CHAPTER 5

Social Welfare Orderings: Requirements and Possibilities

1 Introduction

The central objective of the study of welfare economics is to provide a framework which permits meaningful statements to be made about whether some economic situations are socially preferable to others. Ultimately we would like to rank all economic situations (social states) so we would like this ranking to be *complete* (so that every social state can be compared and ranked to another) and *consistent* (so that the ranking is reflexive and transitive). We shall call such a complete and consistent ranking of social states a *social welfare ordering* (SWO). Just as with household orderings, if a continuity assumption is made the SWO can be represented by a *social welfare function* (SWF) that assigns a number to each social state.

States cannot be socially ordered without someone making prior value judgments, although sometimes such value judgments are implicit. Value judgments are statements of ethics which cannot be found to be true or false on the basis of factual evidence. The value judgments contained in a SWO may be weak (i.e. broadly accepted) or strong (i.e. controversial). An example of a relatively strong value judgment is Rawls's (1971) difference principle, which states that inequalities are 'just' if and only if they work to the advantage of the least-well-off household. A far weaker value judgment is the weak Pareto principle, which states that a social state x is socially preferred to y if x is unanimously preferred to y by all households in the economy.

Another weak value judgment that is called *individualism* requires that the preferences of the individual households should matter when determining the SWO. This value judgment, commonly made throughout welfare economics, imposes certain informational requirements on the choice of an SWO. Specifically, information about each household's preference over social states and about how a given level of utility for any household compares with that of another household may be required. These requirements are called the *measurability* and *comparability* requirements, respectively. In this chapter we shall examine how value requirements and informational requirements limit the set of SWO possibilities from which the planner can choose.

In order to be concrete, we shall identify a social state with an allocation of N goods over the H households;[1] that is, an allocation $[x]$ is an $N \times H$ vector where element x_i^h is the amount of good i consumed by household h. In chapter 3 we saw that allocations can be partially ordered in terms of efficiency criteria. This partial SWO is based on two weak value judgments – the Pareto principle and individualism. The Pareto partial ordering is also informationally undemanding. Household utility need only be ordinally measurable and utility comparisons across households are unnecessary.

The most serious drawback of the Pareto partial ordering is that it is not a complete ordering. The usefulness of the partial ordering may be enhanced by allowing the definition of a social state to include hypothetical lump-sum transfers of goods (or generalized purchasing power) among households, as discussed in chapter 3. Nevertheless it remains a partial ordering. Only those situations where all households are made better or worse off (the strong Pareto principle does allow some households to be indifferent) can be ranked. If some households are made better off and others worse off in moving from x to y, even when lump-sum redistribution is assumed to be possible, the Pareto principle cannot tell us anything. Thus, whenever there is a utility conflict among the households, such as created by a move along a utility possibilities frontier, we need more than the Pareto principle. Ideally we should have an SWO. Although such an SWO need not incorporate the Pareto principle, most SWOs commonly used do incorporate it, since it is not the Pareto principle *per se* that is the problem but rather the fact that orderings based on it alone are incomplete.

In reality, utility conflict among the households in the economy is resolved by some means or other, whether it involves property rights and market prices, collective bargaining or the redistributive powers of the government. In this chapter we are not concerned with such 'positive' distribution theory. Rather we are concerned with the ethical problem of resolving the conflict inherent in finding the normative solution to the distribution problem. Specifically, some households in the economy may prefer state x to y whereas others prefer y to x. Given that the household preferences should be taken into account, how should the policy-maker aggregate such conflicting preferences into a single SWO? This is the central question of normative social choice theory.

Before considering the determination of SWO possibilities, it is useful to describe the broad framework of normative social choice theory and introduce some commonly encountered terminology. Unfortunately, the nomenclature in this realm is as extensive as it is distressing. We shall try to avoid introducing unnecessary terminology and explain, in intuitive language, those terms that we do introduce.

[1] Supplies of factors could be introduced as negative elements in the allocation vector.

2 The Framework of Normative Social Choice Theory

The objective is to derive an SWO over social states from the households' orderings of the social states. The means of aggregating the household orderings into the SWO is called the *social choice rule* (SCR) (following Sen, 1970).[2] If the household orderings are continuous they can be represented by household utility functions, and if the SWO is continuous it can be represented by an SWF. In this case, the SCR is a social welfare *functional* (SWFL) which is defined over the set of possible household utility functions.

