $p^{*\prime}(t) \to -1$, and the price received by producers falls in \$1 for every extra dollar in taxes, shifting to producers all the tax burden. In contrast, consumers pay $p^*(t)+t$. A marginal increase in taxes hence produces an increase in consumer's price of $p^{*\prime}(t)+1=-1+1=0$. That is to say, consumers do not bear the tax burden at all. This is illustrated in figure 6.9, where consumers' cost of the good does not increase, from

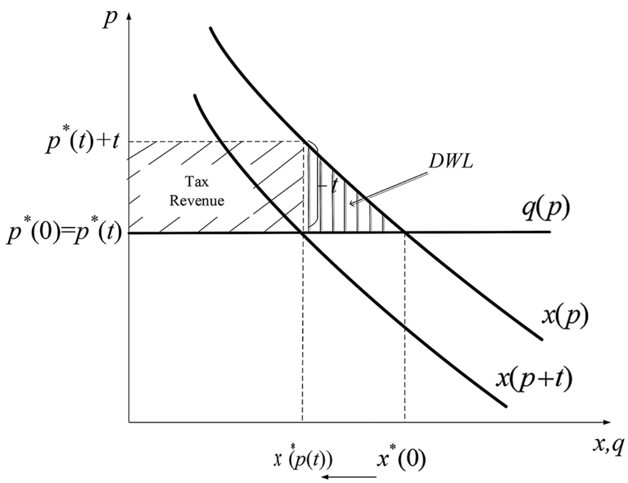

Figure 6.8
Supply curve when very responsive to price changes

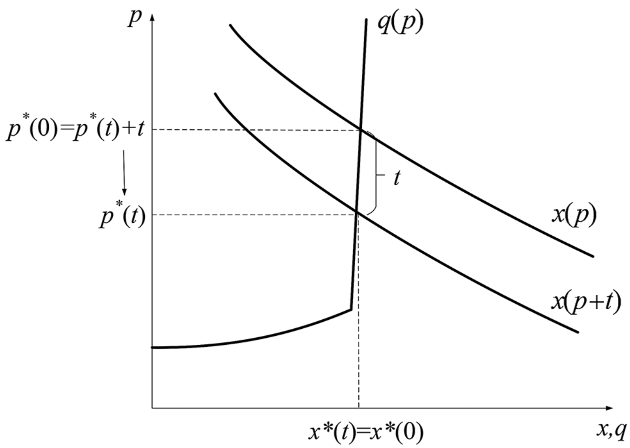

Figure 6.9
Supply curve when not responsive to price changes

$p^*(0)$ before the tax to $p^*(t)+t$ after the tax, whereas the price received by producers falls by \$1 for every extra dollar in taxes, that is, from $p^*(0)$ before the tax to $p^*(t)$ after the tax. (For more on tax incidence and comparative statics, see appendixes A and B at the end of the chapter.) ■

Example 6.3: *Ad valorem taxes* Consider a competitive market in which the government will be imposing an ad valorem tax t. Aggregate demand curve is $x(p)=Ap^{\varepsilon}$, where .. and $\varepsilon<0$, and aggregate supply curve $q(p)=ap^{\gamma}$, where $a>0$ and $\gamma>0$.

To compute the change in the price received by producers, we use the equation measuring a marginal increase in taxes $p^{*\prime}(0)$ that we found above:

$$p^{*\prime}(0)=\frac{x'(p_*)}{x'(p_*)-q'(p_*)}=-\frac{A\varepsilon p_*^{\varepsilon-1}}{A\varepsilon p_*^{\varepsilon-1}-a\gamma p_*^{\gamma-1}}=-\frac{A\varepsilon p_*^{\varepsilon}}{A\varepsilon p_*^{\varepsilon}-a\gamma p_*^{\gamma}}=$$

$$=-\frac{\varepsilon x(p^*)}{\varepsilon x(p^*)-\gamma q(p^*)}=-\frac{\varepsilon}{\varepsilon-\gamma}.$$

(We have multiplied both the numerator and the denominator by p^* and used the fact that p^* is an equilibrium price, which entails $x(p^*)=q(p^*)$.) The price paid by consumers is $(p^*)+t$, and its derivative with respect to t at $t=0$ is

$$p'(0)+1=-\frac{\varepsilon}{\varepsilon-\gamma}+1=-\frac{\gamma}{\varepsilon-\gamma}.$$

Using this expression, we can obtain the following effects on prices:

- When $\gamma=0$ (supply is perfectly inelastic), the price paid by consumers is unchanged, but the price received by producers decreases by the amount of the tax. That is, producers bear the full effect of the tax while consumers are essentially unaffected.
- When $\varepsilon=0$ (demand is perfectly inelastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax. That is, consumers bear the full burden of the tax.
- When $\varepsilon\to-\infty$ (demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax. In contrast, when $\gamma\to\infty$ (supply is perfectly elastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax.

6.3 Welfare Analysis

When evaluating how a change in the competitive equilibrium allocation due to a change in some parameters (e.g., after the introduction of a tax) modifies aggregate social welfare, we use aggregate surplus. This surplus captures the difference between the total benefit from consumption and the total cost of production:

$$S = \sum_{i=1}^{I} v_i(x_i) - \sum_{j=1}^{J} c_j(q_j) .$$

After taking a differential change in the quantity of one of the goods such that aggregate output of this commodity is unaffected, $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j$, we find that the change in the aggregate surplus is

$$dS = \sum_{i=1}^{I} v_i'(x_i)dx_i - \sum_{j=1}^{J} c_j'(q_j)dq_j .$$

Since the marginal benefit from additional units of consumption $v_i'(x_i)$ coincides with the inverse demand function $p(x)$ for all consumers (i.e., every individual consumes until his marginal benefit from additional units is equal to the market price), and $c_j'(q_j) = c'(q)$ for all firms (i.e., every firm j's marginal cost of its equilibrium production coincides with the aggregate marginal cost), we can rewrite the expression as

$$dS = \sum_{i=1}^{I} p(x)dx_i - \sum_{j=1}^{J} c'(q)dq_j ,$$

and after rearranging, we obtain

$$dS = p(x)\sum_{i=1}^{I} dx_i - c'(q)\sum_{j=1}^{J} dq_j .$$

But because $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j = dx$, and $x = q$ by market feasibility, we have

$$dS = \left[p(x) - c'(x)\right]dx .$$

Hence the change in surplus of a marginal increase in consumption (and production) reflects the difference between the consumers' additional utility and firms' additional cost of production. This intuition is graphically represented in figure 6.10, where the differential change in surplus produced by a marginal increase in x, from x_0 to x_1, is depicted in the vertical distance between the marginal benefit that consumers obtain from additional units the good and the marginal cost that firms incur in order to produce those additional units.

We can integrate the same expression to eliminate the differentials, and obtain the total surplus for an aggregate consumption level x, as follows:

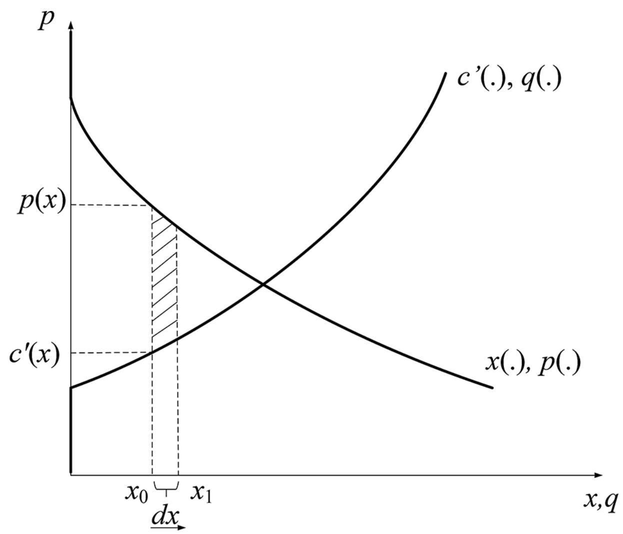

Figure 6.10
Differential change in surplus

$$S(x) = S_0 + \int_0^x p(s) - c'(s)ds ,$$

where $S_0 = S(0)$ is the constant of integration, and the aggregate surplus when aggregate consumption is zero, $x = 0$.[6] Figure 6.11 shows the aggregate surplus for a given aggregate consumption level x.

A natural question at this point is: For which consumption level is aggregate surplus $S(x)$ maximized? By differentiating the expression of $S(x)$ with respect to$x$, we obtain the first-order necessary condition

$$S'(x^*) = p(x^*) - c'(x^*) \leq 0,$$

or after rearranging,

$$p(x^*) \leq c'(x^*).$$

6. Many economics applications consider that consumers' utility from consuming zero units is zero, and that the cost of producing zero units is zero, and thus omit this constant of integration in their analysis, which is $S_0 = 0$.

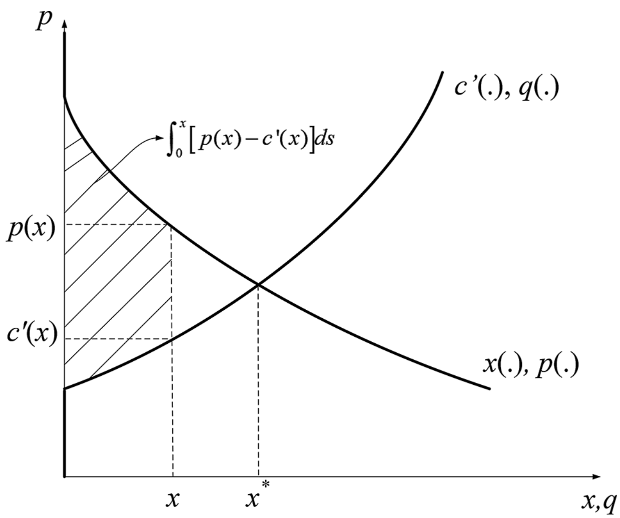

Figure 6.11
Surplus at aggregate consumption x

Then we write the second-order (sufficient) condition as

$$S''(x^*) = \underbrace{p'(x^*)}_{-} - \underbrace{c''(x^*)}_{+} < 0.$$

The expression above is negative, since $p'(x^*) < 0$, given that the inverse demand function decreases in output, and $c''(x^*) \geq 0$, since firms' costs are convex in output (and therefore aggregate production costs are convex as well). Hence $S''(x) < 0$ and the surplus $S(x^*)$ is concave in output, implying that the level of output x^* that we found in the first-order condition constitutes a maximum of $S(x)$. In addition, when $x^* > 0$ (interior solutions) aggregate surplus $S(x)$ is maximized for an output level where $p(x^*) = c'(x^*)$. This implies that the aggregate surplus $S(x)$ is maximized at the competitive equilibrium allocation, where $p(x^*)$ crosses $c'(x^*)$. This could be anticipated by a visual examination of figure 6.11, where the shaded region representing $S(x)$ increases until output reaches $x = x^*$. Therefore the CE allocation maximizes aggregate surplus, which is to say, a benevolent planner would allocate production resources and consumption decisions in the exact same way that the perfectly competitive market did in the CE allocation. (This result is often referred to as the "first welfare theorem," and we describe it in more detail in the section on general equilibrium that we study next.)

Example 6.4: ***Aggregate surplus*** Consider a market with aggregate demand $x(p) = a - bp$ and aggregate supply curve$y(p) = J(p/2)$, where $a, b > 0$ and $J > 1$ denotes the number of firms in the industry. The CE price solves

$$a - bp = J\frac{p}{2}$$

or $p^* = 2a/(2b + J)$, which increases in the vertical intercept of aggregate demand but decreases in the number of firms. Therefore the equilibrium output is $x^* = a - b(2a/(2b + J) = aJ/(2b + J)$. In this context, the surplus is

$$S(x^*) = \int_0^{x^*} p(x) - c'(x)dx,$$

where $p(x)$ is the inverse aggregate demand function. We solve for p in $x(p) = a - bp$ to obtain the indirect demand $p(x) = (a - x)/b$. Then, to find the aggregate marginal cost $c'(x)$, we solve for p in $y(p) = J(p/2)$ and get $p(x) = c'(x) = 2x/J$. Substituting these values yields

$$S(x^*) = \int_0^{aJ/(2b+J)} \left(\frac{a - x}{b} - \frac{2x}{J}\right)dx = \frac{a^2 J}{4b^2 + 2bJ},$$

which is increasing in the number of firms J, since $\partial S(x^*)/\partial J = a^2(2b + J)^2 > 0$. ■

6.4 General Equilibrium

We now extend our discussion of equilibrium conditions in markets with a representative consumer to markets with multiple consumers (each consumer with potentially different preferences). We seek to evaluate under which price conditions the agents' demands for different goods are compatible with one another given the initial endowment of goods in the economy. For simplicity, we start with equilibrium allocations in economies without production (called "barter equilibrium," since consumers exchange units of the goods they are initially endowed with), and subsequently analyze economies with production. At the end of the chapter we test our equilibrium results in large economies, and finally explore some comparative statics.

6.4.1 Economies without Production

Consider an economy with two goods and two consumers, $i = \{1, 2\}$, each initially endowed with $\mathbf{e}^i \equiv (e_1^i, e_2^i)$ units of good 1 and 2, respectively. Figure 6.12a depicts the a.

b.

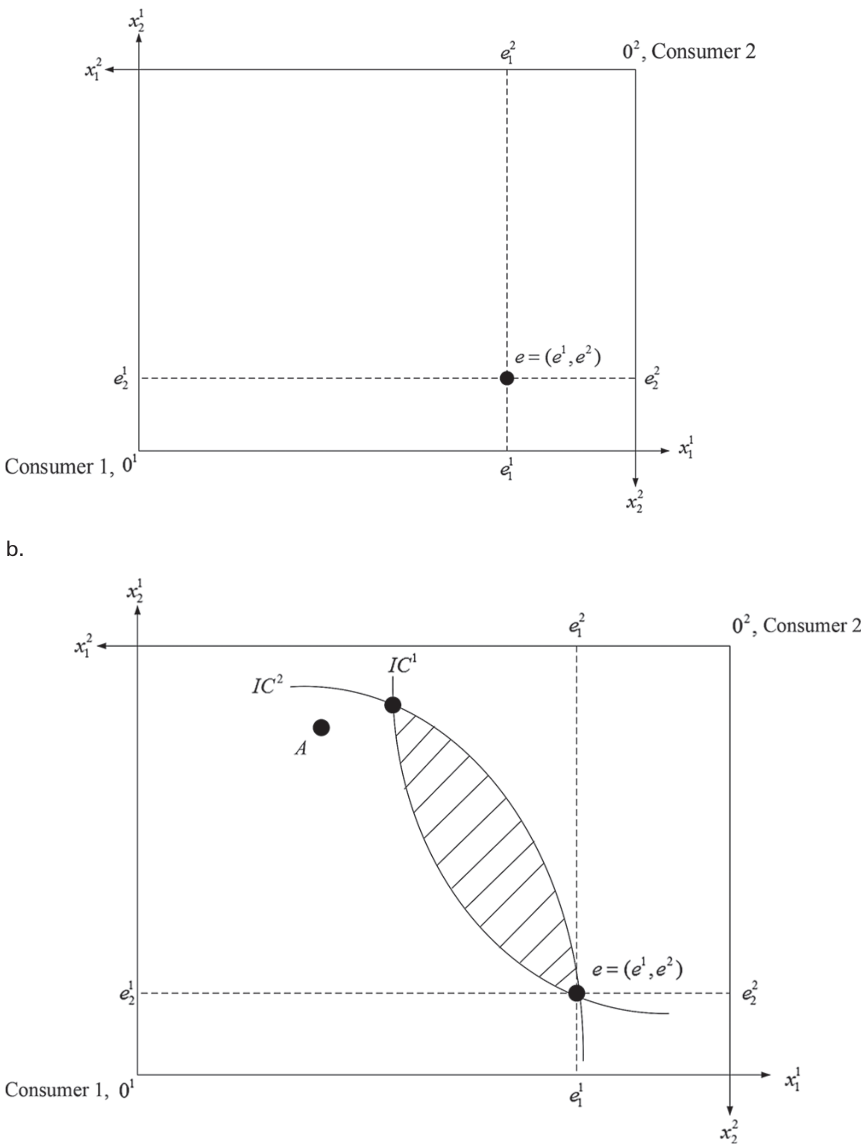

Figure 6.12
Two types of Edgeworth boxes

so-called Edgeworth box, with consumer 1's origin in the lower left-hand corner and consumer 2's origin at the opposite point of the box (the upper right-hand corner; if you cannot see that rotate the page 180 degrees). The figure also includes the initial endowment $\mathbf{e} \equiv (\mathbf{e}^1, \mathbf{e}^2)$, while any other allocation $\mathbf{x} \equiv (\mathbf{x}^1, \mathbf{x}^2)$ could similarly be depicted as a point in the box.

Figure 6.12b adds IC^1, the indifference curve of consumer 1 passing through his endowment point $\mathbf{e}^1$, thus depicting bundles in the box that yield the same utility level as $\mathbf{e}^1$ for consumer 1. The figure also includes the indifference curve through the endowment point for consumer 2, IC^2, and shades the region of bundles in the lens-shaped area between both consumers' indifference curves. More formally, the shaded area represents the set of bundles (x_1^1, x_2^1) for consumer 1 and (x_1^2, x_2^2) for consumer 2, satisfying

$$u^1(x_1^1, x_2^1) \geq u^1(e_1^1, e_2^1) \quad \text{and}$$

$$u^2(x_1^2, x_2^2) \geq u^2(e_1^2, e_2^2).$$

Hence a movement from the initial endowment $\mathbf{e}$ to allocation A (which lies outside the lens-shaped area) cannot be a barter equilibrium, since consumer 1 is worse off at$A$; thus he would oppose a proposal to exchange $\mathbf{e}$ for A. Does that imply that any point in the lens-shaped area is a barter equilibrium? Not necessarily. Consider bundle B in figure 6.13. Despite lying inside the lens-shaped area, and thus yielding a higher utility level than the initial endowment $\mathbf{e}$ for both consumers, individuals could still find other points, such as D, that would make both of them better off than at B. Generally, any point on the cc curve depicted in figure 6.13 (often referred to as the "contract curve" in which indifference curves are tangent to one another) would be an equilibrium, since Pareto improvements are no longer possible. As we show in the next sections, while the contract curve depicts Pareto efficient allocations, only its portion lying inside the lens-shaped area constitutes a barter equilibrium.

The graphical presentation in figure 6.13 helped us in our initial search of a definition of equilibrium allocations. Nonetheless, before providing such a definition, we first need to define some additional ingredients. In particular, since allocations can only be part of an equilibrium if they are feasible, we still need to clarify which allocations are feasible, as well as which allocations can be blocked by one or more individuals in the economy.

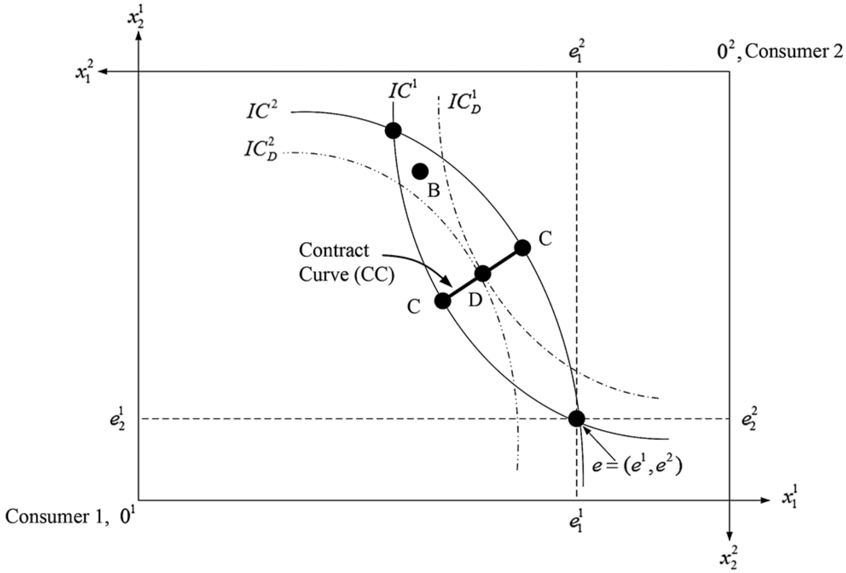

Figure 6.13
Contract curve

Feasible allocation An allocation $\mathbf{x} \equiv (\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^I)$ is feasible if it satisfies $\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i$. That is, the aggregate amount of goods in allocation $\mathbf{x}$, when summing over all individuals $i = 1, 2, \dots, I$, does not exceed the aggregate initial endowment $\mathbf{e} \equiv \sum_{i=1}^{I} \mathbf{e}^i$.

Pareto efficient allocations A feasible allocation $\mathbf{x}$ is Pareto efficient if there is no other feasible allocation $\mathbf{y}$ that is weakly prefered by all consumers, meaning $\mathbf{y}^i \succsim \mathbf{x}^i$ for all $i \in I$ and is strictly preferred by at least one consumer, $\mathbf{y}^i \succ \mathbf{x}^i$.