The most general form of the SWF (over social states) is the so-called *Bergson–Samuelson* (B–S) SWF, expressed as

$$W(x) = F((u^1(x), u^2(x), \dots, u^H(x))$$

The function $W(x)$ may take any form, although it is usually assumed to satisfy at least three properties. Firstly, it is assumed that it can be defined over utility space; that is, $W(x)$ can be evaluated from an H vector of utility values. In this case, the SWF can be written as $W(u)$ and represented by a social welfare indifference curve map as in figure 5.1. If the social welfare depends only on the utility outcomes of the social state in this way, it is said to satisfy *welfarism* (Sen, 1977). This will be discussed further in the next section. Secondly, the B–S SWF is usually assumed to incorporate a version of the *Pareto principle* known as the strong Pareto principle. This means the SWF is increasing in each household's utility *ceteris paribus*. Thus, the social welfare indifference curves are negatively sloped and those further from the origin correspond to higher levels of social welfare, so $W_3 > W_2 > W_1$ in figure 5.1. Finally, the B–S SWF is often assumed to be strictly quasi-concave so that social welfare indifference curves have the shape shown in figure 5.1. This assumption reflects the egalitarian ethic that inequality in utilities among households, *per se*, is socially undesirable.

In figure 5.1, the B–S SWF is combined with the utility possibilities frontier (UPF) discussed in chapter 3 and labelled UPF. The social welfare maximum occurs at point E which corresponds to the particular allocation of goods and resources that is Pareto optimal and maximizes social welfare. The social welfare optimum could be attained in principle by a combination of perfectly competitive markets combined with lump-sum redistribution, although neither are likely to exist in practice. At the social welfare optimum the slope of the UPF is equal to the slope of an SWF indifference curve. As discussed in chapter 3, the absolute value of the slope of the UPF is given by $\lambda^h(\ )/\lambda^g(\ )$ where $\lambda^h(\ )$ is the marginal utility of income of household h. The absolute value of the slope of the SWF

[2] Arrow (1963) called the means of aggregating household preferences a social welfare function. In order to avoid confusion with the conventional Bergson–Samuelson definition of a social welfare function, we adopt Sen's terminology.