That is, allocation $\mathbf{x}$ is Pareto efficient if there is no other feasible allocation $\mathbf{y}$ making all individuals at least as well off as under $\mathbf{x}$ and making one or more individual strictly better off. Intuitively, we cannot rearrange the bundles each consumer has in order to make at least one of them better off than under $\mathbf{x}$, without making others worse off. Mathematically, we can define the set of Pareto efficient allocations as the vector $(\mathbf{x}^1, \dots, \mathbf{x}^I)$ that solves

$$\max_{\mathbf{x}^1,\ldots,\mathbf{x}^I \geq 0} \; u^1(\mathbf{x}^1)$$

subject to $\quad u^j\left(\boldsymbol{x}^j\right) \geq \bar{u}^j \quad$ for every individual $j \neq i$, and

$$\sum_{i=1}^{I} \boldsymbol{x}^i \leq \sum_{i=1}^{I} \boldsymbol{e}^i \qquad \text{(feasibility)}$$

where $\mathbf{x}^i = (x_1^i, x_2^i)$. That is, an allocation $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$ is Pareto efficient if it maximizes individual 1's utility without reducing the utility of all other individuals below a given level $\bar{u}^j$ and satisfying feasibility (which in a two-consumer economy implies that $\mathbf{x}^1 + \mathbf{x}^2 \leq \mathbf{e}^1 + \mathbf{e}^2$). (Generally, such a problem can be specified as maximizing the utility level of any individual i without reducing the utility level of any other individual j.) The Lagrangian associated to this maximization problem is

$$L(\mathbf{x}^1,\ldots,\mathbf{x}^I;\lambda^2,\ldots,\lambda^I,\mu) = u^1(\mathbf{x}^1) + \lambda^2[u^2(\mathbf{x}^2) - \bar{u}^2] + \ldots + \lambda^I[u^I(\mathbf{x}^I) - \bar{u}^I]$$
$$+\mu\left[\sum_{i=1}^{I}\mathbf{e}^i - \sum_{i=1}^{I}\mathbf{x}^i\right].$$

Taking first-order conditions with respect to $\mathbf{x}^1 = (x_1^1, x_2^1)$ yields $\partial L/\partial x_k^1 = \left(\partial u^1(\mathbf{x}^1)/\partial x_k^1\right) - \mu \leq 0$ for every good $k = \{1,2\}$ of consumer 1, whereas when we take first-order conditions with respect to and $\mathbf{x}^j = (x_1^j, x_2^j)$ for any individual $j \neq 1$, we obtain $\partial L/\partial x_k^j = \lambda^j\left(\partial u^j(\mathbf{x}^j)/\partial x_k^j\right) - \mu \leq 0$. Finally, taking first-order conditions with respect to Lagrange multipliers λ^j and μ yields the constraints $u^j(\mathbf{x}^j) \geq \bar{u}^j$ and $\sum_{i=1}^{I}\mathbf{x}^i \leq \sum_{i=1}^{I}\mathbf{e}^i$, respectively. In the case of interior solutions, the combination of these first-order conditions produces a compact condition for Pareto efficiency

$$\frac{\partial u^1(\mathbf{x}^1)/\partial x_1^1}{\partial u^1(\mathbf{x}^1)/\partial x_2^1} = \frac{\partial u^j(\mathbf{x}^j)/\partial x_1^j}{\partial u^j(\mathbf{x}^j)/\partial x_2^j}, \quad \text{or} \quad MRS_{1,2}^1 = MRS_{1,2}^j$$

for every consumer $j \neq 1$. That is, the marginal rate of substitution between goods 1 and 2 ($MRS_{1,2}$) must coincide across all individuals in this economy. (The result above easily extends to the case of economies with more than two goods, so that the *MRS* between any two goods k and l must coincide across all individuals in the economy, $MRS_{k,l}^1 = MRS_{k,l}^j$.) Graphically, their indifference curves become tangent to one another at the Pareto efficient allocations (PEAs). Intuitively, if we tried to increase the utility of any consumer, we would need to make other consumer/s worse off. The next example applies this result to a setting where individual preferences are of the Cobb–Douglas type.

Example 6.5: *Finding Pareto efficient allocations* Consider a barter economy with two goods, 1 and 2, and two consumers, A and B, each with the following initial endowments: $\mathbf{e}^A=(100, 350)$ and $\mathbf{e}^B=(100, 50)$. For simplicity, assume both consumers' utility function is a Cobb–Douglas type given by $u^i(x_1^i, x_2^i) = x_1^i x_2^i$ for all individual $i=\{A, B\}$. Let us find the set of PEAs. Given regular preferences, such allocations are reached at points where the indifference curves of both consumers are tangent to one another, which is where their slopes satisfy $MRS^A = MRS^B$. In this context, $MRS^A = MRS^B$ implies $x_2^A / x_1^A = x_2^B / x_1^B$, or $x_2^A x_1^B = x_2^B x_1^A$. Using the feasibility requirement, $e_1^A + e_1^B = x_1^A + x_1^B$ for good 1 and $e_2^A + e_2^B = x_2^A + x_2^B$ for good 2, we obtain $x_1^B = e_1^A + e_1^B - x_1^A$ and $x_2^B = e_2^A + e_2^B - x_2^A$. Combining the tangency condition, $x_2^A x_1^B = x_2^B x_1^A$, and feasibility yields

$$x_2^A(\underbrace{e_1^A + e_1^B - x_1^A}_{x_1^B}) = (\underbrace{e_2^A + e_2^B - x_2^A}_{x_2^B})x_1^A,$$

which can be rewritten as

$$x_2^A = \frac{e_2^A + e_2^B}{e_1^A + e_1^B} x_1^A = \frac{350+50}{100+100} x_1^A = \frac{400}{200} x_1^A ,$$

or, more compactly, $x_2^A = 2x_1^A$ for all $x_1^A \in [0, 200]$. Figure 6.14 depicts the line representing the set of PEAs (the contract curve), $x_2^A = 2x_1^A$. ■

We are now ready to use these definitions in order to identify what we mean by an individual (or group of individuals) blocking a given allocation, that is, the formation of a blocking coalition of S individuals in an economy with I individuals.

Blocking coalitions Let $S \subset I$ denote a coalition of consumers. We say that S blocks the feasible allocation $\mathbf{x}$ if there is an allocation $\mathbf{y}$ meeting two conditions:

1. *Allocation is feasible for S.* The aggregate amount of goods that individuals in S enjoy in allocation $\mathbf{y}$ coincides with their aggregate initial endowment, $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$.
2. *Allocation is Pareto superior for S.* Allocation $\mathbf{y}$ makes all individuals in the coalition weakly better off than under $\mathbf{x}$, $\mathbf{y}^i \succsim \mathbf{x}^i$ where $i \in S$, and it makes at least one individual strictly better off, $\mathbf{y}^i \succ \mathbf{x}^i$.

The following definitions form the " building blocks" of our definition of equilibrium in a barter economy:

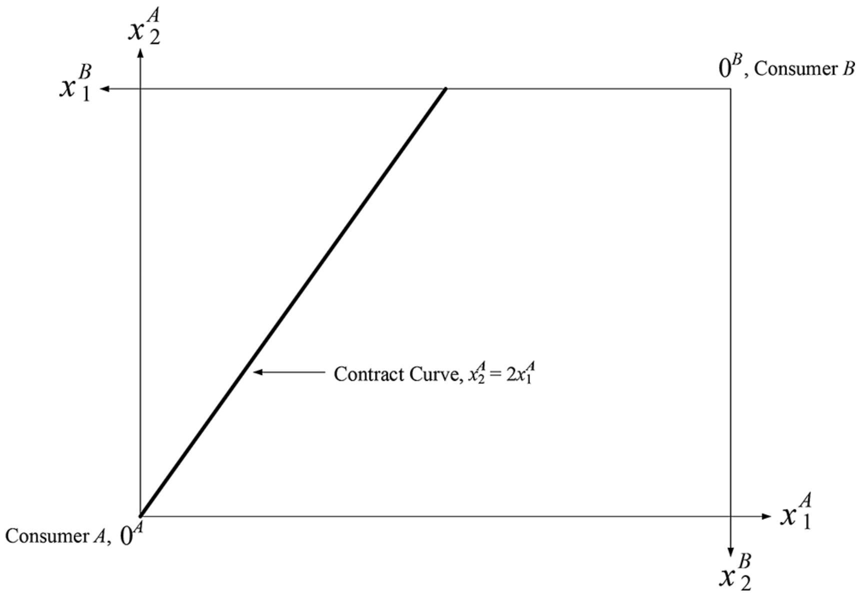

Figure 6.14
Contract curve $x_2^A = 2x_1^A$

Equilibrium A feasible allocation **x** is an equilibrium in the exchange economy with initial endowment **e** if **x** is *not blocked* by any coalition of consumers.

Intuitively, we can claim that a feasible allocation **x** is an equilibrium if there is no group of individuals S that could form a blocking coalition against **x** by finding a feasible allocation **y** that makes one of its members strictly better without harming any of the other members in S. Hence we can group together all equilibrium allocations in what is called the "core" of an exchange economy.

Core The core of an exchange economy with endowment **e**, denoted $C(\mathbf{e})$, is the set of all unblocked feasible allocations.

Intuitively, it represents those allocations that are unblocked: (1) they are mutually beneficial for all individuals (i.e., they lie in the lens-shaped area), and (2) they do not allow for further Pareto improvements (i.e., lie on the contract curve). As depicted in figure 6.15, the set of core allocations is therefore the segment of the contract curve

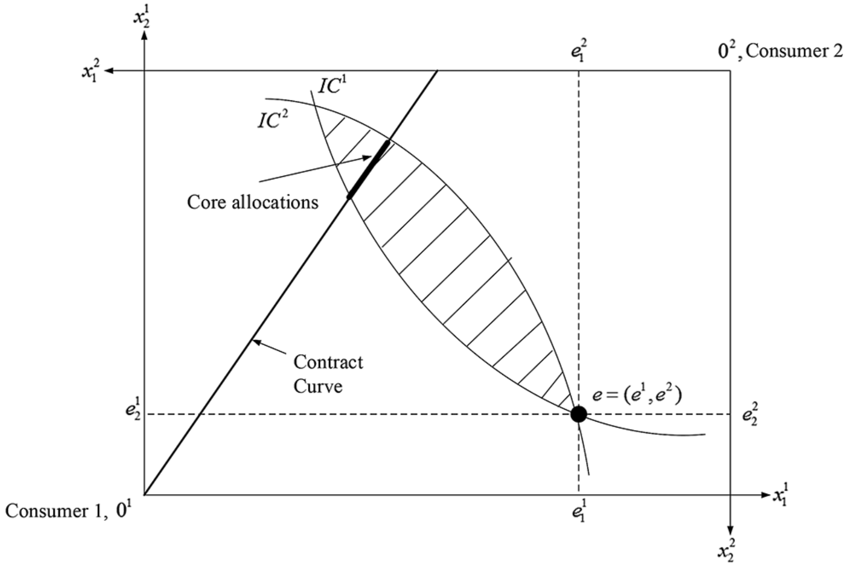

Figure 6.15
Core of an exchange economy

that lies within the lens-shaped area. Remarkably, this set of allocations coincides with those in equilibria, as we show next.

6.4.2 Competitive Markets

In the previous barter economy we did not require prices. Let us now explore the notion of equilibrium in economies where we do allow prices to emerge. For presentation purposes, we first describe consumers' preferences, next the excess demand function that results from comparing the aggregate demand under a specific market price against the total endowment of each good, and then we define equilibrium allocations in competitive markets (which we refer as Walrasian equilibrium allocations) and explore conditions under which they exist.

Consumers We consider consumers' utility functions to be continuous, strictly increasing, and strictly quasi-concave in $\mathbb{R}_+^n$ (recall that strictly quasi-concavity entails strictly convex indifference curves).

As a consequence the UMP of every consumer i, when facing a budget constraint

$$\mathbf{p}\cdot\mathbf{x}^i \leq \mathbf{p}\cdot\mathbf{e}^i \quad \text{for all price vectors } \mathbf{p} >> \mathbf{0},$$

yields a unique solution, denoted as the Walrasian demand $\mathbf{x}(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i)$. In addition, $\mathbf{x}(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i)$ is continuous in the price vector $\mathbf{p}$. Intuitively, note that individual i's income comes from selling his endowment $\mathbf{e}^i$ at market prices $\mathbf{p}$, producing $\mathbf{p}\cdot\mathbf{e}^i = p_1 e_1^i + \ldots + p_k e_k^i$ dollars to be used in the purchase of allocation $\mathbf{x}^i$.

We can add the Walrasian demand $\mathbf{x}(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i)$ for good k of every individual in the economy, obtaining the aggregate demand for good k, and compare it against the aggregate endowment of that good, which yields the *excess demand of good k*:

$$z_k(\mathbf{p}) \equiv \sum_{i=1}^{I} x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) - \sum_{i=1}^{I} e_k^i, \quad \text{where } z_k(\mathbf{p}) \in \mathbb{R}.$$

Hence, when $z_k(\mathbf{p}) > 0$, the aggregate demand for good k exceeds its aggregate endowment, and we say that there is excess demand of good k; in contrast, when $z_k(\mathbf{p}) < 0$, the opposite argument applies, and we say there is excess supply of good k. Figure 6.16 depicts the difference $\sum_{i=1}^{I} x_u^i(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i) - \sum_{i=1}^{I} e_k^i$ for a given good k in the left panel, and the resulting excess demand function $z_k(\mathbf{p})$ in the right panel.

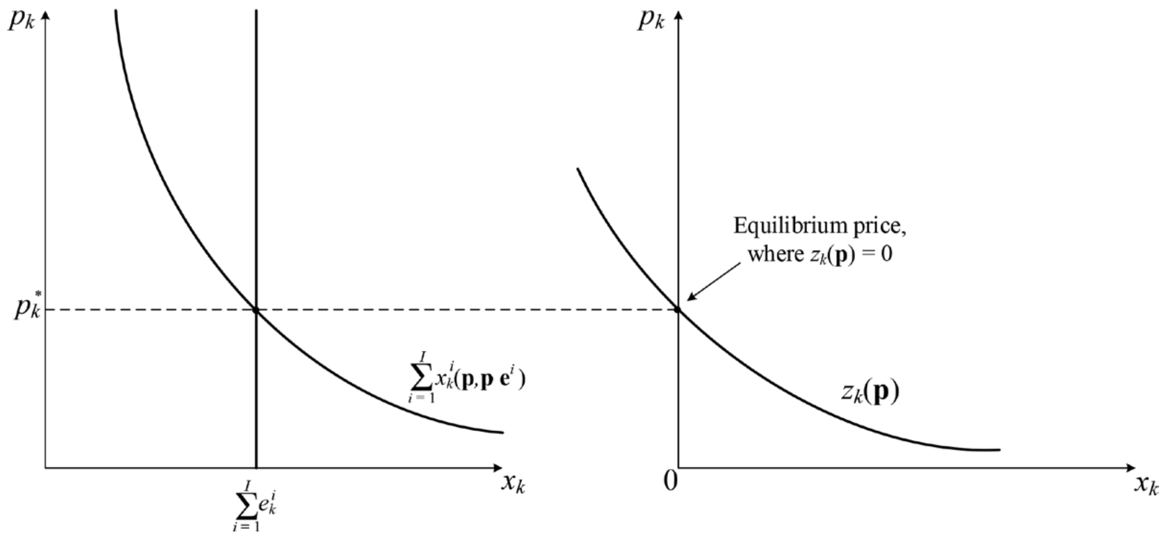

Figure 6.16
Difference in demand and supply, and excess demand

The excess demand function $\mathbf{Z}(\mathbf{p}) \equiv (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_k(\mathbf{p}))$ satisfies some interesting properties:

1. *Walras's law,* $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ This follows from the property of strictly increasing utility function: the budget constraint in the UMP will be binding for every consumer $i \in I$. In particular, since every consumer $i \in I$ exhausts all his income,

$$\sum_{k=1}^{L} p_k x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) = \sum_{k=1}^{L} p_k e_k^i \Leftrightarrow \sum_{k=1}^{L} p_k \left[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i \right] = 0.$$

Summing over all individuals gives

$$\sum_{i=1}^{I} \sum_{k=1}^{L} p_k \left[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i \right] = 0.$$

Since the order of summation is inconsequential, we can rewrite the expression above as

$$\sum_{k=1}^{L} \sum_{i=1}^{I} p_k \left[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i \right] = 0,$$

which, in turn, is equivalent to

$$\sum_{k=1}^{L} p_k \underbrace{\left(\sum_{i=1}^{I} x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i=1}^{I} e_k^i \right)}_{z_k(\mathbf{p})} = 0$$

$$\Leftrightarrow \sum_{k=1}^{L} p_k z_k(\mathbf{p}) = \mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0\,.$$

In a two-good economy, Walras's law implies that $p_1 z_1(\mathbf{p}) = -p_2 z_2(\mathbf{p})$ indicating that, if there is excess demand in market 1, $z_1(\mathbf{p}) > 0$, then there must be excess supply in market 2, $z_2(\mathbf{p}) < 0$. Similarly, if market 1 is in equilibrium, $z_1(\mathbf{p}) = 0$, then so is market 2, $z_2(\mathbf{p}) = 0$. More generally, if the markets of $L - 1$ goods are in equilibrium, then so is the Lth market.

2. *Continuity,* $\mathbf{z}(\mathbf{p})$ *is continuous at* $\mathbf{p}$ This property follows from individual Walrasian demands being continuous in prices.
3. *Homegeneity,* $z(\lambda \mathbf{p}) = z(\mathbf{p})$ *for all* $\lambda > 0$ This property follows from individual Walrasian demands being homogeneous of degree zero in prices, as described in previous chapters, which is to say, they were unaffected by an increase (or decrease) in all prices by a common factor $\lambda > 0$.

We can now use excess demand $\mathbf{z}(\mathbf{p})$ to define a Walrasian equilibrium allocation.

Walrasian equilibrium A price vector $\mathbf{p}^* \gg 0$ is a Walrasian equilibrium if aggregate excess demand is zero at that price vector, $\mathbf{z}(\mathbf{p}^*)=0$. In words, price vector $\mathbf{p}^*$ clears all markets.

Alternatively, $\mathbf{p}^* \gg 0$ is a Walrasian equilibrium if

1. each consumer solves his UMP, and
2. aggregate demand equals aggregate supply (i.e., markets clear)

$$\sum_{i=1}^{I} x^i(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i) = \sum_{i=1}^{I} \mathbf{e}^i.$$

Let us next explore existence and uniqueness of a Walrasian equilibrium.

Existence of a Walrasian equilibrium A Walrasian equilibrium price vector $\mathbf{p}^* \gg 0$, where $\mathbf{z}(\mathbf{p}^*)=0$, exists if the excess demand function $\mathbf{z}(\mathbf{p})$ satisfies continuity and Walras's law (and both of these properties hold given the initial assumptions we imposed on utility functions). (For a proof of that result, see Varian 1992, 321–22.)

Uniqueness A desirable property of equilibrium prices is their uniqueness, as satisfied by the excess demand function depicted in figure 6.17a but violated in figure 6.17b. As we next show, gross substitutability of all goods is a sufficient condition for equilibrium price to be unique.

Proof By contradiction, suppose that there is another equilibrium price vector $\mathbf{p}'$, that is, $z(\mathbf{p}')=0$, where $\mathbf{p}', \mathbf{p}^* \gg 0$. Let us define price vector $\mathbf{p}'$ to be an increase in the price of all goods $j \neq k$ in vector $\mathbf{p}^*$ (except for the price of good k) as follows:

$$\mathbf{p}' = \left(mp_1^*, \ldots, p_k^*, \ldots, mp_L^*\right),$$

where $m>1$. Hence, since the price of all other goods is increasing, the excess demand of good k must be positive; as prescribed by gross substitutability (p_j and z_k must move in the same direction). But then $z_k(\mathbf{p}')>0$, implying that price vector $\mathbf{p}'$ does not clear the market of good k. Since all markets are not in equilibrium at $\mathbf{p}'$, price vector $\mathbf{p}'$ is not a WEA. (A similar argument applies if, rather than increasing the price of all other goods, we decrease it. That is, if $0<m<1$, we obtain that the excess demand of good k satisfies $z_k(\mathbf{p}')<0$.) ■

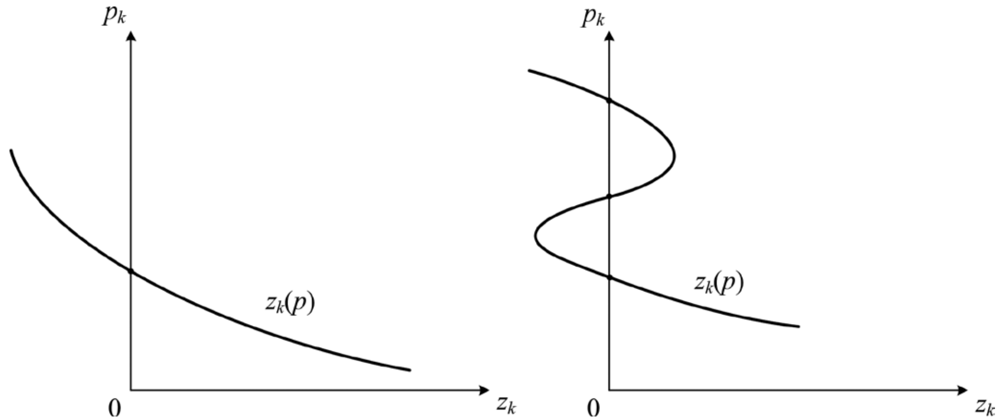

Figure 6.17
Unique and non-unique WEAs

Example 6.6: *Finding Walrasian equilibrium allocation* Continuing our example 6.5, we already determined that

$$MRS^A = MRS^B = \frac{p_1}{p_2},$$

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{p_1}{p_2}.$$

From these two equations we can find the Walrasian demands of each good for each consumer. Starting with consumer A, we can rearrange the first and third terms of the preceding equation to obtain $p_1x_1^A = p_2x_2^A$. Plugging this into consumer A's budget constraint yields

$$p_1x_1^A + p_1x_1^A = p_1(100) + p_2(350) \Rightarrow x_1^A = 50 + 175\frac{p_2}{p_1},$$

which is consumer A's demand for good 1. Plugging this value back into $p_1x_1^A = p_2x_2^A$ yields

$$p_1\left(50 + 175\frac{p_2}{p_1}\right) = p_2x_2^A \Rightarrow x_2^A = 175 + 50\frac{p_1}{p_2},$$

which is consumer A's demand for good 2. For consumer B, the process is similar except that we rearrange the second and third terms of our initial equation to obtain $p_1x_1^B = p_2x_2^B$. Substituting this into consumer B's budget constraint yields

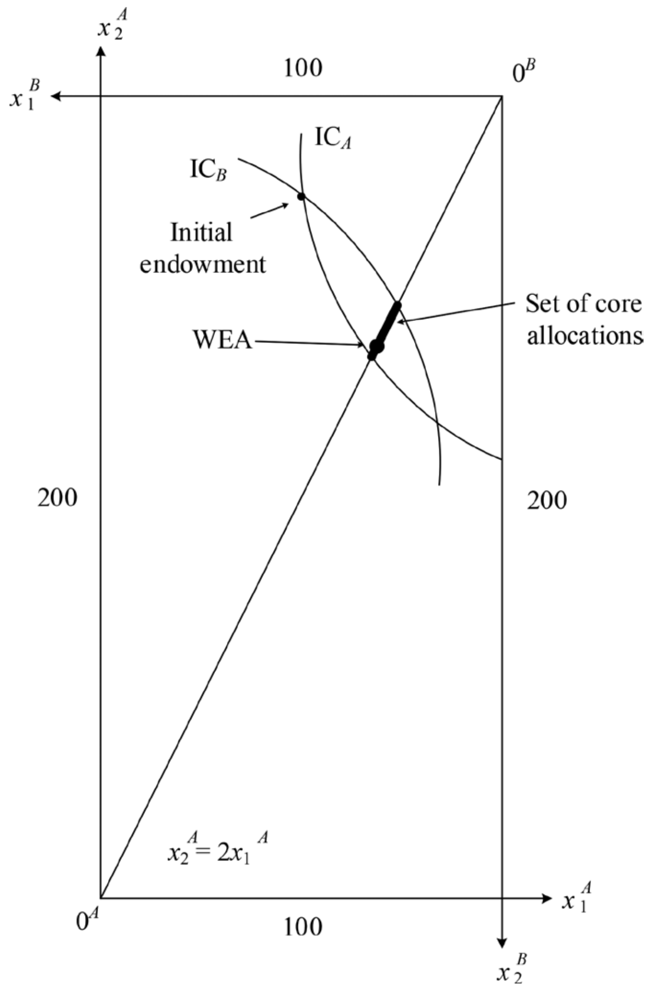

Figure 6.18
Initial allocation, core allocation, and WEA of example 6.6

$$p_1 x_1^B + p_1 x_1^B = p_1(100) + p_2(50) \Rightarrow x_1^B = 50 + 25\frac{p_2}{p_1},$$

which is consumer B's demand for good 1. Substituting this value back into $p_1 x_1^B = p_2 x_2^B$ yields our final demand

$$p_1\left(50 + 25\frac{p_2}{p_1}\right) = p_2 x_2^A \Rightarrow x_2^A = 25 + 50\frac{p_1}{p_2}.$$

All that remains is to substitute each of our demands into our feasibility constraints and solve for relative prices. For good 1, the feasibility constraint is

$$x_1^A + x_1^B = 100 + 100,$$

$$50 + 175\frac{p_2}{p_1} + 50 + 25\frac{p_2}{p_1} = 200\frac{p_2}{p_1} = \frac{1}{2}.$$

(Note that using the feasibility constraint for good 2 will produce the same result.) Substituting the relative prices back into our Walrassian demands yields our Walrasian equilibrium,

$$\left(x_1^{A,*}, x_2^{A,*}; x_1^{B,*}, x_2^{B,*}; \frac{p_1}{p_2}\right) = (137.5, 275; 62.5, 125; 2). \blacksquare$$

Equilibrium Allocations Must Be in the Core Our previous discussion suggested that Walrasian equilibrium allocations (WEAs) are mutually beneficial for all individuals. That is, an allocation cannot be blocked by any coalition of individuals, or, in other words, must be in the *core* of the economy. Let us next show that, if each consumer's utility function is strictly increasing, then every WEA is in the core, which we express as $W(\mathbf{e}) \subset C(\mathbf{e})$.

Proof Assume, by contradiction, a WEA, $\mathbf{x}(\mathbf{p}^*)$ with equilibrium price $\mathbf{p}^*$, that does not belong to the core, that is, $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$. Because $\mathbf{x}(\mathbf{p}^*)$ is a WEA, it must be feasible (as all equilibrium allocations must be feasible by definition). However, if such allocation in not part of the core, $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, we can find a coalition of individuals S and another allocation $\mathbf{y}$ such that

$$u^i(\mathbf{y}^i) \geq u^i\left(\mathbf{x}^i\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right)\right) \quad \text{for all } i \in S,$$

with strict inequality for at least one individual in the coalition, and that is feasible for the coalition, $\sum_{i\in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$. We can now multiply both sides of the feasibility condition by $\mathbf{p}^*$ to obtain

$$\mathbf{p}^* \sum_{i\in S} \mathbf{y}^i = \mathbf{p}^* \sum_{i\in S} \mathbf{e}^i.$$

However, if $\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)$ is a WEA, the preferable vector $\mathbf{y}^i$ must be more costly than $\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)$, that is,

$$\mathbf{p}^*\mathbf{y}^i \geq \mathbf{p}^*\mathbf{x}^i\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) = \mathbf{p}^*\mathbf{e}^i$$

with strict inequality for at least one individual. Summing over all consumers in the coalition S, we obtain

$$\mathbf{p}^* \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \sum_{i \in S} \mathbf{x}^i \left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) = \mathbf{p}^* \sum_{i \in S} \mathbf{e}^i,$$

contradicting $\mathbf{p}^* \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \sum_{i \in S} \mathbf{e}^i$. Therefore all WEAs must be in the core, such that $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$. ■

We can immediately infer two conclusions from the previous result. On one hand, the core $C(\mathbf{e})$ will at least contain the WEA (or WEAs), implying that the core will always be nonempty. On the other hand, since all core allocations are Pareto efficient, we cannot increase the welfare of one consumer without decreasing that of other consumers, implying that all WEAs (which are part of the core) are also Pareto efficient. This last result is often referred to as the *first welfare theorem,* as we compactly describe next.

First welfare theorem Every WEA is Pareto efficient.

As depicted in figure 6.19, the WEA lies on the core (the segment of the contract curve within the lens-shaped area), and the core is a subset of all PEAs (as illustrated in the contract curve). For a numerical example of the first welfare theorem, see example 6.6, where the WEA we found $(x_1^{A,*}, x_2^{A,*}; x_1^{B,*}, x_2^{B,*}; p_1/p_2) = (137.5, 275; 62.5, 125; 2)$, lies on the contract curve $x_2^A = 2x_1^A$, and hence it is Pareto efficient.

Consider the setting depicted in figure 6.20 and, starting from initial endowment $\mathbf{e}$, assume that the WEA is $\mathbf{x}'$, which also belongs to the set of core allocations $C(\mathbf{e})$. However, suppose that society would prefer allocation $\bar{\mathbf{x}}$ to $\mathbf{x}'$ according to some social welfare function that aggregates individual preferences over bundles.[7] A natural

7. Most of the social welfare functions we describe aggregate individual utility functions, $W=f(u_1, \ldots, u_I)$.

Common examples are (1) the "utilitarian," $W = \sum_{i=1}^{I} a_i u_i$, where $a_i \geq 0$ denotes the weight that society

assigns to individual i; (2) the Cobb–Douglas type, $W = \prod_{i=1}^{I} u_i^{b_i}$ or applying logs $\sum_{i=1}^{I} b_i \log u_i$,

where $b_i \geq 0$; and (3) the "Benthamian" $W = \min\{u_1, \ldots, u_I\}$, which is concerned about the welfare of the individual in the worst position of society.

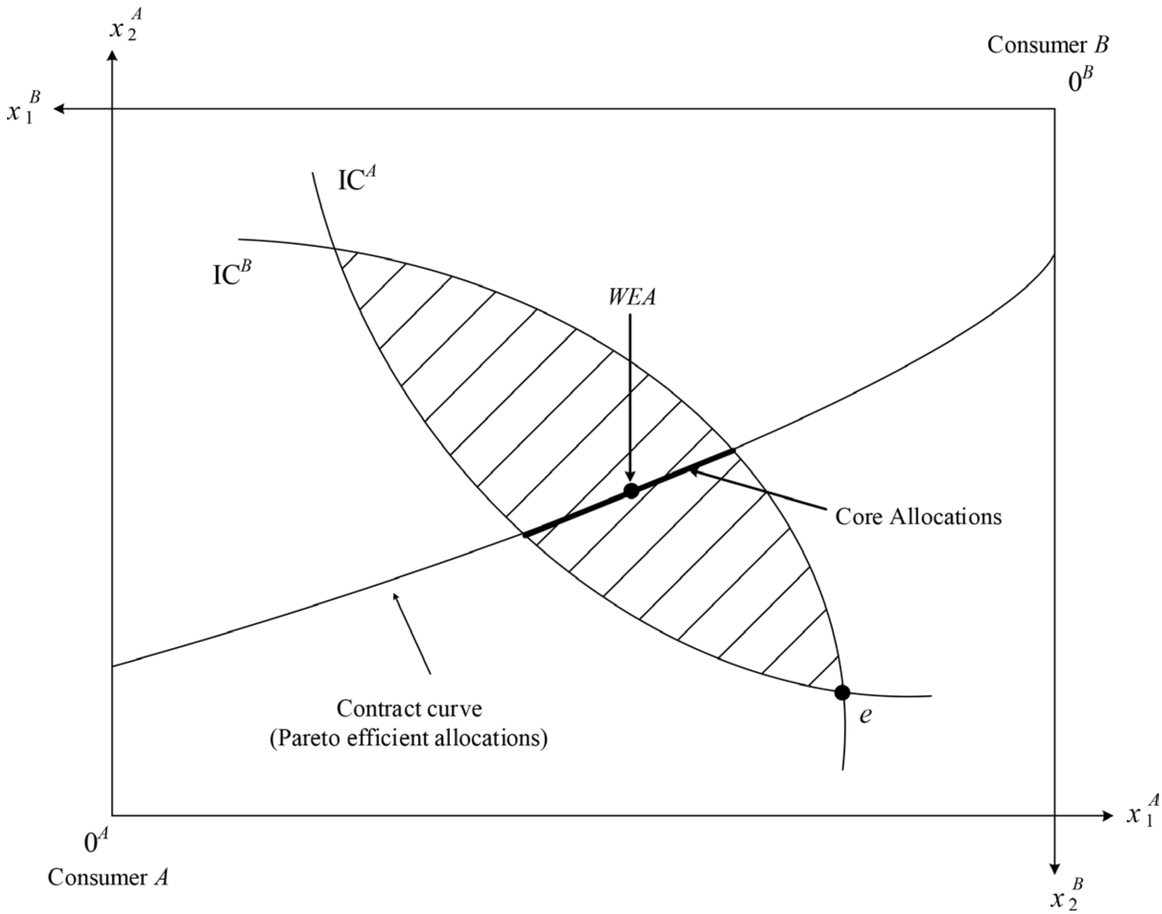

Figure 6.19
First welfare theorem

question is whether society could simply alter the initial endowment, from **e** to **e″** (or generally, to any point $\mathbf{e}^{*i}$ on the budget line, satisfying $\mathbf{p}^* \cdot \mathbf{e}^{*i} = \mathbf{p}^* \cdot \bar{\mathbf{x}}^i$), and then "let the market work" (i.e., allow each individual consumer to independently solve his own UMP). Would that variation in the initial endowment (followed by letting the market work) lead to the desired WEA $\bar{\mathbf{x}}$? As shown by the second welfare theorem, the answer to this question is yes.

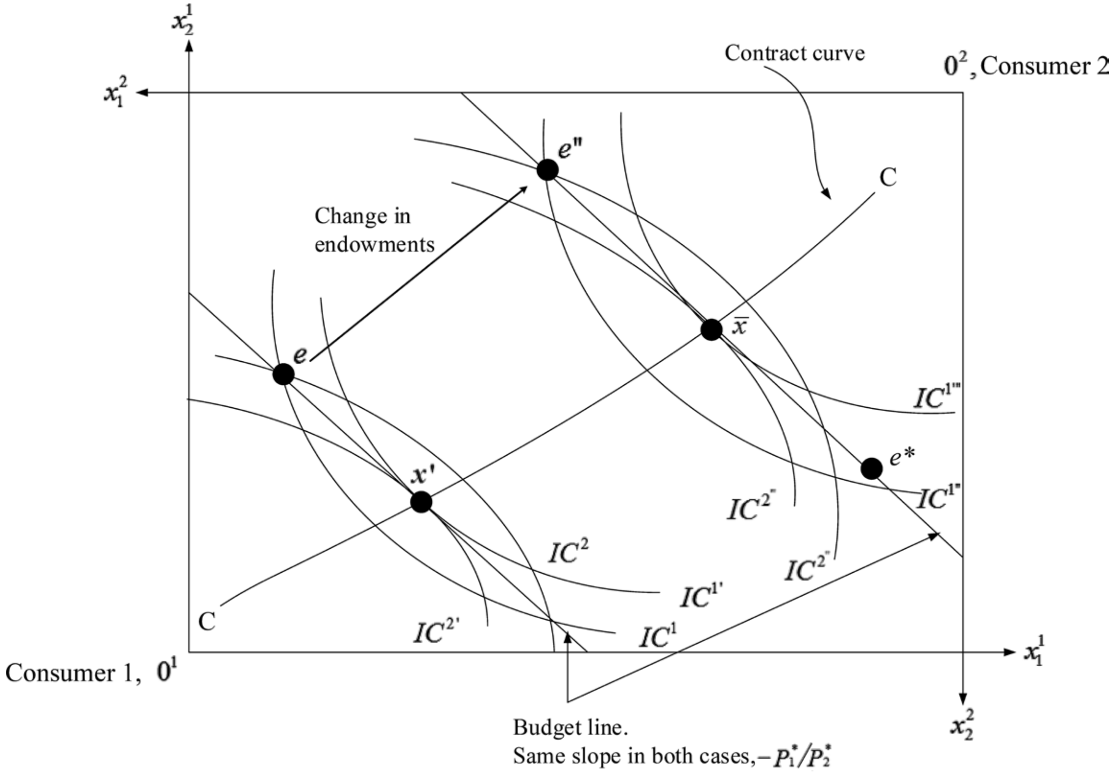

Figure 6.20
Second welfare theorem

> **Second welfare theorem** A Pareto efficient allocation $\bar{\mathbf{x}}$ (i.e., $\bar{\mathbf{x}}$ lies on the contract curve), and the endowments, are redistributed so that the new endowment vector $\mathbf{e}^{*i}$ lies on a line satisfying $\mathbf{p}^* \cdot \mathbf{e}^{*i} = \mathbf{p}^* \cdot \bar{\mathbf{x}}^i$ for every consumer i. Then the Pareto-efficient allocation $\bar{\mathbf{x}}$ is a WEA given the new endowment vector $\mathbf{e}^*$.

The first and second welfare theorems provide sharp results about the relationship between WEAs and PEAs when markets operate without distortions. However, when market failures exist, such as market power, externalities (in consumption or production), and public goods, or when some agents have access to information that other agents cannot accurately observe, these two theorems do not necessarily apply, as we examine in subsequent chapters. Likewise our previous analysis assumed that consumers have similar bargaining power when negotiating a price for each good. However, in some settings one consumer can sustain all the bargaining power if, for instance, he makes a take-it-or-leave-it price offer to other consumers, in which case consumer A could announce a price ratio to individual B, who either accepts it or rejects it (which yields every consumer with his initial endowment). As in our previous discussion, consumer B would take the price ratio announced by A as given, and solve his UMP in order to find his offer curve. Consumer A, in contrast, would anticipate B's offer curve, and use it as the constraint of his UMP to reach the highest possible utility level. As one can expect, the WEA that emerges in this context is not Pareto optimal, as we illustrate in exercise 14 at the end of the chapter.

Example 6.7: ***WEA and second welfare theorem*** Consider an economy with utility functions $u^A = x_1^A x_2^A$ for consumer A and $u^B = \min\{x_1^B, x_2^B\}$ for consumer B, where initial endowments are $\mathbf{e}^A = (3,1)$ and $\mathbf{e}^B = (1,3)$. First, let us find the set of PEAs (similarly as in example 6.5), then we will find the set of WEA, where we use good 2 as the numeraire, $p_2 = 1$. Finally, assuming that society seeks to implement allocation $\hat{x}^A = (1,1)$ for consumer A and $\hat{x}^B = (3,3)$ for consumer B, we will determine the initial endowments that would achieve that this allocation becomes the WEA.

- *PEAs* Starting with consumer B, it is clear that calculus cannot be used to determine his marginal rate of substitution. However, due to his preferences being perfect complements, it is known that consumer B will want to consume at the kink of his indifference curves, that is, by consuming goods 1 and 2 in equal quantities, $x_1^B = x_2^B$. From this information, and the following feasibility constraints, we have $x_1^A + x_1^B = 4$, $x_2^A + x_2^B = 4$; then we can substitute x_2^B for x_1^B in the first feasibility condition, $x_1^B = x_2^B$, and solve it for x_2^B, yielding $x_2^B = 4 - x_1^A$. Substituting this value into the second feasibility condition gives $x_2^A + \underbrace{(4 - x_1^A)}_{x_2^B} = 4 \Rightarrow x_2^A = x_1^A$, which defines our contract curve, the set of PEAs, as depicted in figure 6.21.
- *WEA* Consumer A's maximization problem is

$$\max_{x_1^A, x_2^A} \; x_1^A x_2^A$$

$$\text{subject to} \quad p_1 x_1^A + x_2^A \le p_1(3) + 1$$

we take first-order conditions to obtain

$$x_2^A - \lambda p_1 = 0$$

$$x_1^A - \lambda = 0,$$

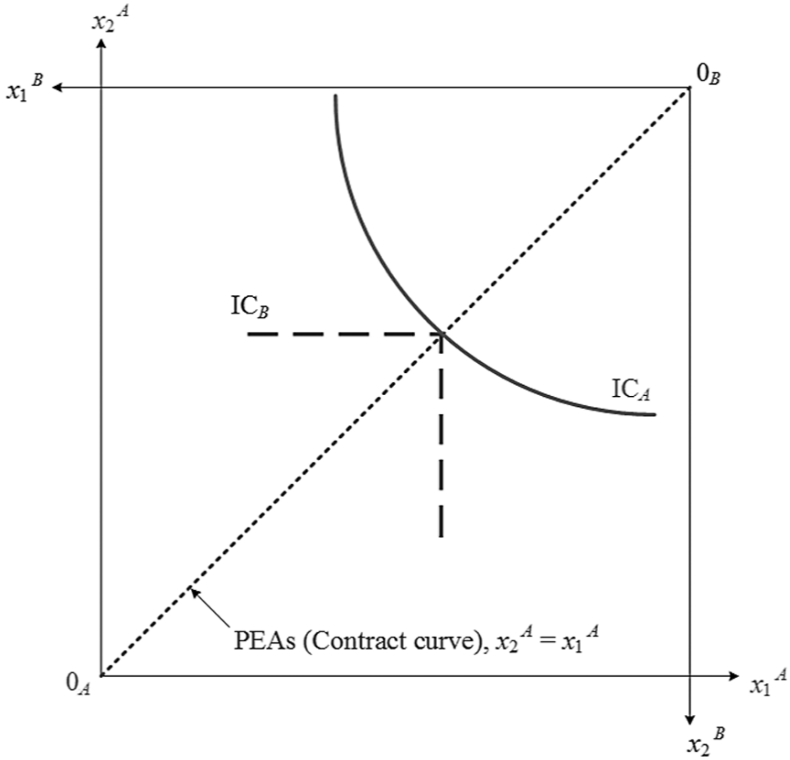

Figure 6.21
Contract curve

$$p_1 x_1^A + x_2^A = 3p_1 + 1,$$

where λ denotes the Lagrange multiplier. Combining the first two equations yields

$$\lambda = \frac{x_2^A}{p_1} = x_1^A, \quad \text{or} \quad p_1 = \frac{x_2^A}{x_1^A}.$$

In addition, for Pareto efficiency, we know that $x_2^A = x_1^A$, implying that $p_1 = x_2^A / x_1^A = 1$. All that remains is to substitute both the price and the PEA requirement back into the budget constraint, which obtains

$$2x_1^A = 4 \implies x_1^{A,*} = x_2^{A,*} = 2.$$

Then, using our feasibility conditions, we have

$$\underbrace{2}_{x_1^A} + x_1^B = 4 \implies x_1^{B,*} = x_2^{B,*} = 2,$$

and thus our WEA is $\left(x_1^{A,*}, x_2^{A,*}; x_1^{B,*}, x_2^{B,*}; p_1/p_2\right) = (2, 2; 2, 2; 1)$, as shown in figure 6.22.