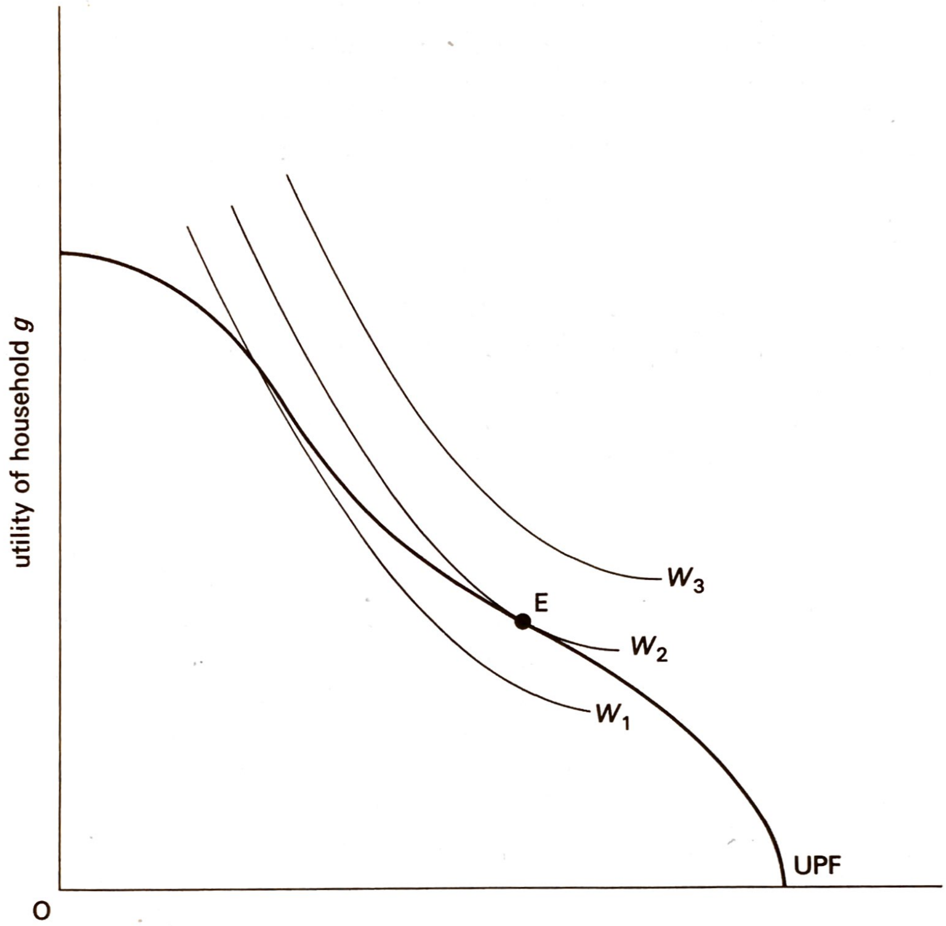

FIGURE 5.1

indifference curve is known as the *marginal rate of social substitution* (MRSS) in utility and is given by W_h/W_g, where $W_h = \partial W(\ )/\partial u^h$. Thus, at the social welfare optimum,

$$\frac{W_h}{W_g} = \frac{\lambda^h}{\lambda^g} \qquad \text{for all } h, g$$

or

$$\frac{W_h}{\lambda^h} = \frac{W_g}{\lambda^g} = \ldots = \phi \qquad (5.1)$$

where ϕ is the common social marginal utility of income for every household.

Analytically this is all well and good, but how can such a framework be utilized in the practice of welfare economics? And under what circumstances does a general B–S SWF exist? The first question is addressed in the second part of this book. The second question will be answered in this chapter. It will be found that the general B–S SWF, although flexible in form, is demanding in terms of informational requirements. Other 'popular' SWFs are found to be less informationally demanding but far more specific in functional form. Perhaps the oldest and best-known form is the simple *utilitarian* (or 'Benthamite') SWF, where

$$W = \sum_{h=1}^{H} u^h \tag{5.2}$$

In this case, social welfare is the unweighted sum of household utilities. Slightly less restrictive is the *generalized utilitarian* or weighted sum SWF, where

$$W = \sum_{h=1}^{H} a_h u^h \tag{5.2'}$$

and a_h, $h = 1, \ldots, H$, are positive constants. Other specific forms are the *Bernoulli-Nash* (B–N) SWF, where

$$W = \prod_{h=1}^{H} u^h \tag{5.3}$$

and the *generalized* B–N SWF, where

$$W = \prod_{h=1}^{H} (u^h)^{a_h} \tag{5.3'}$$

In this case, social welfare is the product (weighted or unweighted) of the household utilities. Note that the B–N SWF is utilitarian in the logarithms of utility. Also there is the *Rawlsian* or *maximin* SWF

$$W = \min\,[u^1, \ldots, u^H] \tag{5.4}$$

where social welfare is identified with the utility of the worst-off household.[3] The social welfare indifference contours for these three SWF forms (utilitarian, B–N and maximin) are shown in figures 5.2(*a*), (*b*) and (*c*), respectively.

All of the five SWF forms described above are special cases of a more general SWF known as the *isoelastic* form or

$$W = \frac{\sum_{h=1}^{H} a_h (u^h)^{1-\rho}}{1-\rho} \tag{5.5}$$

[3] This SWF is termed 'maximin' because it involves maximizing the minimum value of the utility vector and is related to the maximin strategy encountered in game theory.