There are several possible allocations in which $\hat{x}^A$ and $\hat{x}^B$ are the equilibrium allocations, such as $\hat{\mathbf{e}}^A = (2, 0)$ and $\hat{\mathbf{e}}^B = (2, 4)$. More generally, any allocation satisfying $\hat{e}_1^B + \hat{e}_2^B = 6$ and $\hat{e}_1^A + \hat{e}_2^A = 2$ will give this solution (assuming that the total amount of each good is still 4). ■

6.4.3 Equilibrium with Production

Let us now extend our previous results to a setting where firms are also active. Specifically, assume J firms in the economy, each with production set Y^j that satisfies (1) possible inaction, $\mathbf{0} \in Y^j$ as depicted in the origin of figure 6.23; (2) closed and bounded Y^j, so that points on the production frontier are part of the production set and thus feasible; and (3) strictly convex Y^j, whereby linear combinations of two production plans also belong to the production set, as depicted in the interior of figure 6.23.

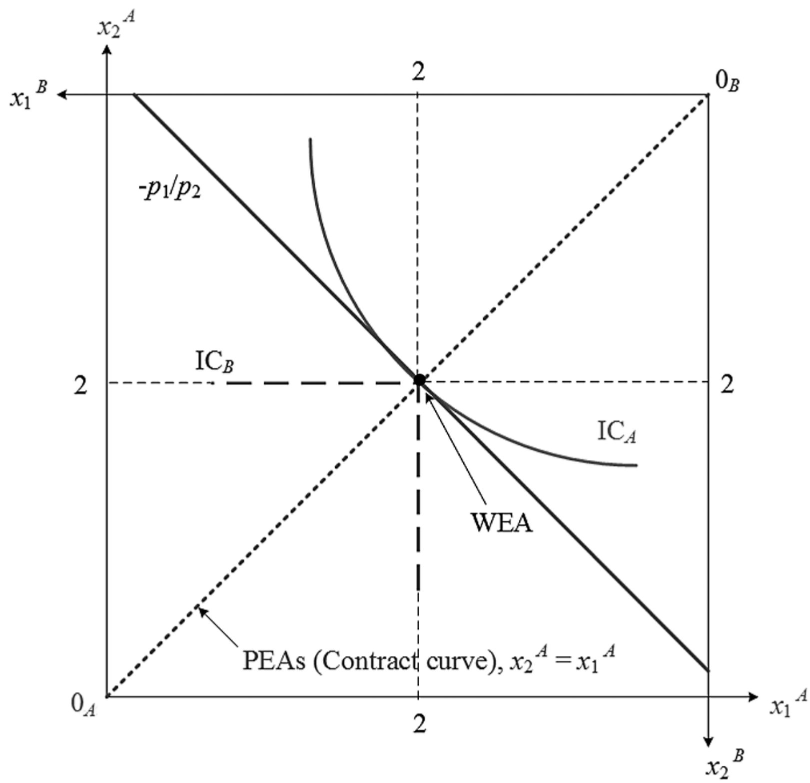

Figure 6.22
Second welfare theorem: WEA and PEAs

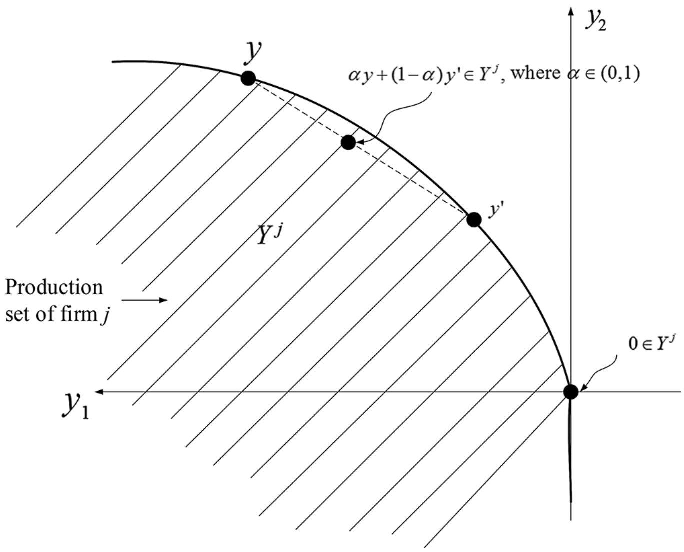

Figure 6.23
Production set Y^j for a representative firm

Similarly to consumers, who independently and simultaneously solve their own UMPs (when facing a fixed price vector $\mathbf{p} \gg \mathbf{0}$), every firm j facing a fixed price vector $\mathbf{p} \gg \mathbf{0}$ independently and simultaneously solves its PMP:

$$\max_{y^j \in Y^j} \mathbf{p} \cdot y^j.$$

From these assumptions on production sets Y^j, a profit-maximizing production plan $y^j(\mathbf{p})$ exists for every firm j, and it is unique, as illustrated in figure 6.24 (for more details, see chapter 4 on production theory). In addition, by the theorem of the maximum, both the argmax, $y^j(\mathbf{p})$, and the value function, $\pi^j(\mathbf{p}) \equiv \mathbf{p} \cdot y^j(\mathbf{p})$, are continuous in p.[8]

Aggregate Production Set We can now define the aggregate production set as the sum of all the J firms' production plans (whether profit maximizing or not), which we express mathematically as follows:

$$Y = \left\{ \mathbf{y} \middle| \mathbf{y} = \sum_{j=1}^{J} y^j, \text{ where } y^j \in Y^j \right\}.$$

8. Since profits are $\pi^0 = p_2 y_2 - p_1 y_1$, solving for y_2 yields the isoprofit line $y_2 = \pi^0/p_2 + (p_1/p_2)y_1$, where π^0/p_2 is the vertical intercept of the isoprofit lines in figure 6.11, while p_1/p_2 represents their slope.

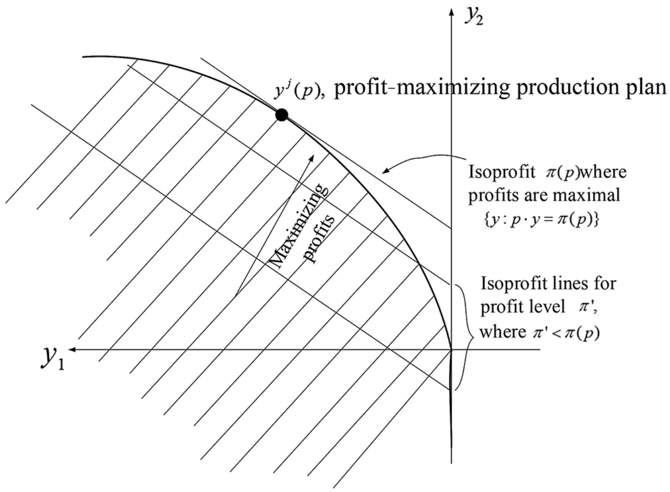

Figure 6.24
Case wherey^j ($\mathbf{p}$) exists and is unique

What about the relationships between the production plans $y^j(\mathbf{p})$ that maximize the individual profits of every firm j and the production plan $\mathbf{y}(\mathbf{p}) \in Y$ that maximize aggregate profits (i.e., one point in the aggregate production set Y we just defined)? As shown in chapter 4, we can express such a joint-profit maximizing production plan $\mathbf{y}(\mathbf{p})$ as the sum of each firm's profit-maximizing plan, $\mathbf{y}(\mathbf{p}) = y^1(\mathbf{p}) + y^2(\mathbf{p}) + \ldots y^J(\mathbf{p})$.

In this economy with J firms, each firm earns $\pi^j(\mathbf{p})$ profits in equilibrium. How are profits distributed? We can assume that each individual i owns a share θ_{ij} of firm j's profits, where$1 \leq \theta_{ij} \leq 1$, and that firm j's profits are distributed across all I consumers, that is, $\sum_{i=1}^{I} \theta_{ij} = 1$. Note that such distribution of profits allows for multiple sharing profiles, from $\theta_{ij} = 1$ where individual i owns all shares of firm j, to $\theta_{ij} \leq 1/I$, so that every individual's share on firm j coincides. In this context, consumer i's budget constraint becomes

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta_{ij} \pi^j(\mathbf{p}) \, ,$$

where only the last term, $\sum_{i=1}^{J} \theta_{ij} \pi^j(\mathbf{p})$, is new relative to the standard budget constraint that we have considered so far in studying economies without production. For compactness, we can express the budget constraint as

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta_{ij} \pi^j(\mathbf{p}) \Rightarrow \mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p}),$$

where $m^i(\mathbf{p})$ denotes all the resources of individual i, either originating from the market value of his initial endowment (if he sells it at the current market prices) or the profits he makes from the firms he owns. Last, note that given our previous assumptions on the production sets Y^j, the profit-maximizing plans entail a positive profit, $m^i(\mathbf{p})>0$.

Equilibrium with Excess Demand In order to characterize equilibrium in economies with production, let us follow a similar approach to that in economies without production, where we started defining excess demand functions and subsequently used such a definition to compactly identify the set of equilibrium allocations:

Excess demand The excess demand function for good k is

$$z_k(\mathbf{p}) \equiv \sum_{i=1}^{I} x_k^i\left(\mathbf{p}, m^i(\mathbf{p})\right) - \sum_{i=1}^{I} e_k^i - \underbrace{\sum_{j=1}^{J} y_k^j(\mathbf{p})}_{\text{New}},$$

where $\sum_{j=1}^{J} y_k^j(\mathbf{p})$ is a new term relative to our analysis of general equilibrium without production, and denotes the profit-maximizing production of good k that all J firms chose as part of their supply correspondence. Hence the aggregate excess demand vector is

$$\mathbf{z}(\mathbf{p}) \equiv \left(z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p})\right).$$

We can now use $\mathbf{z}(\mathbf{p})$ to define a WEA with production.

WEA with Production If the price vector is strictly positive in all of its components, $\mathbf{p}^* \gg 0$, a pair of consumption and production bundles $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ is a WEA if

1. each consumer i solves his UMP, which becomes the ith entry of $\mathbf{x}(\mathbf{p}^*)$, that is, $\mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*))$;
2. each firm j solves its PMP, which becomes the jth entry of $\mathbf{y}(\mathbf{p}^*)$, that is, $\mathbf{y}^j(\mathbf{p}^*)$; and
3. demand equals supply

$$\sum_{i=1}^{I} \mathbf{x}^i\left(\mathbf{p}^*, m^i(\mathbf{p}^*)\right) = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j(\mathbf{p}^*),$$

which states the market-clearing condition (or feasibility when expressed for any price vector $\mathbf{p}$).

From every consumer i simultaneously solving his UMP, we obtain that the marginal rate of substitution between any goods 1 and 2 satisfies $MRS^i_{1,2} = MRS^j_{1,2} = p_1/p_2$ for every $j \neq i$ (we also found this result in the barter economies analyzed in previous sections). Similarly, from every firm j simultaneously solving its PMP, we obtain that

$$p_1 F_{1K} = r \quad \text{and} \quad p_1 F_{1L} = w \quad \text{for firm 1,}$$

$$p_2 F_{2K} = r \quad \text{and} \quad p_2 F_{2L} = w \quad \text{for firm 2.}$$

Dividing these expressions yields

$$MRTS^1_{L,K} \equiv \frac{F_{1L}}{F_{1K}} = \frac{w}{r} \quad \text{for firm 1,}$$

$$MRTS^2_{L,K} \equiv \frac{F_{2L}}{F_{2K}} = \frac{w}{r} \quad \text{for firm 2.}$$

Therefore $MRTS^1_{L,K} = MRTS^2_{L,K}$. Similarly we can divide $p_1 F_{1K} = r$ and $p_2 F_{2K} = r$ to find

$$\frac{p_1 F_{1K}}{p_2 F_{2K}} = \frac{r}{r} = 1,$$

or after rearranging,

$$\frac{p_1}{p_2} = \frac{F_{2K}}{F_{1K}} \equiv MRT^K_{1,2}.$$

A similar result emerges when we divide $p_1 F_{1L} = w$ and $p_2 F_{2L} = w$, that is, $p_1/p_2 = F_{2L}/F_{1L} \equiv MRT^L_{1,2}$. Overall, combining the equilibrium conditions for every consumer i, $MRS^i_{1,2} = p_1/p_2$, and for every input $m = \{K, L\}$, $p_1/p_2 = MRT^m_{1,2}$, yields

$$MRS^i_{1,2} = MRT^m_{1,2} = \frac{p_1}{p_2},$$

where $MRS^i_{1,2} \equiv MU^i_1/MU^i_2$ is increasing in good 1 (as x_1 increases, MU^i_1 decreases while MU^i_2 decreases). In contrast, $MRT^m_{1,2} \equiv F_{2m}/F_{1m}$ is decreasing in good 1. Intuitively, in order to increase x_1, we need to move units of input m from firm 2 to firm 1, thus increasing the marginal product of this input for firm 2 and lowering it for firm 1. Figure 6.25 plots $MRS^i_{1,2}$ and $MRT^m_{1,2}$ as a function of x_1, crossing each other at the equilibrium level of good 1, x_1^*, at a height of p_1/p_2.

Existence Assume that consumers' utility functions are continuous, strictly increasing, and strictly quasi-concave (as considered in previous sections), and that they are

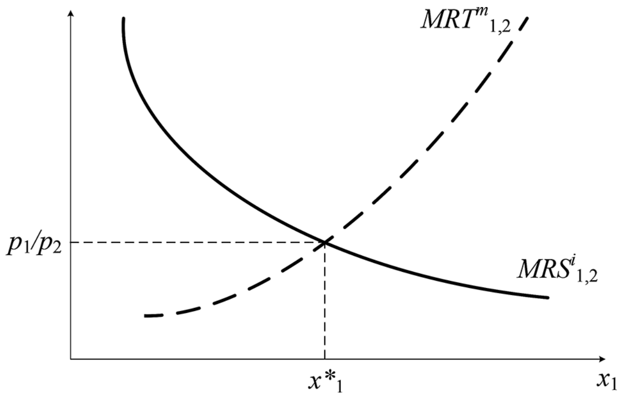

Figure 6.25
Equilibrium with production

initially endowed with positive units of at least one good, so the sum $\sum_{i=1}^{I} \mathbf{e}^i \gg 0$. Additionally every firm j's production set Y^j is closed and bounded,[9] strictly convex, and satisfies the property of inaction being possible (as stated at the beginning of this section). In this economy, there is a price vector $\mathbf{p}^* \gg 0$ such that a WEA exists, that is $z(\mathbf{p}^*)=0$. (For a proof of this result, see Varian 1992.) Let us now consider a numerical example of equilibrium with production with two consumers and two firms.

Example 6.8: *Finding WEAs with production* In a two-consumer, two-good economy every consumer $i=\{1,2\}$ has utility function $u^i = x_1^i x_2^i$. There are two firms in this economy, each using capital and labor as inputs to produce one of the consumption goods. Firm 1 produces good 1 with production function $y_1 = K_1^{0.75} L_1^{0.25}$, and firm 2 produces good 2 with production function $y_2 = K_2^{0.25} L_2^{0.75}$. Consumer A is endowed with $(K^A, L^A)=(1,1)$, while consumer 2 is endowed with $(K^B, L^B)=(2,1)$. Let us find a WEA in this economy with production.

UMPs Starting on the consumer side of this problem, consumer A's utility maximization problem is

$$\max_{x_1^A, x_2^A} \; x_1^A x_2^A$$

$$\text{subject to} \quad p_1 x_1^A + p_2 x_2^A = rK^A + wL^A,$$

9. Recall from chapter 4 that bounded production sets allow for efficient production plans (graphically, those on the production frontier).

where r and w are the prices for capital and labor, respectively. Taking first-order conditions gives us the tangency condition for utility maximization under regular preferences

$$\frac{p_1}{p_2} = MRS_{1,2}^A \Rightarrow \frac{p_1}{p_2} = \frac{x_2^A}{x_1^A} \Rightarrow p_1 x_1^A = p_2 x_2^A \tag{6.1}$$

Likewise consumer B's utility maximization problem is

$$\max_{x_1^B, x_2^B} \; x_1^B x_2^B$$

$$\text{subject to } \; p_1 x_1^B + p_2 x_2^B = rK^B + wL^B,$$

with first-order conditions giving

$$\frac{p_1}{p_2} = MRS_{1,2}^B \;\Rightarrow\; \frac{p_1}{p_2} = \frac{x_2^B}{x_1^B} \;\Rightarrow\; p_1 x_1^B = p_2 x_2^B. \tag{6.2}$$

Now, taking equations (6.1) and (6.2) and adding them together, we have

$$p_1(x_1^A + x_1^B) = p_2(x_2^A + x_2^B).$$

But recall that $x_1^A + x_1^B$ is the left side of our feasibility condition, so $x_1^A + x_1^B = y_1 = K_1^{0.75} L_1^{0.25}$. Substituting both feasibility conditions into this problem, and rearranging gives

$$\frac{p_1}{p_2} = \frac{K_2^{0.25} L_2^{0.75}}{K_1^{0.75} L_1^{0.25}}. \tag{6.3}$$

PMPs Next we move to the production side of the economy. Firm 1's PMP is

$$\max_{K_1, L_1} p_1 K_1^{0.75} L_1^{0.25} - rK_1 - wL_1$$

with first-order conditions

$$r = 0.75 p_1 K_1^{-0.25} L_1^{0.25},$$

$$w = 0.25 p_1 K_1^{0.75} L_1^{-0.75}.$$

Combining these conditions gives the tangency condition for profit maximization under regular technologies:

$$\frac{r}{w} = MRTS_{L,K}^1 \Rightarrow \frac{r}{w} = 3\frac{L_1}{K_1}. \tag{6.4}$$

Likewise firm 2's PMP gives the following first-order conditions

$$r = 0.25 p_2 K_2^{-0.75} L_2^{0.75},$$

$$w = 0.75 p_2 K_2^{0.25} L_2^{-0.25},$$

and combining them gives the tangency condition

$$\frac{r}{w} = MRTS_{L,K}^2 \;\Rightarrow\; \frac{r}{w} = \frac{1}{3}\frac{L_2}{K_2}. \tag{6.5}$$

Combining both *MRTS* gives

$$3\frac{L_1}{K_1} = \frac{1}{3}\frac{L_2}{K_2} \;\Rightarrow\; \frac{K_1}{L_1} = 9\frac{K_2}{L_2}.$$

Intuitively, this result implies that firm 1 is more capital intensive than firm 2, since its capital to labor ratio is higher. Using both firm's price of capital, *r*, and setting them equal to each other gives

$$0.75 p_1 K_1^{-0.25} L_1^{0.25} = 0.25 p_2 K_2^{-0.75} L_2^{0.75} \;\Rightarrow\; \frac{p_1}{p_2} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75}, \tag{6.6}$$

and likewise, setting both firms' price of labor equal to each other gives

$$0.25 p_1 K_1^{0.75} L_1^{-0.75} = 0.75 p_2 K_2^{0.25} L_2^{-0.25} \;\Rightarrow\; \frac{p_1}{p_2} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25}. \tag{6.7}$$

Now setting (6.3)—from both consumers' UMPs—equal to (6.6)—from both firms' PMPs—gives

$$\frac{K_2^{0.25} L_2^{0.75}}{K_1^{0.75} L_1^{0.25}} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75} \;\Rightarrow\; K_1 = 3K_2,$$

and by our feasibility conditions, we know that $K_1 + K_2 = K^A + K^B) = 3$, or $K_2 = 3 - K_1$. Substituting, we find the profit-maximizing demands for capital use by firms 1 and 2,

$$K_1 = 3(3 - K_1) \;\Rightarrow\; K_1^* = \frac{9}{4},$$

$$K_2^* = \frac{1}{3}K_1^* = \frac{3}{4}.$$

Next we set (6.3)—from both consumers' UMPs—equal to (6.7)—from both firms' PMPs—as

$$\frac{K_2^{0.25}L_2^{0.75}}{K_1^{0.75}L_1^{0.25}} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25} \Rightarrow L_1 = \frac{1}{3}L_2,$$

and by our feasibility condition, we know that $L_1+L_2=L^A+L^B)=2$, or $L_2=2-L_1$. Substituting, we find the labor demands for firm 1 and 2,

$$L_1 = \frac{1}{3}(2-L_1) \Rightarrow L_1^* = \frac{1}{2},$$

$$L_2^* = 3L_1^* = \frac{3}{2}.$$

From here, we can substitute these values into equation (6.3) to find that the equilibrium price ratio is

$$\frac{p_1}{p_2} = \frac{(3/4)^{0.25}(3/2)^{0.75}}{(9/4)^{0.75}(1/2)^{0.25}} = \sqrt{2/3},$$

and normalizing the price of good 2, $p_2=1$, gives $p_1 = \sqrt{2/3} \cong 0.82$. Furthermore we can substitute our calculated values into the price of capital and labor to find

$$r^* = 0.75(0.82)\left(\frac{9}{4}\right)^{-0.25}\left(\frac{1}{2}\right)^{0.25} = 0.42,$$

$$w^* = 0.25(0.82)\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{-0.75} = 0.63.$$

Last, we return to the consumer side of the market. Using consumer A's tangency condition (equation 6.1), we know that

$$x_2^A = \frac{p_1}{p_2}x_1^A \Rightarrow x_2^A = 0.82x_1^A,$$

and substituting this value into consumer A's budget constraint gives

$$p_1x_1^A + p_2(0.82x_1^A) = rK^A + wL^A.$$

Putting in our calculated values and solving this expression for x_1^A yields

$$x_1^{A,*} = 0.64,$$

$$x_2^{A,*} = 0.82x_1^{A,*} = 0.53.$$

Performing the same process with the tangency condition of consumer B (equation 6.2) yields

$$x_1^{B,*} = 0.90$$

$$x_2^{B,*} = 0.74\ ,$$

which completes our WEA:

$$(x_1^A, x_2^A; x_1^B, x_2^B; \frac{p_1}{p_2}; L_1, L_2, K_1, K_2) = \left(0.64, 0.53; 0.90, 0.74; 0.82; \frac{1}{2}, \frac{3}{2}, \frac{9}{4}, \frac{3}{4}\right). \blacksquare$$

Equilibrium with Production—Welfare In this subsection we seek to extend the first and second welfare theorems to economies with production, connecting WEA and PEAs. Before stating the first welfare theorem, let us define what we mean by a PEA in economies with production.