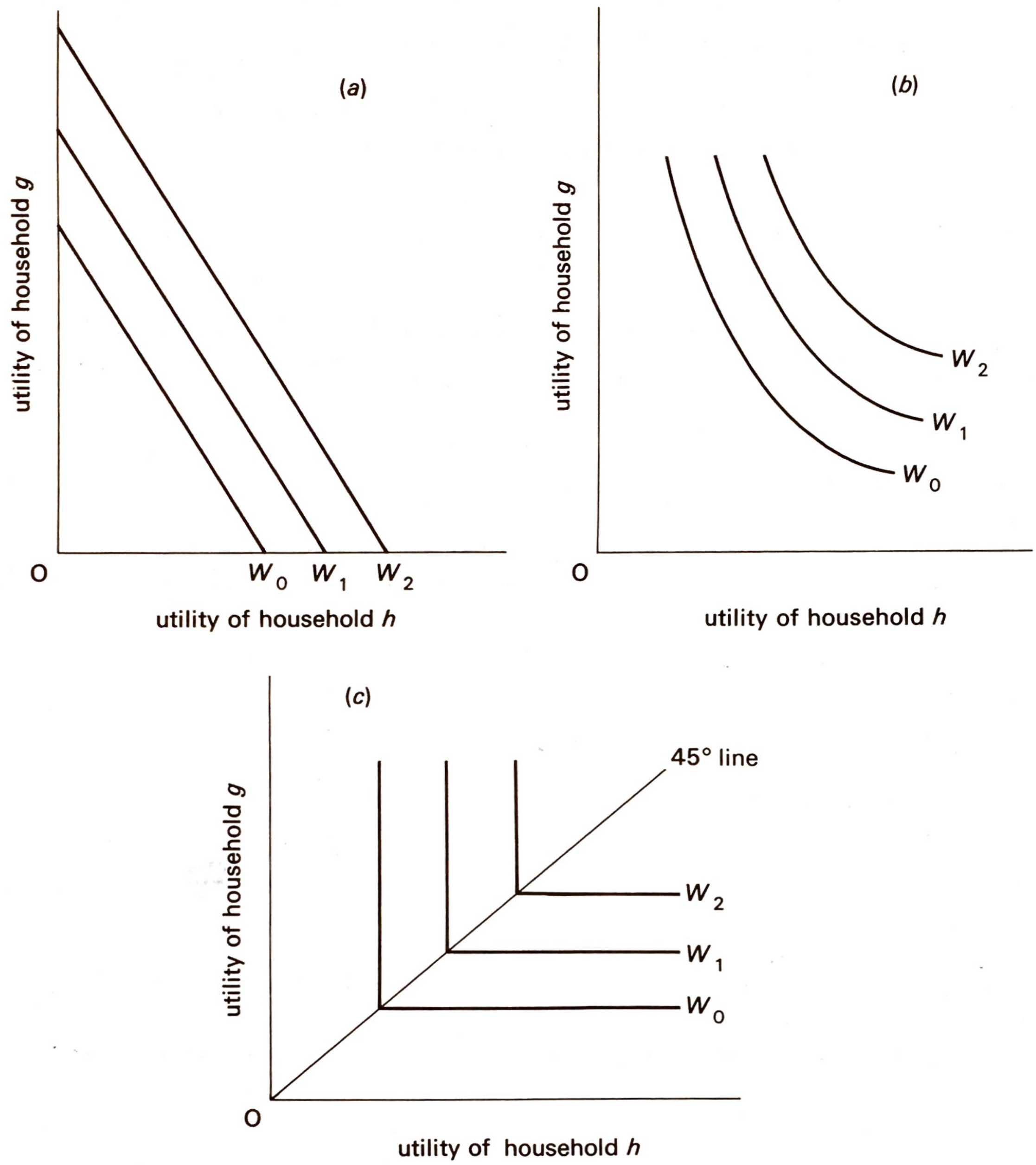

FIGURE 5.2

where $1/\rho$ is the (constant) elasticity of substitution of an SWF indifference contour. If $\rho = 0$ and $a_h = 1$ for all h, (5.5) reduces to the utilitarian case. As $\rho \to 1$ and $a_h = 1$ the limiting expression for (5.5) is the B–N SWF. As $\rho \to \infty$, (5.5) reduces to the maximin form.[4] It should be noted that since the SWO is an ordering, the SWF representing it will be an ordinal function. Therefore, an SWF formed by taking an increasing function of any one of the above functional forms is also a legitimate representation.

[4] Multiplying (5.5) by $1-\rho$ and taking the $(1-\rho)$th root yields the CES functional form. Since this is just a monotonic transformation of W it is permitted by the ordinality of W. We can now use the well-known limiting cases of the CES function to obtain the results in the text. A good proof of the limiting case of the CES function is found in Varian (1978, p. 18).

3 Informational Restrictions and the Social Welfare Ordering

We have said that a *social welfare ordering* (SWO) that completely and transitively orders all social states (say, allocations of goods across households) is a desirable objective in the study of welfare economics. In this section we begin an examination of SWO possibilities, a topic that will concern us for much of this chapter. An important point here is that we wish to restrict the choice of an SWO to those that satisfy certain requirements. If we are able to choose any SWO, out of the air so to speak, then the SWO possibilities are unlimited. With such liberty, however, the SWO concept may not be very interesting. For this reason we constrain the SWO to satisfy certain requirements. Surprisingly, imposing particular combinations of requirements, each of which seems reasonable in other contexts, is found to restrict the SWO possibilities rather drastically.

We shall examine the SWO possibilities under two sorts of restriction. Both sorts of restriction pertain to the information that policy-makers are permitted to utilize when deriving a social ordering. The first set of restrictions implies a property that Sen (1977) has called *welfarism* (W) or *strong neutrality*. Basically this restricts the information that can be utilized in ranking social states to utility information corresponding to those social states. The second set of restrictions, which are called *invariance requirements*, are informational requirements pertaining to the measurability and interpersonal comparability of the individual utilities.

3.1 *Welfarism*

An SWO has the property of welfarism if the ranking of social states depends only on the utility levels of the households. Specifically, information about how the utility levels are obtained is irrelevant for determining how the social states should be ordered. That is, states having the same welfare consequences are indistinguishable for social welfare purposes. This is a strong requirement for it implies that social welfare depends solely upon the numerical value of utility attained by each individual regardless of the measurement conventions by which numerical utility levels are arrived at. +

Three conditions are sufficient for welfarism. We will state (non-formally) each in turn.

Universality or unrestricted domain (condition U) This condition requires that any logically possible H vector of individual utility functions is admissible in determining the social ranking. That is, the same SWO must be used to aggregate individual utilities regardless of what the individual utility functions happen to be. The only thing asked of the households' preferences is that each household be able to order consistently (i.e. reflexively and transitively) all social states. It seems reasonable to require the SWO to be universally applicable in this sense.

Pareto indifference (condition PI) If all households are indifferent between two social states, the SWO must rank the two states equivalently.

Independence of irrelevant alternatives (condition I) This condition requires that the social ranking of any two social states x and y be the same whenever the utility levels attached to x and y by the individual households are the same. This implies that the social ranking must be unchanged if any or all households' indifference curves are renumbered in a way that leaves the indifference curve numbers associated with states x and y unchanged. This also means that the social ranking of x and y must be independent of the availability of other social states and of the households' preferences over social states other than those being ranked.

A proof that conditions U, PI and I imply welfarism is given by Sen (1979). Intuitively it can be seen how welfarism is implied for states which are socially equivalent through the PI condition. This condition requires that x and y be ranked as equivalent if all households are indifferent between them. In other words, all other information about x and y is irrelevant, and this is the heart of welfarism. Conditions U and I generalize this informational parsimony to strict rankings of x and y.

3.2 *Invariance requirements*

These requirements limit the measurability and comparability of household utility functions. *Measurability* refers to the sense in which the real numbers attached to a given household's utility levels are meaningful (i.e. convey information). *Comparability* refers to the sense in which the real numbers attached to different households' utility levels can be meaningfully compared. Comparability in this sense is a statement about utility information that is commensurable among households, and should not be confused with the welfare judgment of how (or whether) to trade off one household's utility against another.

Assumptions about measurability and comparability can be formalized by considering the set of transformations that can be applied to an H household utility vector without changing the SWO. Following Sen, we let $\psi(\ ) = [\psi^1(\ ), \ldots, \psi^H(\ )]$ be a vector of transformation functions with one element for each household's utility function.

Measurability concerns the transformations applicable to the individual household's utility function. The most restrictive measurability assumption is that the household's utility function is fully measurable or measurable with an absolute scale (AS). In this case a unique real number is attached to each indifference curve of a household. Alternatively, the only admissible transformation of scale is the identity transformation. That is, $v^h(\ ) = u^h(\ )$ where $u^h(\ )$ is a utility representation of the preferences of household h (i.e. a numbering of its indifference curves) and $v^h(\ )$ is the admitted transformation of that utility representation.

The least restrictive measurability assumption is that utility is measurable only with an *ordinal scale* (OS).[5] In this case indifference curves can be numbered in any arbitrary manner, but higher indifference curves must be given higher numbers in order that the numerical scale preserves the ranking of the indifference curves. Formally, this permits the utility function of a household h to be rescaled by taking any monotonic increasing transformation of it. That is, a transform of u^h, $v^h(\ ) = \psi^h(u^h)$ for any $\psi^h(u^h)$ with $\partial\psi^h/\partial u^h > 0$, conveys the same information as u^h, and therefore the SWO should be the same if u^h is replaced with v^h.