Pareto efficiency A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no other feasible allocation $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ such that

$$u^i(\overline{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$$

for every consumer $i \in I$, with $u^i(\overline{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.

That is, a feasible allocation of bundles to consumers and production plans to firms is Pareto efficient if there is no other feasible allocation that makes at least one consumer strictly better off and no consumer worse off.

As in section 6.4.1 analyzing barter economies, let us describe how to mathematically find the set of PEAs. In particular, in an economy with two goods, two consumers, two firms, and two inputs (labor and capital), the set of PEAs solves

$$\max_{x_1^1, x_2^1, x_1^2, x_1^2, L_1, K_1, L_2, K_2 \geq 0} u^1(x_1^1, x_2^1)$$

$$\text{subject to } u^2(x_1^2, x_2^2) \geq \bar{u}^2,$$

$x_1^1 + x_2^1 \leq F_1(L_1, K_1)$ and $x_1^2 + x_2^2 \leq F_2(L_2, K_2)$ (technological feasibility), and

$L_1 + L_2 \leq \overline{L}$ and $K_1 + K_2 \leq \overline{K}$ (input feasibility).

The Lagrangian associated with this maximization problem is

$$\mathcal{L} = u^1(x_1^1, x_2^1) + \lambda\left[u^2(x_1^2, x_2^2) - \bar{u}^2\right] + \mu_1\left[F_1(L_1, K_1) - x_1^1 - x_1^2\right]$$
$$+\mu_2\left[F_2(L_2, K) - x_2^1 - x_2^2\right] + \delta_L\left[\bar{L} - L_1 - L_2\right] + \delta_K\left[\bar{K} - K_1 - K_2\right].$$

In the case of interior solutions, the set of first-order conditions yield a condition for efficiency in consumption that we also found in barter economies, $MRS_{1,2}^1 = MRS_{1,2}^2$. The first-order conditions with respect to inputs $L_j$and K_j yield a condition for efficiency that we encountered in the chapter on production theory,

$$\frac{\partial F_j/\partial L}{\partial F_j/\partial K} = \frac{\partial F_m/\partial L}{\partial F_m/\partial K} \quad \text{for every two firms } j \neq m .$$

That is, the marginal rate of technical substitution, $MRTS_{L,K}$, must coincide across firms. Otherwise, welfare could be increased by assigning more labor to the firm with the highest $MRTS_{L,K}$. Finally, combining the two conditions above for efficiency in consumption and production, we obtain

$$\frac{\partial U^i/\partial x_1^i}{\partial U^i/\partial x_2^i} = \frac{\partial F_2/\partial L}{\partial F_1/\partial L} .$$

Or, more completely, $MRS_{1,2}^i$ must coincide with the rate at which units of good 1 can be transformed into units of good 2, namely the marginal rate of transformation $MRT_{1,2}$. Indeed, if we move labor from firm 2 to firm 1, the production of good 2 increases by $\partial F_2/\partial L$ while that of good 1 decreases by $\partial F_1/\partial L$. Hence, in order to increase the total output of good 1 by one unit, we need $(\partial F_2/\partial L)/(\partial F_1/\partial L)$ units of good 2. Intuitively, for an allocation to be efficient, we need that the rate at which consumers are willing to substitute goods 1 and 2 coincides with the rate at which good 1 can be transformed into good 2. We can now use this definition of Pareto efficiency to state the first welfare theorem in economies with production.

First Welfare Theorem with Production If the utility function of every individual i, u^i, is strictly increasing, then every WEA is Pareto efficient.

Proof We will prove this result by contradiction. In particular, suppose that $(\mathbf{x}, \mathbf{y})$ is a WEA at prices $\mathbf{p}^*$ but is *not* Pareto efficient. Because $(\mathbf{x}, \mathbf{y})$ is a WEA, it is feasible that is,

$$\sum_{i=1}^{I} \mathbf{x}^i = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j . \qquad (A)$$

In addition, because $(\mathbf{x}, \mathbf{y})$ is *not* Pareto efficient, there exists some other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$ for every consumer $i \in I$, with $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer. That is, the alternative allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ makes at least one consumer strictly better off than WEA $(\mathbf{x}, \mathbf{y})$without making others worse off. Since utility function $u^i(\cdot)$ is increasing, this implies that bundle $\hat{\mathbf{x}}^i$ contains more of at least one good than $\mathbf{x}^i$, and thus is more costly than $\mathbf{x}^i$, meaning $\mathbf{p}^* \cdot \hat{\mathbf{x}}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i$ for every individual i (with at least one strict inequality). Summing over all consumers yields

$$\mathbf{p}^* \cdot \sum_{i=1}^{I} \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \sum_{i=1}^{I} \mathbf{x}^i. \qquad (B)$$

Combining inequalities A and B with the feasibility of allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ yields

$$\mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \hat{\mathbf{y}}^j \right) > \mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j \right),$$

or, after rearranging,

$$\mathbf{p}^* \cdot \sum_{j=1}^{J} \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \sum_{j=1}^{J} \mathbf{y}^j.$$

However, this result implies that $\mathbf{p}^* \cdot \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \mathbf{y}^j$ for some firm j, indicating that production plan $\mathbf{y}^j$ was *not* profit maximizing and, as a consequence, it cannot be part of a WEA. We therefore reached a contradiction, implying that the original statement was true: if an allocation $(\mathbf{x}, \mathbf{y})$ is a WEA, it must also be Pareto efficient. ■

Example 6.9: *WEA and PEA with production* Consider the setting described in example 6.8. The set of PEAs must satisfy

$$MRS^A_{1,2} = MRS^B_{1,2} \quad \text{and} \quad MRTS^1_{L,K} = MRTS^2_{L,K}.$$

From equation (6.1) we have

$$MRS^A_{1,2} = \frac{x_2^A}{x_1^A} = \frac{0.53}{0.64} = 0.82,$$

and from equation (6.2),

$$MRS^B_{1,2} = \frac{x_2^B}{x_1^B} = \frac{0.74}{0.90} = 0.82,$$

which implies that $MRS^A_{1,2} = MRS^B_{1,2}$. Likewise from equation (6.4) we have

$$MRTS^1_{L,K} = 3\frac{L_1}{K_1} = 3\frac{1/2}{9/4} = \frac{2}{3},$$

and from equation (6.5),

$$MRTS^2_{L,K} = \frac{1}{3}\frac{L_2}{K_2} = \frac{1}{3}\frac{3/2}{3/4} = \frac{2}{3},$$

which implies that $MRTS^1_{L,K} = MRTS^2_{L,K}$. Since both of these conditions hold, our WEA from example 6.8 is Pareto efficient.

Second Welfare Theorem with Production Consider, again, the assumptions on consumers and producers described above. For every PEA $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ we can now find: (1) a profile of income transfers $(T_1, T_2, \ldots T_I)$ redistributing income among consumers, namely satisfying $\sum_{i=1}^{I} T_i = 0$, and (2) a price vector $\bar{\mathbf{p}}$.

1. Bundle $\hat{\mathbf{x}}^i$ solves the UMP

$$\max_{\mathbf{x}^i} \quad u^i(\mathbf{x}^i) \quad \text{subject to} \quad \bar{\mathbf{p}} \cdot \mathbf{x}^i \le m^i(\bar{\mathbf{p}}) + T_i \quad \text{for every } i \in I,$$

where individual i's original income $m^i(\bar{\mathbf{p}})$ is increased (if the transfer he receives T_i is positive) or decreased (if it is negative).

2. Production plan $\hat{\mathbf{y}}^j$ solves the PMP

$$\max_{\mathbf{y}^j} \; \bar{\mathbf{p}} \cdot \mathbf{y}^j \quad \text{subject to} \quad \mathbf{y}^i \in Y^j \quad \text{for every firm } j \in J.$$

Example 6.10: ***Second welfare theorem with production*** Consider an alternative allocation in the set of PEAs identified in example 6.9, such as $(\hat{x}^A_1, \hat{x}^A_2; \hat{x}^B_1, \hat{x}^B_2) = (0.75, 0.61; 0.79, 0.65)$. This allocation can be a WEA with the appropriate transfers. Consumer A's budget constraint becomes

$$p_1\hat{x}^A_1 + p_2\hat{x}^A_2 = rK^A + wL^A + T_1.$$

Recall that $(p_1, p_2;\ K^A, L^A;\ r, w) = (0.82, 1;\ 1, 1;\ 0.42, 0.63)$ remains unchanged from our WEA in example 6.8 due to our allocation being Pareto efficient. Substituting these values into consumer A's budget constraint gives

$$0.82\hat{x}^A_1 + \hat{x}^A_2 = 1.05 + T_1;$$

also recall that

$$\frac{p_1}{p_2} = \frac{\hat{x}_2^A}{\hat{x}_1^A} \Rightarrow \hat{x}_2^A = 0.82\hat{x}_1^A.$$

After substituting these results into the equation above, we obtain

$$2(0.82)\underbrace{(0.75)}_{\hat{x}_1^A} = 1.05 + T_1 \Rightarrow T_1 = 0.17.$$

Likewise for consumer B his budget constraint becomes

$$p_1\hat{x}_1^B + p_2\hat{x}_2^B = rK^B + wL^B + T_2,$$

and after substituting our unchanged values $(p_1, p_2;\ K^B, L^B;\ r, w) = (0.82, 1;\ 2, 1;\ 0.42, 0.63)$, we obtain

$$0.82\hat{x}_1^B + \hat{x}_2^B = 1.47 + T_2.$$

Similarly, for consumer B, we can write

$$\frac{p_1}{p_2} = \frac{\hat{x}_2^B}{\hat{x}_1^B} \Rightarrow \hat{x}_2^B = 0.82\hat{x}_1^B,$$

and then, by substituting, obtain

$$2(0.82)\underbrace{(0.79)}_{\hat{x}_1^B} = 1.47 + T_2 \Rightarrow T_2 = -0.17.$$

Clearly, $T_1 + T_2 = 0$, and thus these transfers allow for our new allocation to be a WEA.

6.5 Comparative Statics

In this section we briefly explore some comparative statics of our equilibrium results at the WEA in economies with production. Specifically, we analyze how equilibrium outcomes are affected by an increase in the price of one good, and then by an increase in the endowment of one input. For both questions, consider a setting with two goods, each being produced by two factors, 1 and 2, under constant returns to scale (CRS). Given CRS, a necessary condition for input prices (w_1^*, w_2^*) to be in equilibrium is that firms produce until their marginal costs equal the price of the good, that is,

$$c_1(w_1, w_2) = p_1 \quad \text{and} \quad c_2(w_1, w_2) = p_2.$$

For compactness, let $z_{1j}(w)$ denote firm j's demand for factor 1, and $z_{2j}(w)$ be its demand for factor 2. (This is equivalent to the factor demand correspondences $z(w,q)$ in the chapter on production theory where, for simplicity, we consider the production of one unit of output $q=1$, which helps us ignore the second argument of $z(w,q)$.) Hence we say that the production of good 1 is *relatively more intense* in factor 1 than is the production of good 2 if

$$\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)},$$

where $z_{1j}(w)/z_{2j}(w)$ represents firm j's demand for input 1 relative to that of input 2.

6.5.1 Changes in the Price of One Good p_j (Stolper–Samuelson Theorem)[10]

Consider an economy with two consumers and two firms satisfying the factor intensity assumption given above. If the price of good j, p_j, increases, then (1) the equilibrium price of the factor more intensively used in the production of good j increases while (2) the equilibrium price of the other factor decreases.

Proof Let us first take the equilibrium conditions about marginal costs being equal to output prices:

$$c_1(w_1,w_2) = p_1 \quad \text{and} \quad c_2(w_1,w_2) = p_2.$$

Differentiating the two prices, we have

$$\frac{\partial c_1(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_1(w_1,w_2)}{\partial w_2} dw_2 = dp_1,$$

$$\frac{\partial c_2(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_2(w_1,w_2)}{\partial w_2} dw_2 = dp_2 .$$

Applying Shephard's lemma, $\partial c_i(w_1,w_2)/\partial w_j = z_{ij}(w)$ results in

$$z_{11}(w)dw_1 + z_{12}(w)dw_2 = dp_1,$$

$$z_{21}(w)dw_1 + z_{22}(w)dw_2 = dp_2 .$$

Hence, if only price p_1 varies, then $dp_2 = 0$. We can rewrite the second expression as $dw_2 = (-z_{21}/z_{22})/dw_1$. We can now use the first expression. In particular, solving for dw_1/dp_1 yields

10. See Stolper and Samuelson (1941).

$$\frac{dw_1}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}}.$$

Solving, instead, for dw_2/dp_1 yields

$$\frac{dw_2}{dp_1} = -\frac{z_{21}}{z_{11}z_{22} - z_{12}z_{21}}$$

From the factor intensity condition, $z_{11}(w)/z_{21}(w) > z_{12}(w)/z_{22}(w)$, we know that $z_{11}z_{22} - z_{12}z_{21} > 0$ (the denominator in both $dw_1/dp_1$and dw_2/dp_1 is positive). Hence, since the numerator is also positive (they are just factor demands), the overall sign of the previous expressions is

$$\frac{dw_1}{dp_1} > 0 \quad \text{and} \quad \frac{dw_2}{dp_1} < 0.$$

Intuitively, if the price of good 1 increases, the price of input 1 (the input more intensively used in the production of good 1), w_1, increases while that of the other input (less intensively used than input 1), w_2, decreases. ■

Example 6.11: ***Stolper–Samuelson theorem*** Returning to our problem in Example 6.8, let us now solve for the input demands:

$$r_1 = p_1 0.75 K_1^{-0.25} L_1^{0.25} \Rightarrow z_{11} = K_1 = \left(\frac{0.75 p_1}{r}\right)^4 L_1,$$

$$w_1 = p_1 0.25 K_1^{0.75} L_1^{-0.75} \Rightarrow z_{21} = L_1 = \left(\frac{p_1}{4w}\right)^{4/3} K_1,$$

$$r_2 = p_2 0.25 K_2^{-0.75} L_2^{0.75} \Rightarrow z_{12} = K_2 = \left(\frac{p_2}{4r}\right)^{4/3} L_2,$$

$$w_2 = p_2 0.75 K_2^{0.25} L_2^{-0.25} \Rightarrow z_{22} = L_2 = \left(\frac{0.75 p_2}{w}\right)^4 K_2.$$

Since firm 1 is more capital intensive than firm 2, $z_{11}z_{22} - z_{12}z_{21} > 0$ must hold, that is,

$$\left(\frac{0.75 p_1}{r}\right)^4 L_1 \left(\frac{0.75 p_2}{w}\right)^4 K_2 - \left(\frac{p_1}{4w}\right)^{4/3} K_1 \left(\frac{p_2}{4r}\right)^{4/3} L_2 > 0.$$

Recall from example 6.8 that $K_1/L_1 = 9(K_2/L_2) \Rightarrow K_1/L_2 = 9K_2L_1$. Substituting these values and simplifying the expression above gives

$$36.33\left(\frac{p_1 p_2}{rw}\right)^{8/3} - 1 > 0 \Rightarrow \frac{p_1 p_2}{rw} > 1.16$$

and in our solution, $p_1 p_2 / rw = 3.08$; hence this condition is satisfied.

Next, observe that both z_{22} and z_{11} are trivially positive. We can apply the Stolper–Samuelson theorem at this point to find

$$\frac{dw_1}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}} > 0\,,$$

$$\frac{dw_2}{dp_1} = -\frac{z_{21}}{z_{11}z_{22} - z_{12}z_{21}} < 0\,.$$

6.5.2 Changes in Endowments (Rybczynski's Theorem)[11]

Let us now examine how the equilibrium output is affected by a change in the endowment of one input. Consider an economy with two consumers and two firms satisfying the factor intensity assumption given above. Additionally assume that this is a small open economy, so output prices are given (no market power). In this setting, if the endowment of a factor increases, the production of the good that uses this factor more intensively increases, whereas the production of the other good decreases.

Proof Consider an economy with two factors, labor and capital, and two goods, 1 and 2. In addition recall that for a firm j, $z_{Lj}(w)$ denotes its factor demand for labor (when producing one unit of output), and similarly $z_{Kj}(w)$ represents its factor demand for capital. Then factor feasibility requires that

$$L = z_{L1}(w) \cdot y_1 + z_{L2}(w) \cdot y_2,$$

where the first (second) term measures the units of labor that firm 1 (2, respectively) demands. A similar condition applies to capital:

$$K = z_{K1}(w) \cdot y_1 + z_{K2}(w) \cdot y_2\,.$$

Differentiating the first condition, we have

$$dL = z_{L1} \cdot \frac{\partial y_1}{\partial L} + z_{L2} \cdot \frac{\partial y_2}{\partial L}\,.$$

11. See Rybczynski (1955).

Then dividing both sides by the aggregate amount of labor yields

$$\frac{dL}{L} = \frac{z_{L1}}{L} \cdot \frac{\partial y_1}{\partial L} + \frac{z_{L2}}{L} \cdot \frac{\partial y_2}{\partial L}.$$

We multiply the first term on the right-hand side by y_1/y_1 and the second term by y_2/y_2 to obtain

$$\frac{dL}{L} = \frac{z_{L1} \cdot y_1}{L} \cdot \frac{\partial y_1/\partial L}{y_1} + \frac{z_{L2} \cdot y_2}{L} \cdot \frac{\partial y_2/\partial L}{y_2},$$

which can be more compactly expressed using $z_{Li}(w) \cdot y_i/L \equiv \gamma_{Li}$, which is the share of labor used by firm i, $(\partial y_i \partial L)/y_i \equiv \%\Delta y_i$, which is the percentage increase in the production of firm i brought by the increase in the endowment of labor; and $dL/L \equiv \%\Delta L$,which is the percentage increase in the endowment of labor in the economy. As a consequence the expression above becomes

$$\%\Delta L = \gamma_{L1} \cdot (\%\Delta y_1) + \gamma_{L2} \cdot (\%\Delta y_2).$$

A similar argument with the endowment of capital yields

$$\%\Delta K = \gamma_{K1} \cdot (\%\Delta y_1) + \gamma_{K2} \cdot (\%\Delta y_2).$$

In addition, note that labor shares $\gamma_{L1}, \gamma_{L2} \in (0,1)$ and that $\gamma_{L1}, \gamma_{L2} = 1$, implying that $\%\Delta L$ is a linear combination of $\%\Delta y_1$ and $\%\Delta y_2$, and a similar argument for capital shares $\gamma_{K1}.\gamma_{K2} \in (0,1)$ and $\%\Delta K$. Finally, since capital is assumed to be more intensively used in firm 1, in that , or $\gamma_{K1} > \gamma_{L1}$ for firm 1 and $\gamma_{K2} > \gamma_{L2}$ for firm 2. As a consequence, observing the expression describing the percentage change in the endowment of capital and labor, if capital becomes relatively more abundant than labor, in that $\%\Delta K > \%\Delta L$, then it must be that $\%\Delta y_1$ is larger than $\%\Delta y_2$. That is,

$$\begin{array}{ccccccccc}
\%\Delta L & = & \gamma_{L1} & \cdot & \underbrace{(\%\Delta y_1)} & + & \gamma_{L2} & \cdot & \underbrace{(\%\Delta y_2)} \\
\wedge & & \wedge & & \shortparallel & & \vee & & \shortparallel \\
\%\Delta K & = & \gamma_{K1} & \cdot & \overbrace{(\%\Delta y_1)} & + & \gamma_{K2} & \cdot & \overbrace{(\%\Delta y_2)}
\end{array}.$$