Lying between AS and OS measurability is a *ratio scale* (RS) and a *cardinal scale* (CS) measurability. RS measurability means that any positive linear transformation of u^h, $v^h(\ ) = b^h u^h(\ )$ where b^h is a positive constant, conveys the same information as u^h. CS measurability means any positive affine transformation of u^h, $v^h(\ ) = a^h + b^h u^h(\ )$ where $b^h > 0$, conveys the same information as u^h. An example of a cardinally measurable entity is temperature. Fahrenheit, Celsius and kelvin scales all convey the same information and are positive affine transformations of each other.

Comparability means the extent to which utility information measured for the individual household can be meaningfully compared across households. The assumption of *non-comparability* (NC) means that none of the information measured for individual utility can be used when making across-household comparisons. *Full comparability* (FC) means that all of the information available for the individual household is available for comparisons across-households. *Partial comparability* (PC) means that only some of the household information is available for comparisons across households.

It should be realized that the assumption about comparability is not necessarily independent from the assumption about measurability. If, for example, utility for a household is measurable only with an ordinal scale, then increments in utility cannot be compared across households since they cannot be compared for a single household. On the other hand, when utility is measurable to an absolute scale for the single household there must be full comparability across households because the utility level of every household is associated with a unique real number, and real numbers are comparable. Another way of looking at this is that the only admissible transformation under AS is the identity transformation which is, trivially, the same for every household.

In the following sections we shall consider the SWO possibilities under different assumptions about measurability and comparability. In general we shall see that, without comparability, SWO possibilities are extremely limited regardless of the degree of measurability of utility. Under full comparability, however, the SWO possibilities are increased as the measurability of individual utility is increased. SWO possibilities are narrowed,

[5] This is the least restrictive case apart from the trivial case of measurability with a *nominal* scale, which allows an arbitrary numbering of the indifference curves.

often to a single case, if only partial comparability is possible or if additional restrictions are imposed.

The additional restrictions we shall consider are drawn from the following. The *weak Pareto principle* (PW) states that social state x must be preferred to y in the SWO if every household strictly prefers x to y. The *strong Pareto principle* (PS) requires x to be socially preferred to y even if some households are indifferent, provided that at least one household strictly prefers x to y and none prefers y to x. *Anonymity* (A) requires that only the utility levels, and not which households get which utilities, should matter in socially ranking the states. In other words, if u' is a vector of utilities associated with state x and u'' is a permutation of the elements of u', then the utility vector of u' and u'' must be ranked the same by the SWO *vis-à-vis* other utility vectors. *Separability* (SE) requires that the social ranking of x and y depends only on the preferences of households that have a strict preference between x and y, and not on the levels of utility of the households which are indifferent between x and y. *Minimal equity* (EM) requires that if all households, except the one in the best-off position, prefer x to y then x is preferred to y in the social ordering. *Strong equity* (ES) requires that the set of utility distributions which are as least as good as the reference utility distribution u_0 be strictly convex, as shown in figure 5.3. This means that if the SWO is a SWF, it is strictly quasi-concave. Finally, *continuity* (CO) requires the 'at least as