In other words, the change in the input endowment produces a more than proportional increase in the good whose production was more intensive in the use of that input, for example, a 1 percent increase in the capital endowment increases y_1 by more than 1 percent. The converse argument applies for labor and the production of good 2. More compactly, $\%\Delta y_1 > \%\Delta K$ and $\%\Delta L < \%\Delta y_2$

Alternative proof We can alternatively prove the above result by explicitly finding $\%\Delta y_1$ and $\%\Delta y_2$. Consider again the expressions above describing the increase in the labor and capital endowments:

$$\%\Delta L = \gamma_{L1} \cdot (\%\Delta y_1) + \gamma_{L2} \cdot (\%\Delta y_2),$$

$$\%\Delta K = \gamma_{K1} \cdot (\%\Delta y_1) + \gamma_{K2} \cdot (\%\Delta y_2),$$

or, more compactly,

$$\begin{bmatrix} \%\text{”}\, L \\ \%\text{”}\, K \end{bmatrix} = \begin{bmatrix} \gamma_{L1} & \gamma_{L2} \\ \gamma_{K1} & \gamma_{K2} \end{bmatrix} \cdot \begin{bmatrix} \%\text{”}\, y_1 \\ \%\text{”}\, y_2 \end{bmatrix}.$$

Applying Cramer's rule to obtain $\%\Delta y_1$ and $\%\Delta y_2$, we find that

$$\begin{bmatrix} \%\text{”}\, y_1 \\ \%\text{”}\, y_2 \end{bmatrix} = \begin{bmatrix} \gamma_{L1} & \gamma_{L2} \\ \gamma_{K1} & \gamma_{K2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \%\text{”}\, L \\ \%\text{”}\, K \end{bmatrix} = \frac{1}{\gamma_{L1}\gamma_{K2} - \gamma_{K1}\gamma_{L2}} \begin{bmatrix} \gamma_{K2} & -\gamma_{L2} \\ -\gamma_{K1} & \gamma_{L1} \end{bmatrix} \begin{bmatrix} \%\text{”}\, L \\ \%\text{”}\, K \end{bmatrix}.$$

In this context, we can define the term in the denominator as $A = \gamma_{L1}\gamma_{K2} - \gamma_{K1}\gamma_{L2}$, which is negative, since firm 1 is more capital intensive than firm 2, given that $\gamma_{K1} > \gamma_{L1}$ and $\gamma_{K2} > \gamma_{L2}$. As a consequence the expression above can be more compactly represented as

$$\%\text{”}\, y_1 = \frac{1}{A}[\gamma_{K2}(\%\text{”}\, L) - \gamma_{L2}(\%\text{”}\, K)],$$

$$\%\text{”}\, y_2 = \frac{1}{A}[-\gamma_{K1}(\%\text{”}\, L) + \gamma_{L1}(\%\text{”}\, K)].$$

For simplicity, consider that only the capital endowment changes, $\%\Delta K > 0$ and $\%\Delta L = 0$, which further simplifies the expressions above to

$$\%\text{”}\, y_1 = -\frac{1}{A}[\gamma_{L2}(\%\text{”}\, K)] > 0,$$

$$\%\text{”}\, y_2 = \frac{1}{A}[\gamma_{L1}(\%\text{”}\, K)] < 0,$$

since $A < 0$, as shown above. Hence $\%\Delta\gamma_1 > 0 > \%\Delta\gamma_2$. In words, as capital endowment increases the production of the capital-intensive good increases, whereas the production of the labor-intensive good decreases. Let us now examine the relative percentage change. Expanding term A in the expression above of $\%\Delta y_1$ yields

$$\%\text{”}\, y_1 = -\frac{\gamma_{L2}}{\gamma_{L1}\gamma_{K2} - \gamma_{K1}\gamma_{L2}}(\%\text{”}\, K) = \frac{\gamma_{L2}}{\gamma_{K1}\gamma_{L2} - \gamma_{L1}\gamma_{K2}}(\%\text{”}\, K).$$

Rearranging, and dividing the right-hand side by γ_{L2}, we obtain

$$\frac{\%\text{''}\, y_1}{\%\text{''}\, K} = \frac{1}{\gamma_{K1} - \gamma_{L1}(\gamma_{K2}/\gamma_{L2})}.$$

Finally, the right-hand side yields a number larger 1. Indeed, the infimum of the right-hand term occurs when the denominator reaches its highest value, which is at $\gamma_{K1}=1$ and $\gamma_{L1}(\gamma_{K2}/\gamma_{L2})=0$. Hence $\%\Delta y_1/\%\Delta K>1$, implying that $\%\Delta y_1>\%\Delta K$, as required. (A similar proof applies if we expand term A in the expression of $\%\Delta y_2$, yielding $\%\Delta y_2>\%\Delta L$, which is left to the reader for practice.) ■

Example 6.12: *Rybczynski theorem* Consider the production decisions of the two firms in example 6.8, where we found that $K_1=3K_2$ and $K_1+K_2=\bar{K}=3$. Assume that total endowment of capital increases to $\bar{K}=5$, which is to say, $K_2=5-K_1$, yielding profit-maximizing demands for capital of

$$K_1 = 3(5-K_1) \Rightarrow K_1^* = \frac{15}{4},$$

$$K_2 = \frac{1}{3}K_1^* = \frac{5}{4}.$$

Similarly for labor we found that $L_1 = \frac{1}{3}L_2$ and $L_1 + L_2 = \bar{L} = 2$. However, we do not alter the aggregate endowment of labor, $\bar{L}=2$, as we seek to increase the endowment of the input more intensively used by firm 1 (capital). We have thus shown that capital use by firm 1 increases from $K_1^* = 9/4$ to 15/4. That is, if firm 1 uses capital more intensively than firm 2 does, meaning $K_1/L_1>K_2/L_2$, since $(9/4)/(1/2)>(3/4)/(3/2)$, an increase in the endowment of capital of $(5-3)/3=0.66=0.66$ percent entails an increase in good 1's output by 100 percent while that of good 2 decreases by 33.33 percent. We can show this by calculating the factor demands for each good, which are given as

$$z_{K1} = \left(\frac{3r}{w}\right)^{-0.75} \quad \text{and} \quad z_{L1} = \left(\frac{3r}{w}\right)^{0.25} \quad \text{for good 1,}$$

$$z_{K2} = \left(\frac{r}{3w}\right)^{-0.25} \quad \text{and} \quad z_{L2} = \left(\frac{r}{3w}\right)^{0.75} \quad \text{for good 2.}$$

Using values from example 6.8, we can assign values of shares of each input used by each firm as $(\gamma_{K1}, \gamma_{L1}, \gamma_{K2}, \gamma_{L2})=(0.75, 0.25, 0.25, 0.75)$. Our two equations then become

$$0 = 0.25 \cdot (\%\Delta y_1) + 0.75 \cdot (\%\Delta y_2)\,,$$

$$0.66 = 0.75 \cdot (\%\Delta y_1) + 0.25 \cdot (\%\Delta y_2)\,,$$

which, solving simultaneously for $\%\Delta y_1$ and $\%\Delta y_2$, yields values of $\%\Delta y_1 = 1 = 100$ percent and $\%\Delta y_2 = 0.3333 = -33.33$ percent.

6.6 Introducing Taxes

6.6.1 Tax on Goods

Assume that as a result of a sales tax t_k imposed on good k, the price paid by consumers, p_k^C, increases by $p_k^C = (1 + t_j)\, p_k^P$, where p_k^P is the price received by producers. In this circumstance, if the tax on goods 1 and 2 coincides (i.e., $t_1 = t_2$), the price ratio consumers and producers face is unaffected because

$$\frac{p_1^C}{p_2^C} = \frac{(1+t_1)\, p_1^P}{(1+t_2)\, p_2^P} = \frac{p_1^P}{p_2^P}\,.$$

Hence the after-tax allocation is still Pareto efficient. However, if only good 1 is affected by the tax (i.e., $t_1 > 0$) while $t_2 = 0$, or more generally, if each good is subject to a different tax, (i.e., $t_1 \neq t_2$), then the allocation will not be Pareto efficient. In this setting, the $MRTS_{L,K}$ is still the same as before the introduction of the tax, since

$$\frac{\partial F_1/\partial L}{\partial F_1/\partial K} = \frac{w_L}{w_K} = \frac{\partial F_2/\partial L}{\partial F_2/\partial K}$$

is unaffected by the tax. Therefore the allocation of inputs still achieves efficiency in production. Similarly the $MRT_{1,2}$ still coincides with the price ratio of goods 1 and 2, that is,

$$\frac{\partial F_2/\partial L}{\partial F_1/\partial L} = \frac{p_1^P}{p_2} = \frac{\partial F_2/\partial K}{\partial F_1/\partial K}\,,$$

where recall that the price received by the producer, p_1^P, is the same before and after introducing the tax. However, while the $MRS_{1,2}$ is equal to the price ratio that consumers face, which is p_1^C / p_2 or $(1+t_1)\, p_1^P / p_2$, it now becomes larger than the price ratio that producers face, p_1^P / p_2.

$$MRS_{1,2} = \frac{p_1^C}{p_2} = \frac{(1+t_1)\, p_1^P}{p_2} > \frac{p_1^P}{p_2}\,.$$

Intuitively, the rate at which consumers are willing to substitute good 1 for 2 is larger than the rate at which firms can transform good 1 for 2. As a consequence the production of good 1 should decrease and that of good 2 increase.

6.6.2 Tax on Inputs

Similar arguments extend to the introduction of taxes on inputs, yielding a price paid by producers (firms hiring the input) of $w_m^P = (1+t_m)w_m^C$ for input $m=\{L,K\}$, where w_m^C represents the price that input owners (consumers) receive. Specifically, if both inputs are subject to the same tax, $t_L=t_K=t$, the input price ratio consumers and producers face coincides, yielding

$$w_L^P / w_K^P = (1+t)w_L^C / (1+t)w_K^C = w_L^C / w_K^C ,$$

so the efficiency conditions remain unaffected. However, when they differ, $t_L \neq t_K$ (or, as a special case, when only one input is subject to taxes), such condition for productive efficiency no longer holds. Indeed, while input consumers satisfy $w_L^C / w_K^C = (\partial F_1/\partial L)/(\partial F_1/\partial K)$ and input producers satisfy $w_L^P / w_K^P = (\partial F_2/\partial L)/(\partial F_2/\partial K)$, the input price ratios they face do not coincide, that is,

$$\frac{\partial F_1/\partial L}{\partial F_1/\partial K} = \frac{w_L^C}{w_K^C} \neq \frac{(1+t_L)w_L^C}{(1+t_K)w_K^C} = \frac{w_L^P}{w_K^P} = \frac{\partial F_2/\partial L}{\partial F_2/\partial K} .$$

For instance, if $t_L > t_K$, the $MRTS_{L,K}$ is larger for firm 1 than 2, implying that the allocation of inputs is inefficient, in that the marginal productivity of additional units of labor (relative to capital) is larger in firm 1 than in 2.

Further reading For a detailed discussion of general equilibrium analysis (including its connections with game theory), see Mas-Colell, Whinston, and Green (1995, chs. 17–20). For the application of the contents in this chapter to computable general equilibrium (CGE) models, see Ginsburgh and Keyter (1997) and the empirical references therein.

Appendix A: Large Economies and the Core

While we know that equilibrium allocations (WEAs) are part of the core, in this appendix we seek to show that, as the economy becomes larger, the core shrinks until exactly coinciding with the set of WEAs.

Let us first consider an economy with I consumers, each with utility function u^i and endowment vector $\mathbf{e}^i$, and next consider this economy's replica by doubling the number of consumers to $2I$, each of them still with utility function u^i and endowment vector

$\mathbf{e}^i$. Intuitively, there are now two consumers of each type, namely "twins," having identical preferences and endowments. We can now define an r-fold replica economy $\boldsymbol{\varepsilon}_r$, having r consumers of each type, for a total of rI consumers. For any consumer type $i \in I$, all r consumers of that type share the common utility function u^i and have identical endowments $\mathbf{e}^i \gg 0$. As a consequence, when comparing two replica economies, the largest will be that having more of every type of consumer.

In the context of replica economies, we need to adapt our notation from previous sections to keep track of consumer types and their number. In particular, allocation $\mathbf{x}^{iq}$ indicates the vector of goods for the qth consumer of type i (you can think about consumer i existing in the original economy, and now having r twins in the r-fold replica economy). Given this notation, we can rewrite feasibility in this setting as follows:

$$\sum_{i=1}^{I}\sum_{q=1}^{r}\mathbf{x}^{iq} = r\sum_{i=1}^{I}\mathbf{e}^{i} ,$$

since each of the r consumers of type i has a endowment vector $\mathbf{e}^i$.

Let us now examine the core of this replica economy $\boldsymbol{\varepsilon}_r$. An important property of the core in the r-fold replica economy, is that not only similar type of consumers start with the same endowment vector $\mathbf{e}^i$, but they also end up with the same allocation at the core; a property often referred to as "equal treatment at the core."

Equal Treatment at the Core

If $\mathbf{x}$ is an allocation in the core of the r-fold replica economy $\boldsymbol{\varepsilon}_r$, then every consumer of type i must have the same bundle, $\mathbf{x}^{iq} = \mathbf{x}^{iq'}$, for every two "twins" q and q' of type i, $q \neq q' \in \{1, 2, \ldots, r\}$, and for every type $i \in I$.

Proof We will prove the result above for a twofold replica economy, ε_2, since the result can be easily generalized to r-fold replicas. Suppose that allocation

$$\mathbf{x} \equiv \left\{\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22}\right\}$$

is an allocation at the core of ε_2 (as required in the premise of the above claim). Since $\mathbf{x}$ is in the core, then it must be feasible, that is,

$$\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^{1} + 2\mathbf{e}^{2}$$

because the two type-1 consumers have identical endowments, and so do the two type-2 consumers. Given this setup, let us prove the "equal treatment at the core" property by contradiction. That is, assume that allocation $\mathbf{x}$, despite being at the core, does not assign the same consumption vectors to the two twins of type-1, namely $\mathbf{x}^{11} \neq \mathbf{x}^{12}$. And, without loss of generality, assume that type-1 consumer weakly prefers $\mathbf{x}^{11}$ to $\mathbf{x}^{12}$, namely $\mathbf{x}^{11} \succsim^1 \mathbf{x}^{12}$, which is true for both type-1 twins because they have the same preferences. (A similar result emerges if we instead assume that $\mathbf{x}^{12} \succsim^1 \mathbf{x}^{11}$ for both type-1 consumers.) Figure A6.1 depicts the two cases embodied in $\mathbf{x}^{11} \succsim^1 \mathbf{x}^{12}$, namely $\mathbf{x}^{11} \sim^1 \mathbf{x}^{12}$ (left panel) and $\mathbf{x}^{11} \succ^1 \mathbf{x}^{12}$ (right panel).

At this point of the proof, let us stop for a second to recall what we look for. Since we are operating by contradiction, we need that when the premise of the claim is satisfied (allocation $\mathbf{x}$ is at the core) but the conclusion is violated (*unequal* treatment at the core, $\mathbf{x}^{11} \neq \mathbf{x}^{12}$), we end up with the original premise being contradicted (i.e., $\mathbf{x}$ is *not* at the core because we can find a blocking coalition). In search of such a blocking coalition, consider that for type-2 consumers we have $\mathbf{x}^{21} \succsim^2 \mathbf{x}^{22}$. (Note that this is done without loss of generality, as the same result would apply if we revert this preference relation, making the first type-2 consumer, 21, worse off.) Hence consumer 12 is the worst-off type-1 consumer, $\mathbf{x}^{11} \succsim^1 \mathbf{x}^{12}$, and consumer 22 is the worst-off type 2 consumer.

Let us now take these two "poorly treated" consumers of each type, and check if they can form a blocking coalition to oppose allocation $\mathbf{x}$. First, define the average bundles

$$\bar{\mathbf{x}}^{12} = \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} \quad \text{and} \quad \bar{\mathbf{x}}^{22} = \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2},$$

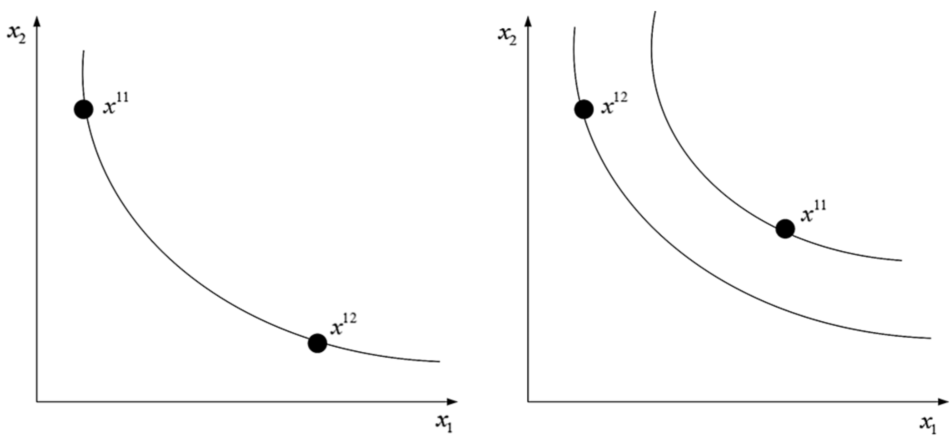

Figure A6.1
Unequal treatment at the core for type-1 consumers

where the first (second) bundle is the average of the bundles going to the type-1 (type-2, respectively) consumers. Figure A6.2 superimposes these average bundles into the indifference curves depicted in figure A6.1.

Desirability Because of preferences being strictly convex, the worst-off type-1 consumer prefers $\bar{\mathbf{x}}^{12} \succ^1 \mathbf{x}^{12}$, since $\bar{\mathbf{x}}^{12}$ is a linear combination between $\mathbf{x}^{11}$ and his original bundle $\mathbf{x}^{12}$; as depicted in figure 6.24. A similar argument applies to the worst-off type-2 consumer, $\bar{\mathbf{x}}^{22} \succ^2 \mathbf{x}^{22}$. As a consequence we have now found a pair of bundles, namely the average bundles $\left(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22}\right)$, which would make both consumers 12 and 22 better off than at the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$.

Feasibility After showing desirability of $\left(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22}\right)$ over $(\mathbf{x}^{12}, \mathbf{x}^{22})$, the only question that remains is whether consumers 12 and 22 can achieve $\left(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22}\right)$, that is, whether it is feasible. In order to show feasibility, we can rewrite the amount of goods consumers 12 and 22 need to achieve $\left(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22}\right)$ as follows:

$$\begin{aligned}\bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{22} &= \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} + \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2} \\ &= \frac{1}{2}\left(\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22}\right)\end{aligned}$$

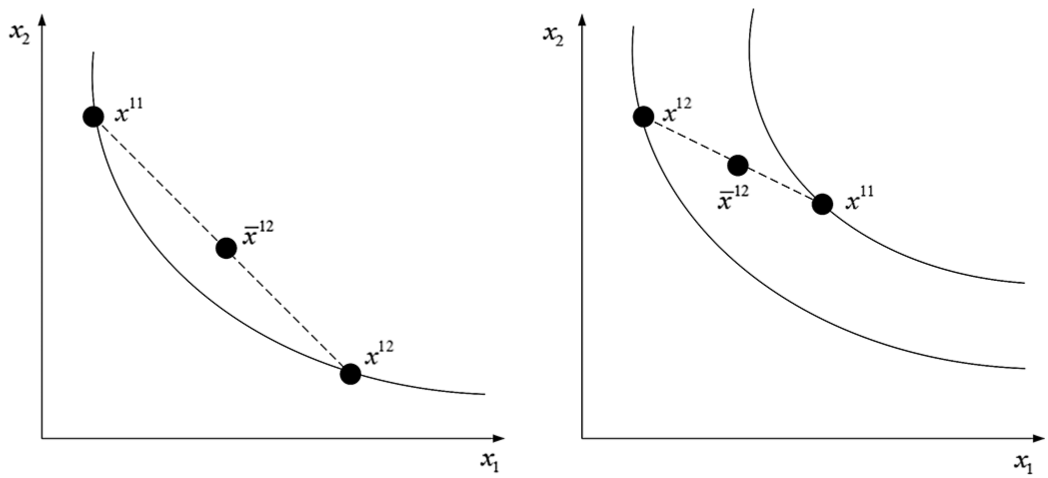

Figure A6.2
Average bundles leading to a blocking coalition

$$= \frac{1}{2}\left(2\mathbf{e}^1 + 2\mathbf{e}^2\right)$$

$$= \mathbf{e}^1 + \mathbf{e}^2 \quad ,$$

which coincides with these consumers' initial endowments. Hence the pair of bundles $\left(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22}\right)$ is feasible.

To sum up, since this pair of bundles makes consumers 12 and 22 better off than at the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$, and $\left(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22}\right)$ is feasible, these consumers can block $(\mathbf{x}^{12}, \mathbf{x}^{22})$. In other words, the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$ *cannot be at the core*, since we found a blocking coalition. Therefore, if an allocation is at the core of the replica economy, it must give consumers of the same type the exact same bundle. ■

The "equal treatment at the core" property we just showed helps us describe core allocations in a r-fold replica economy ε_r by reference to a similar allocation in the original (unreplicated) economy ε_1. In particular, if $\mathbf{x}$ is in the core of a r-fold replica economy $\boldsymbol{\varepsilon}_r$, $\mathbf{x} \in C_r$, then by the equal treatment property, allocation $\mathbf{x}$ must be of the form

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r\text{times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r\text{times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r\text{times}} \right)$$

because all consumers of the same type must receive the same bundle. Therefore core allocations in ε_r are just r-fold copies of allocations in ε_1, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$.

After proving the "equal treatment at the core" property, we can continue with our main goal of this section: to show that, as the economy becomes larger (r increases), the core shrinks, and if r is sufficiently large, then the core converges to the set of WEAs.

The Core Shrinks as the Economy Enlarges

The sequence of core sets C_1, C_2, … is decreasing. That is, the core of the original (unreplicated) economy, C_1, is a superset of that in the twofold replica economy, C_2. Similarly the core in the twofold replica economy, C_2, is a superset of the threefold replica economy, C_3; as depicted in figure A6.3.

Proof Since we seek to show that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots \supseteq C_{r-1} \supseteq C_r \supseteq \ldots$ it suffices to find that, for any $r > 1$, $C_{r-1} \supseteq C_r$. First, suppose that allocation $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}^I) \in C_r$. Intuitively, we cannot find any blocking coalition to $\mathbf{x}$ in the r-fold replica economy ε_r. We then need to show that $\mathbf{x}$ cannot be blocked by any coalition in the $(r-1)$-fold replica

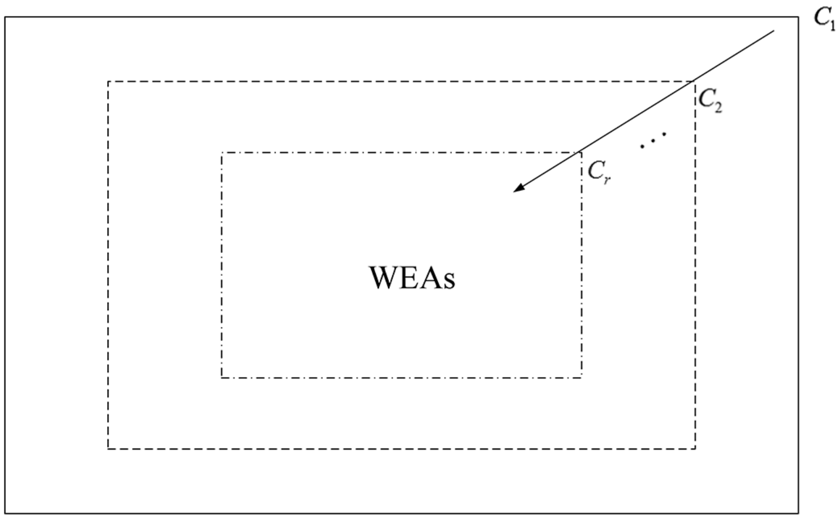

Figure A6.3
Core shrinks as r increases

economy ε_{r-1} either. This is easy! If we could find a blocking coalition to $\mathbf{x}$ in ε_{r-1}, then we could also find a blocking coalition in ε_r (and we could not, as $\mathbf{x}$ in the core C_r). Indeed, all members in ε_{r-1} are also present in the larger economy ε_r and their endowments have not changed.

While the result above shows that the core satisfies $C_{r-1} \supseteq C_r$, it does not exclude the possibility that the core does not shrink (keeping its size unaffected as the economy is replicated). That is, we still need to show that, as r increases, the core shrinks. We will do this by demonstrating that allocations at the frontier of C_1 do not belong to the core of the twofold replica economy, C_2. Consider figure A6.4, which depicts an unreplicated economy ε_1. The line between points $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is part of the core, but some point in this line are WEAs and others are not. For instance, $\tilde{\mathbf{x}}$ is not a WEA since the price line through $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is not tangent to the consumer's indifference curve at $\tilde{\mathbf{x}}$. (In addition, note that allocation $\tilde{\mathbf{x}}$, despite being at the core, yields the same utility level as endowment $\mathbf{e}$ for consumer 1. That is, is the "worst" admissible allocation for consumer 1 among all core allocations.)

If the core shrinks as the economy enlarges, we should be able to show that allocation $\tilde{\mathbf{x}}$ (lying at the "frontier" of the core C_1) does not belong to the core of the twofold replica economy C_2. In order to demonstrate that allocation $\tilde{\mathbf{x}} \notin C_2$, let us build a blocking coalition against $\tilde{\mathbf{x}}$, by finding that an alternative allocation is (1) *desired* by all coalition members and (2) *feasible* for coalition members.

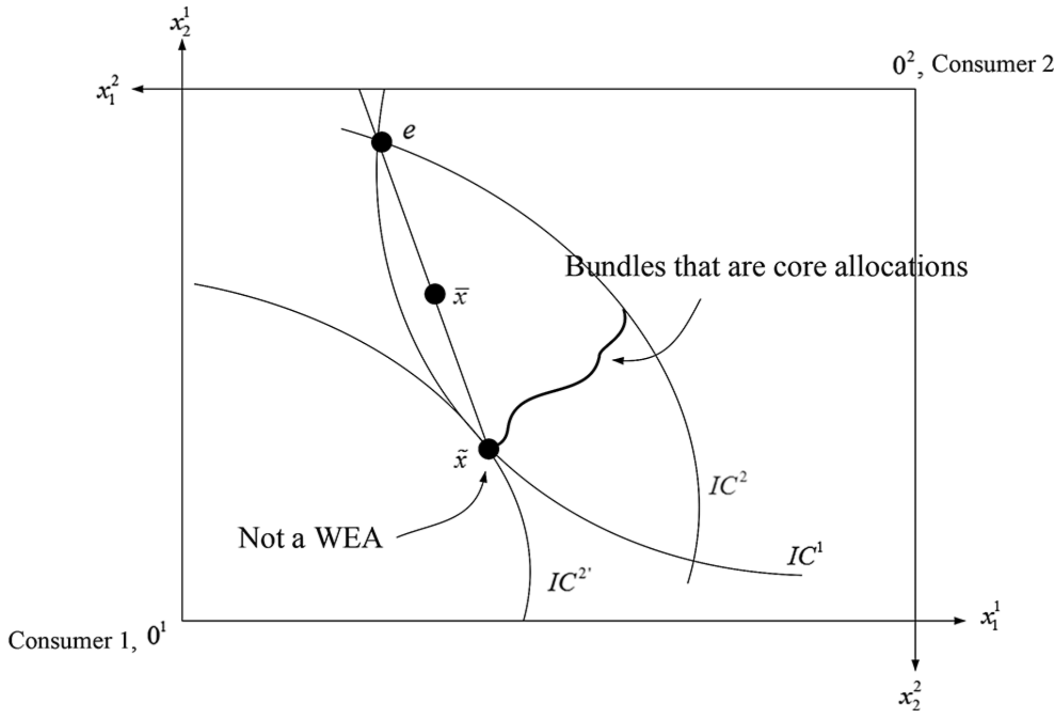

Figure A6.4
Unreplicated economy

Desirability Consider the midpoint allocation $\bar{\mathbf{x}}$ and the coalition $S = \{11, 12, 21\}$. As suggested in figure A6.4, such a midpoint in the line connecting $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is strictly preferred by both types of consumer 1 (recall that type-1 consumer now has a twin in the twofold replica economy). Specifically, if the midpoint allocation $\bar{\mathbf{x}}$ is offered to both types of consumers 1, 11, and 12, and to one of the consumer 2 types, they will all accept it.

$$\bar{\mathbf{x}}^{11} \equiv \frac{1}{2}\left(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}\right) \succ^1 \tilde{\mathbf{x}}^{11},$$

$$\bar{\mathbf{x}}^{12} \equiv \frac{1}{2}\left(\mathbf{e}^1 + \tilde{\mathbf{x}}^{12}\right) \succ^1 \tilde{\mathbf{x}}^{12},$$

$$\bar{\mathbf{x}}^{21} \sim^2 \tilde{\mathbf{x}}^{21}.$$

Feasibility Let us now check that the suggested allocation $\bar{\mathbf{x}} \equiv \left\{\bar{\mathbf{x}}^{11}, \bar{\mathbf{x}}^{12}, \tilde{\mathbf{x}}^{21}\right\}$ is feasible for coalition S. Since $\bar{\mathbf{x}}^{11} = \bar{\mathbf{x}}^{12}$, the sum of the suggested allocation yields

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = 2\frac{1}{2}\left(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}\right) + \tilde{\mathbf{x}}^{21}$$

$$= \mathbf{e}^1 + \tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21}.$$

Recall now that $\tilde{\mathbf{x}}$ was part of the unreplicated economy ε_1. It then must be feasible, meaning $\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$. Hence $\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \mathbf{e}^2$, which allows us to rewrite the equality above as

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \underbrace{\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21}}_{\mathbf{e}^1 + \mathbf{e}^2}$$

$$= \mathbf{e}^1 + \mathbf{e}^1 + \mathbf{e}^2$$

$$= 2\mathbf{e}^1 + \mathbf{e}^2 \quad ,$$

thus confirming feasibility. Hence the frontier allocation $\tilde{x}$ in the core of the unreplicated economy does not belong to the core of the twofold economy, $\tilde{x} \subsetneq C_2$, since we could identify a blocking coalition $S = \{11, 12, 21\}$ and an alternative feasible allocation $\bar{x} = \{\bar{x}^{11}, \bar{x}^{12}, \tilde{x}^{21}\}$ which they would prefer to $\tilde{x}$. ■

WEA in Replicated Economies

Consider a WEA in the unreplicated economy ε_1, $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$. Then an allocation $\mathbf{x}$ is a WEA for the r-fold replica economy if and only if it is of the form

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r\text{times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r\text{times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r\text{times}} \right).$$

Indeed, if $\mathbf{x}$ is a WEA for $\boldsymbol{\varepsilon}_r$, then it also belongs to the core of that economy. By the "equal treatment at the core" property, the result follows. We are now ready to present the main result of this section.

A Limit Theorem on the Core

If an allocation $\mathbf{x}$ belongs to the core of all r-fold replica economies then such allocation must be a WEA of the unreplicated economy $\boldsymbol{\varepsilon}_1$.

Proof Let us work by contradiction by considering that an allocation $\tilde{\mathbf{x}}$ belongs to the core of the r-fold replica economy C_r but is *not* a WEA. Figure A6.5 depicts a core

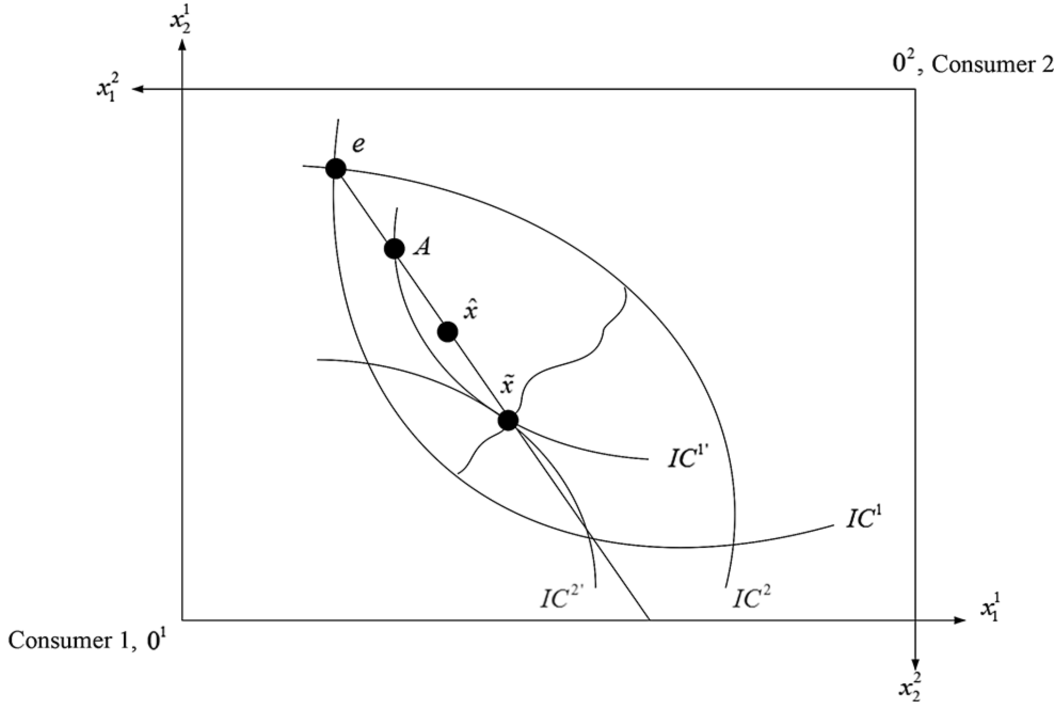

Figure A6.5
Core allocation $\tilde{\mathbf{x}}$ that is not WEA

allocation for the unreplicated economy ε_1, $\tilde{\mathbf{x}} \in C_1$, thus satisfying $\tilde{\mathbf{x}} \in C_r$ since $C_1 \supset C_r$. Allocation $\tilde{\mathbf{x}}$ must then be within the lens-shaped area and on the contract curve.

Consider now the line connecting $\tilde{\mathbf{x}}$ and $\mathbf{e}$. Since $\tilde{\mathbf{x}}$ is not a WEA, the budget line cannot be tangent to both consumers' indifference curves, implying that either $p_1/p_2 > MRS$ or $p_1/p_2 < MRS$. (The figure depicts the first case, as the budget line is steeper than the indifference curves at $\tilde{\mathbf{x}}$; the second case is analogous.) The question we now pose is: Can allocation $\tilde{\mathbf{x}}$ be at the core C_r and yet not be a WEA?

In order to show that such allocation must be a WEA if it is in the core C_r, let us work by contradiction, that is, by showing that if $\tilde{\mathbf{x}}$ is *not* a WEA it *cannot* be part of the core C_r either. To demonstrate that $\tilde{\mathbf{x}} \notin C_r$, let us find a blocking coalition. Specifically, by the convexity of preferences, we can find a set of bundles, such as those between A and $\tilde{\mathbf{x}}$ in figure A6.5, that consumer 1 prefers to $\tilde{\mathbf{x}}$. One example of such bundle is the linear combination

$$\hat{\mathbf{x}} \equiv \frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1$$

for some $r>1$, where $1/r+(r-1)/r=1$. Hence consider a coalition S with all r type-1 consumers and $r-1$ type-2 consumers. Let us separately show that allocation $\hat{\mathbf{x}}$ satisfies the properties of acceptance and feasibility for the blocking coalition S.

Acceptance If we give every type-1 consumer the bundle $\hat{\mathbf{x}}^1$, $\hat{\mathbf{x}}^1 \succ^1 \tilde{\mathbf{x}}^1$. Similarly, if we give every type-2 consumer in the coalition the bundle $\hat{\mathbf{x}}^2 = \tilde{\mathbf{x}}^2$, then $\hat{\mathbf{x}}^2 \sim^2 \tilde{\mathbf{x}}^2$.

Feasibility Summing over the consumers in coalition S, their aggregate allocation is

$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = r\left[\frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1\right] + (r-1)\tilde{\mathbf{x}}^2$$

$$= \mathbf{e}^1 + (r-1)\left(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2\right).$$

Since $\tilde{\mathbf{x}} \equiv \left(\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2\right)$ is in the core of the unreplicated economy $\boldsymbol{\varepsilon}_1$, it must be feasible, that is, $\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$. Combining the above two results, we find that

$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = \mathbf{e}^1 + (r-1)\underbrace{\left(\mathbf{e}^1 + \mathbf{e}^2\right)}_{\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2} = r\mathbf{e}^1 + (r-1)\mathbf{e}^2,$$

thus confirming feasibility.

Hence r type-1 consumers and $r-1$ type-2 consumers can get together in coalition S and block allocation $\tilde{\mathbf{x}}$. We have therefore shown that if $\tilde{\mathbf{x}}$ is not a WEA, then $\tilde{\mathbf{x}}$ cannot be in the core of the r-fold replica economy $\boldsymbol{\varepsilon}_r$. As a consequence, if $\tilde{\mathbf{x}} \in C_r$ for all $r>1$, then $\tilde{\mathbf{x}}$ must be a WEA. ■

Appendix B: Marshall–Hicks Four Laws of Derived Demand

Consider a production function $q=f(K,L)$, with positive marginal products, $f_L, f_K>0$. Assume that the supply of each input is positively sloped, $w(L)$ where $w'(L)>0$ and $r(K)$ where $r'(K)>0$.Demand for output is given by $q=g(p)$, which satisfies the law of demand, $g'(p)<0$ Total cost is then $w(L)L+r(K)K$. In addition assume that the capital market is perfectly competitive, but let us allow for the labor and output market to not necessarily be competitive.

In this context, define $\varepsilon_{q,p} \equiv (\partial q/\partial p)(p/q)$ as the price elasticity of output; $s_{K,r} \equiv (\partial K/\partial r)(r/K)$ as the elasticity of capital supply to a change in its price, r; $s_{L,r} \equiv (\partial L/\partial r)(r/L)$ as the elasticity of labor supply to a change in the price of capital, r; $\varepsilon_{L,w} \equiv (\partial L/\partial w)(w/L)$ as the elasticity of labor supply to a change in its own price, w; and let σ be the elasticity of substitution between inputs. We will use superscript i in the to refer to the elasticity that an individual firm faces, for example, $\varepsilon^{i}_{q,p}$, while industry elasticities will not include superscripts, such as $\varepsilon_{q,p}$. Additionally let $\theta_L \equiv wL/pq$ and $\theta_K \equiv rK/pq$ be the cost of labor and capital, respectively, relative to total sales, which implies that $\theta_L = 1 - \theta_K$. Finally, for compactness, let us define $A \equiv 1 - \left(1/\varepsilon^{i}_{q,p}\right)$ and $B \equiv 1 + \left(1/\varepsilon^{i}_{L,w}\right)$.

In this setting, Marshall, Hicks, and Allen separately analyze how the input demand of a perfectly competitive input, such as capital, is affected by a marginal change in the price of capital r, finding the following expression[12]:

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} A + \left(\sigma \varepsilon_{q,p}/s_{L,w}\right) A^2 + \theta_L AB\sigma}{\left(\theta_K + \theta_L B\right)^2 + \theta_K \left(\sigma/s_{L,w}\right) A + \theta_L \left(\sigma/s_{L,w}\right) AB}.$$

Marshall–Hicks's four laws of input demand (also known as "derived demand") state that an input demand becomes more elastic, whereby $s_{K,r}$ *decreases*, in (1) the elasticity of substitution between inputs, σ; (2) the price-elasticity of output demand, $\varepsilon_{q,p}$; (3) the cost of the input relative to total sales, $\theta_K = (rK/pq)$;and (4) the elasticity of the other input's supply to a change in its price, $s_{L,w}$. For simplicity, we analyze these four comparative statics under two common market structures considered in the literature: (1) the Marshall's presentation, which assumes that $\varepsilon^{i}_{q,p} = s^{i}_{L,w} = \infty$ and that inputs cannot be substituted in the production process, $\sigma = 0$, and (2) the Hick's presentation, which assumes that $\varepsilon^{i}_{q,p} = s^{i}_{L,w} = \infty$ but does not impose assumptions on the elasticity of substitution, σ.[13]

Marshall's Presentation

$\varepsilon^{i}_{q,p} = s^{i}_{L,w} = \infty$ for every firm i and $\sigma = 0$, which simplifies the expression of $s_{K,r}$ to

$$s_{k,r} = -\frac{\theta_K\, \varepsilon_{q,p}\, s_{L,w}}{s_{L,w} + \theta_L\, \varepsilon_{q,p}}.$$

Hence the derivatives testing the laws we stated above are

$$\frac{\partial s_{k,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K \left(s_{L,w}\right)^2}{\left[s_{L,w} + \theta_L \varepsilon_{q,p}\right]^2},$$

12. See Bronfenbrenner (1961) and Berra and Porto (1971) for a detailed analysis of input derived demands, and these elasticities.

13. See Hicks (1957), Allen (1967), andMarshall (1997).

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{s_{L,w} \cdot \varepsilon_{q,p}(s_{L,W} + \varepsilon_{q,p})}{\left[s_{L,w} + \theta_L \varepsilon_{q,p}\right]^2},$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L \left(\varepsilon_{q,p}\right)^2}{\left[s_{L,W} + \theta_L \varepsilon_{q,p}\right]^2}.$$

When labor is a "normal" input, $s_{L,w} > 0$, implying that the three derivatives above are all negative and thus the three laws hold. If labor is inferior, $s_{L,w} < 0$, $s_{K,r}$ is still decreasing in $\varepsilon^i_{q,p}$ and in $s^i_{L,w}$, but not necessarily in θ_K.

Hick's Presentation

Like Marshall, Hicks assumes that output and input markets are competitive, $\varepsilon^i_{q,p} = s^i_{L,W} = \infty$, but he does not impose condition $\sigma = 0$ on the substitutability of inputs. In this context, $s_{K,r}$ becomes

$$s_{K,r} = -\frac{\theta_K\, \varepsilon_{q,p}\, s_{L,W} - \sigma\varepsilon_{q,p} - \theta_L \sigma s_{L,W}}{s_{L,W} + \theta_K \sigma + \theta_L \varepsilon_{q,p}}.$$

Differentiating with respect to $\varepsilon_{q,p}$, θ_K, $s_{L,w}$, and σ, we obtain

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K \left(s_{L,w} + \sigma\right)^2}{\left[s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right]^2},$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{\left(\varepsilon_{q,p} + s_{L,w}\right) + (s_{L,w} + \sigma)(\varepsilon_{q,p} - \sigma)}{\left[s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right]^2},$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L \left(\varepsilon_{q,p} - \sigma\right)^2}{\left[s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right]^2},$$

$$\frac{\partial s_{K,r}}{\partial \sigma} = -\frac{\theta_L \left(\varepsilon_{q,p} + s_{L,w}\right)^2}{\left[s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right]^2}.$$

Hence $s_{K,r}$ decreases in $\varepsilon_{q,p}, s_{L,w}$, and σ (confirming three laws), and it also decreases in θ_K if the input is "normal," $s_{L,w} > 0$, and inputs are not extremely easy to substitute, that is, $\varepsilon_{q,p} > \sigma$.

Exercises

1. **Equilibrium number of firms in perfectly competitive markets** Consider a perfectly competitive industry with N symmetric firms, each with cost function $c(q) = F + cq$, where $F, c > 0$. Assume that the inverse demand is given by $p(Q) =$

$a - bQ$, where $a > c$, $b > 0$, and where Q denotes aggregate output. What is the short-run equilibrium price in this market?

a. *Short-run equilibrium* If exit and entry are not possible in the industry, (assuming N firms remain active), find the individual production level of each firm.

b. *Long-run equilibrium* Consider now that firms have enough time to enter the industry (if economic profits can be made) or to exit (if they make losses by staying in the industry). Find the long-run equilibrium number of firms in this perfectly competitive market.

2. **Equilibrium allocations insensitive to a common shock in all prices** Consider a competitive market with L goods, N consumers and J firms. In this setting, assume that we find an equilibrium price vector $p^* \in \mathbb{R}^L_+$ and equilibrium allocation $\left(x_1^*, x_2^*, \ldots, x_N^*; y_1^*, y_2^*, \ldots, y_J^*\right)$, where $x_i^* \in \mathbb{R}^L$ for every consumer i and $y_j^* \in \mathbb{R}^L$ for every firm j. Show that, if we were to scale price vector p^* to λp^*, where $\lambda > 0$, then allocation $\left(x_1^*, x_2^*, \ldots, x_N^*; y_1^*, y_2^*, \ldots, y_J^*\right)$ is still the equilibrium allocation.

3. **Per unit taxes versus ad valorem taxes** A tax is to be levied on a commodity bought and sold in a competitive market. Two possible forms of tax may be used. In one case, a *specific* tax is levied, where an amount t is paid per unit bought or sold. In the other case, an *ad valorem* tax is levied, where the government collects a tax equal to τ times the amount the seller receives from the buyer. Assume that a partial equilibrium approach is valid.

a. Show that, with a specific tax, the ultimate cost of the good to consumers and the amounts purchased are independent of whether the consumers or the producers pay the tax. As guidance, let us use the following steps:

i. *Consumers* Let p^c be the competitive equilibrium price when the consumer pays the tax. Note that, when the consumer pays the tax, he pays $p^c + t$, whereas the producer receives p^c. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the consumer pays the tax.

ii. *Producers* Let p^p be the competitive equilibrium price when the producer pays the tax. Note that, when the producer pays the tax, he receives $p^p - t$,. whereas the consumer pays *pp*. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the producer pays the tax.

b. Show that if an equilibrium price p solves your equality in part (a1), then $p + t$ solves the equality in (ii). Show that, as a consequence, equilibrium amounts are independent of whether consumers of producers pay the tax.

c. Show that this is not generally true with an ad valorem tax. In this case, which collection method leads to a higher cost to consumers? [*Hint*: Use the same steps as above, first for the consumer and then for the producer, but taking into account that now the tax increases the price to $(1 + \tau)p$. Then construct the excess demand function for the case of the consumer and the producer.]

d. Are there any special cases in which the collection method is irrelevant with an ad valorem tax? [*Hint*: Think about cases in which the tax introduces the same wedge on consumers and producers (inelasticity). Then prove your statement by using the above argument on excess demand functions.]

4. **Distribution of tax burden** Consider a competitive market in which the government will be imposing an ad valorem tax of τ. Aggregate demand curve is$x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve$q(p) = \alpha p^{\gamma}$, where $\alpha > 0$ and $\gamma >$ 0. Denote $\kappa = (1 + \tau)$. Assume that a partial equilibrium analysis is valid.

   a. Evaluate how the equilibrium price is affected by a marginal increase in the tax, that is, by a marginal increase in K.

   b. Describe the incidence of the tax when $\gamma = 0$.

   c. What is the tax incidence when, instead, $\varepsilon = 0$?

   d. What happens when each of these elasticities approach ∞ in absolute value?

5. **Perfect competition with heterogeneous goods** In our discussion of perfectly competitive markets, we considered that all firms produced a homogeneous good. However, our analysis can be easily extended to settings in which goods are heterogeneous. In particular, consider that every firm $i \in N$ faces an inverse demand function

$$p_i(q_i, q_{-i}) = \frac{\theta q_i^{\beta-1}}{\sum_{j=1}^{N} q_j^{\beta}},$$

where q_i denotes firm i's output, q_{-i} the output decisions of all other firms, namely$q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N)$, θ is a positive constant, and parameter $\beta \in (0, 1]$ captures the degree of substitutability. In addition assume that every firm faces the same cost function $c(q_i) = F + cq_i$, where $F > 0$ denotes fixed costs and $c >$ 0 represents marginal costs. Find the individual production level of every firm i, q_i^*, as a function of β. Interpret.

6. **Linear and Leontief preferences** Consider an economy in which preferences are

$$\text{Consumer 1:} \quad U^1 = x_1^1 + x_2^1,$$

$$\text{Consumer 2:} \quad U^2 = \min\{x_1^2, x_2^2\}.$$

a. Given the endowments $\omega^1 = (1,2)$ and $\omega^2 = (3,1)$, find the set of PEAs and the contract curve.

b. Which allocations are competitive equilibria?

7. **Finding offer curves for different preferences** Consider a two-good economy, where every person has the endowment $\omega = (0,20)$. For each of the following preferences, solve the individual's UMP in order to find his demand curve. Then use the endowment to identify his offer curve.

a. Cobb–Douglas type: $\alpha \log(x_1) + (1-\alpha)\log(x_2)$, where $\alpha \in (0,1)$.

b. Perfect substitutes: $ax_1 + x_2$.

c. Perfect complements: $\min\{ax_1, x_2\}$.

d. Consider now an economy where all individuals have the Cobb–Douglas preferences of part a. In addition there are two individuals: consumer A with $\alpha = \frac{1}{2}$ and endowment $\omega = (10,0)$, and consumer B with $\alpha = \frac{3}{4}$ and $\omega = (0,20)$. Find the WEA.

8. **Barter economies** Consider the following indirect utility functions for consumers A and B:

$$v^A(\mathbf{p}, m) = \ln m - \frac{1}{2}\ln p_1 - \frac{1}{2}\ln p_2,$$

$$v^B(\mathbf{p}, m) = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)m.$$

Initial endowments coincide across consumers, $\mathbf{e}^A = \mathbf{e}^B = (5.8, 2.1)$. Assuming that good 1 is the numeraire, $p_1 = 1$, find the equilibrium price vector $\mathbf{p}^*$.

9. **Pure exchange economy** Consider a pure exchange economy with two individuals, A and B, each with utility function $u^i(x^i, y^i)$ where $i = \{A, B\}$, whose initial endowments are $e^A = (10,0)$ and $e^B = (0,10)$, that is, individual A (B) owns all units of good x (y, respectively).

a. Assuming that utility functions are $u^i(x^i, y^i) = \min\{x^i, y^i\}$ for all individuals $i = \{A, B\}$, find the set of PEAs and the set of WEAs.

b. Assuming utility functions of $u^A(x^A, y^A) = x^A, y^A$ and $u^B(x^B, y^B) = \min\{x^B, y^B\}$, find the set of PEAs and WEAs.

10. **Gross substitutes** Consider an economy with two individuals, Amelia and Bernardo, with utility functions $u^A(x^A, y^A) = \min\{x^A, 2y^A\}$ for Amelia and $u^B(x^B, y^B) = \min\{2x^B, y^B\}$ for Bernardo, and initial endowments given by $\mathbf{e}^A = (1, 0)$ and $\mathbf{e}^B = (0, 1)$.
    a. Find the Walrasian demands of each individual.
    b. Find the excess demand functions, $z_x(p_x, p_y)$ and $z_y(p_x, p_y)$.
    c. Check that Walras' law holds.
    d. Check if goods are gross substitutes, that is, for any two goods $k \neq j$ where $k, j = \{x, y\}$ their excess demand functions satisfy $\partial z_k(p_x, p_y) / \partial p_j > 0$.
11. **Gross substitutability and uniqueness of equilibrium** Show that in a pure exchange economy with I individuals and J goods. Show that if the excess demand functions of all J goods satisfy the gross substitution property,

$$\frac{\partial z_k(\mathbf{p})}{\partial p_j} > 0 \text{ for any two goods } k \neq j,$$

the equilibrium price vector must be unique.
12. **Core in unreplicated and replicated economies** Consider an economy with two individuals with utility functions $u^A(x^A, y^A) = \min\{x^A, y^A\}$ and $u^B(x^B, y^B) = x^B, y^B$ with initial endowments $e^A = (1, 0)$ and $e^B = (0, 1)$. First find the set of Pareto efficient (PE) allocations, then the set of core allocations in the unreplicated economy, C_1, and finally in the twofold replica, C_2.
13. **Pareto allocations with externalities** Consider an economy with two consumers, Ann and Bob, with utility functions

$$u^A(x^A, y^A) = x^A + \left(y^A + \frac{1}{4}\right)^{1/2} \quad \text{and} \quad u^B(x^B, y^A) = x^B + y^A + \frac{1}{4},$$

where y^A enters Bob's utility (this is not a typo!). Initial endowments satisfy $\mathbf{e}^A = \mathbf{e}^B = (1, 1)$. Find the set of PEAs.
14. **WEAs with market power** Consider an exchange economy with two consumers, A and B, whose utility functions are

$$u_A\left(x_1^A, x_2^A\right) = x_1^A x_2^A,$$
$$u_B\left(x_1^B, x_2^B\right) = x_1^B \left(x_2^B\right)^2,$$

with endowments $\mathbf{e}^A=(80,150)$ and $\mathbf{e}^B=(210,180)$ respectively. Assume that consumer A is price setter, meaning he makes a take-it-or-leave-it price offer to consumer B.

a. Find the Walrasian equilibrium allocation (WEA) in this economy.
b. Find the Pareto efficient allocation (PEA) in this economy, and check if the WEA from part a is a PEA.

15. When goods are bads An exchange economy consists of two consumers, A and B, with utility function

$$u^i(x_1^i, x_2^i) = x_1^i(4 - x_2^i) \quad \text{for consumer } i = \{A, B\}.$$

So the first commodity is a "good" for each consumer, whereas the second commodity is a "bad" for each consumer. Their initial endowments are $\omega^A=(4,3)$ and $\omega^B=(1,0)$.

a. Find the consumers' Walrasian demand functions.
b. Show that an allocation is Pareto optimal if and only if $x_1^A + x_2^A = 4$.
c. Draw the Edgeworth box.
d. Find the competitive equilibria in this economy (remembering that good 2 is a bad.)
e. What happens to the set of competitive equilibria in the economy if consumer A is given the right to dump her endowment of the second good on consumer B without compensating consumer B?

16. Concave/convex contract curve Consider an economy with two consumers, A and B, with utility functions

$$u^A(x^A, y^A) = (x^A)^\alpha (y^A)^{1-\alpha},$$
$$u^B(x^B, y^B) = (x^B)^\beta (y^B)^{1-\beta},$$
where $\alpha, \beta > 0$.

a. Find their contract curve, expressing it as a function of x^A, that is, $y^A=f(x^A)$.
b. Show that such contract curve is convex if $\alpha>\beta$ but concave otherwise.

17. Excess demand in Cobb–Douglas preferences Consider an economy with two consumers, A and B, and two goods, 1 and 2. The utility function of A is

$$U^A = \gamma \log(x_1^A) + (1-\gamma)\log(x_2^A),$$

where x_i^A is consumption of good i by A. A has endowments $\omega^A = (\omega_1^A, \omega_2^A) = (2,1)$. For consumer B,

$$U^B = \gamma \log(x_1^B) + (1-\gamma)\log(x_2^B)\,.$$

$\omega^B = (\omega_1^B, \omega_2^B) = (3, 2)$, where $\gamma \in (0, 1)$.

a. Find the Walrassian demands of consumers A and B.

b. Choosing good 2 as the numeraire, graph the excess demand for good 1 as a function of p_1.

c. Calculate the competitive equilibrium allocation. Verify that this is the point where excess demand is zero.

18. More on excess demands Consider a two-commodity exchange economy, with two agents $i = \{A, B\}$ whose utility functions are

$$U^A(x^A) = \log(x_1^A) + 2\log(x_2^A),$$

$$U^B(x^B) = 2\log(x_1^B) + \log(x_2^B).$$

Initial endowments are $\omega^A = (9, 3)$ and $\omega^B = (12, 6)$.

a. Find the excess demand function for each good. Verify that Walras's law holds.

b. Find the equilibrium price ratio.

c. What is the WEA?

d. Assuming that the aggregate endowment remains fixed at $\omega = \omega^A + \omega^B = (21, 9)$, find the contract curve.

19. Excess demand functions: Homogeneity and Walras's law Excess demand functions must satisfy homogeneity of degree zero in prices; that is, increasing all prices by a common factor $\lambda > 0$ does not affect the excess demand function, $z_k(\mathbf{p}) = z_k(\lambda\mathbf{p})$ for all $\lambda > 0$, and Walras's law, $\mathbf{p} \cdot z(\mathbf{p}) = 0$. Check if the following functions satisfy these two properties, and thus are/are not legitimate excess demand functions:

a. $z_1(\mathbf{p}) = -p_2 + 10/p_1$, $z_2(\mathbf{p}) = p_1$, and $z_3(\mathbf{p}) = -10/p_3$.

b. $z_1(\mathbf{p}) = (p_2 + p_3)/p_1$, $z_2(\mathbf{p}) = (p_1 + p_3)/p_2$, and $z_3(\mathbf{p}) = (p_1 + p_2)/p_3$.

c. $z_1(\mathbf{p}) = p_3/p_1$, $z_2(\mathbf{p}) = p_3/p_2$, and $z_3(\mathbf{p}) = -2$.

20. Excess demand and stability of equilibria Consider a two-commodity economy where the price of commodity 1 is normalized in terms of commodity 2, whereby $p_1/p_2 = p$. Suppose the excess demand function for commodity 1 is given by

$$z_1(p) = 1 - 4p + 5p^2 - 2p^3$$

a. How many equilibria can you find? Are they stable or unstable?

b. Which of the equilibrium price ratios you found are stable?

c. Consider now that the aggregate endowment of good 1 increases. How are your results from parts a and b affected?

21. Production economy Consider an economy with two consumers $i = \{A, B\}$, one firm and two goods $l = \{1, 2\}$. The individual endowments of individuals A and B are $\omega^A = \omega^B = (\frac{1}{2}, \frac{1}{2})$. The utility functions are

$$u^A(x_1^A, x_2^A) = \ln(x_1^A) + \ln(x_2^A),$$

$$u^B(x_1^B, x_2^B) = (x_1^B)^{1/4}(x_2^B)^{3/4}.$$

The firm produces good 2 using good 1 as input, the production function is $y_2 = \sqrt{y_1}$. The consumer B owns the firm (denote π the firm's profit). Good 2 is the numeraire good (i.e., $p_2 = 1$).

a. Determine the demand for good 1 of the consumers and the firm.

b. Show that there is a unique equilibrium price p_1.

c. Assume that the production function is now $y_2 = y_1$, and thus satisfies constant returns to scale. Determine the equilibrium price and allocation (i.e., the WEA).

d. Consider the exchange economy consisting of consumers A and B (i.e., eliminate the firm). Determine the equilibrium (price and allocation).

22. Production economy with CRTS Consider an economy with two consumers $i = \{A, B\}$, one firm (that produces good 2 using good 1 as input) and two goods $l = \{1, 2\}$. Consumer B owns the firm. Good 2 is the numeraire good (i.e., $p_2 = 1$). Consider that consumers' preferences are given by

$$u^A(x_1^A, x_2^A) = x_1^A + 4\sqrt{x_2^A} \quad \text{and} \quad u^B(x_1^B, x_2^B) = x_1^B + 2\sqrt{x_2^B}$$

while their endowments are

$$\omega^A = (4,12) \quad \text{and} \quad \omega^B = (8,8).$$

The production function is $y_2 = 3y_1$. Compute the equilibrium price and allocation.

23. WEAs and PEAs in the household Consider an economy with two individuals, Ann and Bartholomew, each with utility function

$$u^A\left(x^A, l^A\right) = x^A l^A \text{ and } u^B\left(x^B, l^B\right) = x^B l^B$$

where x denotes a consumption good while l represents hours of leisure. Additionally Ann owns the only firm in this economy and has 20 hours to dedicate to either work (L^A) or leisure (l^A), or $20 = L^A + l^A$; whereas Bartholomew does not own any assets in this economy (poor husband!), but has 30 hours to spend, or $30 = L^B + l^B$. Ann's firm produces units of good x with labor hours using a Cobb–Douglas production technology $x = \sqrt{L}$, where $L \equiv L^A + L^B$.

a. Find the set of PEAs.

b. Find the set of WEAs.

c. Is the WEA you found in part b part of the set of PEAs?

24. Equilibrium with production Consider an economy with two goods, 1 and 2, both of them being produced by using capital and labor. Firms are price takers, and output prices are determined in the international market. The output factors of goods 1 and 2 are

$$q_1 = (K_1)^{1/4}(L_1)^{3/4},$$
$$q_2 = (K_2)^{3/4}(L_2)^{1/4}.$$

a. Find the marginal cost for each firm.

b. Use the results from part a to connect your result with the Stopler–Samuelson theorem.

c. Show that if $p_1 = 2p_2$, then in equilibrium $w_L = 4w_K$.

25. Effect of distortionary taxes Consider an economy with two individuals, $i = \{A, B\}$, each with identical Cobb–Douglas utility function $u(x_1^i, x_2^i) = x_1^i x_2^i$, and initial endowments $e_A = (200, 100)$ and $e_B = (100, 200)$.

a. Find the Pareto optimal allocation (PEA).

b. Find the WEA. (For simplicity, you can assume that$p_1 = p_2 = 1$.)

c. Assume that the government sets a tax t on purchases of good 1, which is refunded to the consumers as a lump-sum payment, $T^i = tx_1^i$. Find the post-tax WEA, and compare it with your results in part b.

d. Show that the WEA when taxes are absent in part b is efficient, whereas the WEA when taxes are present found in part c is not necessarily efficient for all values of t.

References

Allen, R. G. D. 1967. *Mathematical Analysis for Economists*. New York: St. Martin's Press.

Berra and Porto. 1971. Notas sobre la teoría de la distribución y la teoría de la demanda de los factores de producción. *Economica* 17: 251–66.

Bosch-Domènech, A., and J. Silvestre. 1997. Credit constraints in general equilibrium: Experimental results. *Economic Journal* 107: 1445–64.

Bronfenbrenner, M. 1961. Notes on the elasticity of derived demand. *Oxford Economic Papers* 13: 254–61.

Ginsburgh, V., and M. Keyter. 1997. *The Structure of Applied General Equilibrium Models*. Cambridge: MIT Press.

Gode, D. K., and S. Sunder. 1993. Allocative efficiency of markets with zero-intelligence traders: Market as a partial substitute for individual rationality. *Journal of Political Economy* 101: 119–37.

Goodfellow, J., and C. R. Plott. 1990. An experimental examination of the simultaneous determination of input prices and output prices. *Southern Economic Journal* 56: 969–83.

Hey, J., and D. Di Cagno. 1998. Sequential markets: An experimental investigation of Clower's dual decision hypothesis. *Experimental Economics* 1: 63–87.

Hicks, J. R. 1957. *The Theory of Wages*. Gloucester, MA: Peter Smith.

Lian, P., and C. R. Plott. 1998. General equilibrium, markets, macroeconomics and money in a laboratory experimental environment. *Economic Theory* 12: 21–75.

Marshall, A. 1997. *Principles of Economics*. Amherst, NY: Prometheus Books.

Mas-Colell, A., M. Whinston, and J. Green. 1995. *Microeconomic Theory*. New York: Oxford University Press.

Rybczynski, T. M. 1955. Factor endowment and relative commodity prices. *Economica* 22: 336–341.

Smith, V. L. 1991. Rational choice: The contrast between economics and psychology. *Journal of Political Economy* 99: 877–97.

Stolper, W. F., and P. A. Samuelson. 1941. Protection and real wages. *Review of Economic Studies* 9: 58–73.

Varian, H. 1992. *Microeconomic Analysis*. New York: Norton.