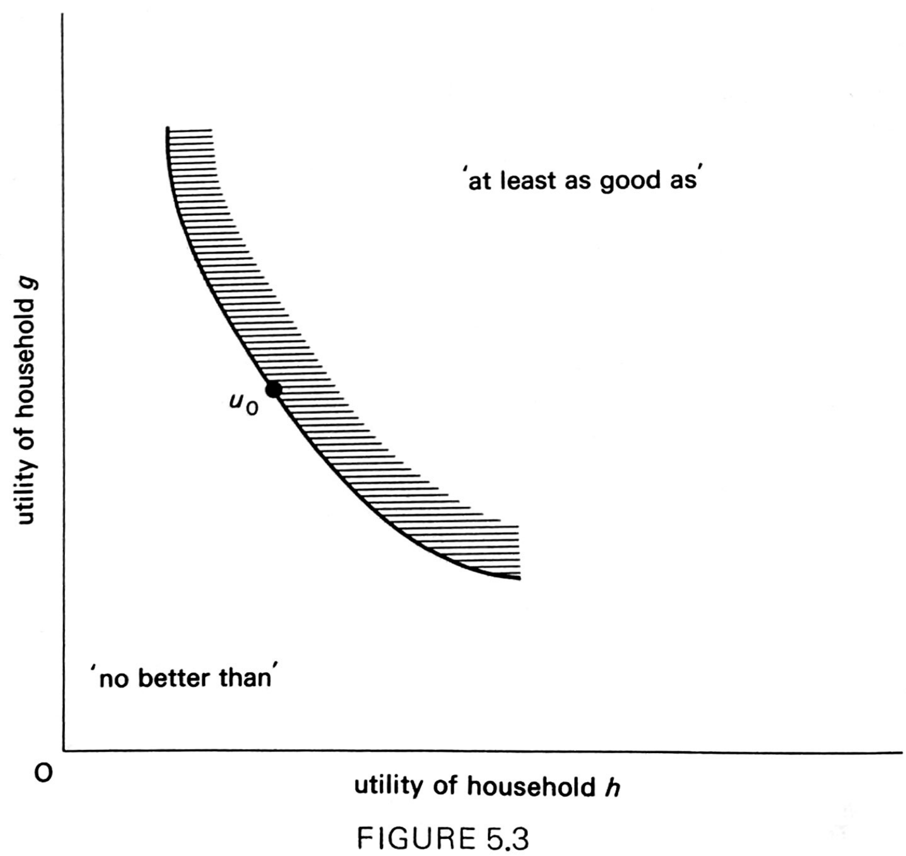

FIGURE 5.3

good as' set be closed so the SWO can be represented by an ordinal social welfare function (SWF).[6] Some SWOs, such as lexicographic orderings, do not satisfy this property and therefore are precluded by the requirement of continuity.

4 Non-comparability and Dictatorship Possibilities

In this section we consider the question addressed by Arrow (1951a) in his celebrated monograph *Social Choice and Individual Values.* In particular, if utility functions are ordinal and non-comparable so that the informational assumptions are OS and NC (which are all that are required to define Pareto optimality), then what SWO possibilities are permitted if one also restricts the SWO to incorporate the weak Pareto principle and welfarism?[7,8] The answer is somewhat surprising: OS–NC, W and PW imply that the only possible SWO is a dictatorship. That is, social orderings must coincide with the preferences of some individual in the economy regardless of the preferences of the others.[9] **

Arrow proved this remarkable theorem by contradiction. In such proofs, one uses the requirements of U, I, PW and the transitivity of the SWO to 'uncover' a dictator. However, with the full welfarism assumptions of this chapter, it is possible to show diagrammatically why the SWO possibility must be a dictatorship in a two-household economy and to give some intuitive meaning to the proof.[10]

To begin with, the welfarism assumption permits us to examine the SWO in terms of the rankings of the two-household utility levels as in figure 5.4, where the utility of household g is measured on the vertical axis and that of household h is measured on the horizontal axis. Consider any utility point, for example $u_0 = [u_0^g, u_0^h]$, as a reference point. We wish to rank all other utility points relative to u_0. We can use u_0 as an origin and divide the utility space into quadrants. Ignoring the boundaries for now, we can immediately rank points in quadrants I and III relative to

[6] Technically, continuity means that the 'at least as good as' set and the 'no better than' set of utility points are closed and contain their own boundaries. Intuitively, this means that, assuming welfarism, for any utility point in the utility space of figure 5.2 and for any ray from the origin there must exist a point on the ray indifferent (in terms of social welfare) to the closer point. In other words, we have social welfare indifference curves. This cannot be the case with a lexicographic SWO. In this case, the only possibility of social welfare indifference occurs if all households are indifferent.

[7] Relaxing the weak Pareto principle simply permits reverse dictatorships, where the SWO is the exact opposite of the 'dictator's' preferences.

[8] Arrow actually used a weaker form of welfarism that applied only to strict rankings. In terms of our definitions, he used U, I and PW. That is, non-welfare desiderata were permitted in the event that all households were indifferent. This subtlety is not important in what follows.

[9] This result is sometimes presented in the form of an impossibilities theorem. In this case, dictatorship is precluded by assumption directly, or indirectly by a stronger assumption such as anonymity.

[10] The following discussion is adapted from a fine paper by Charles Blackorby, David Donaldson and John Weymark (1983) which introduced this diagrammatic framework for analysing social choice questions.

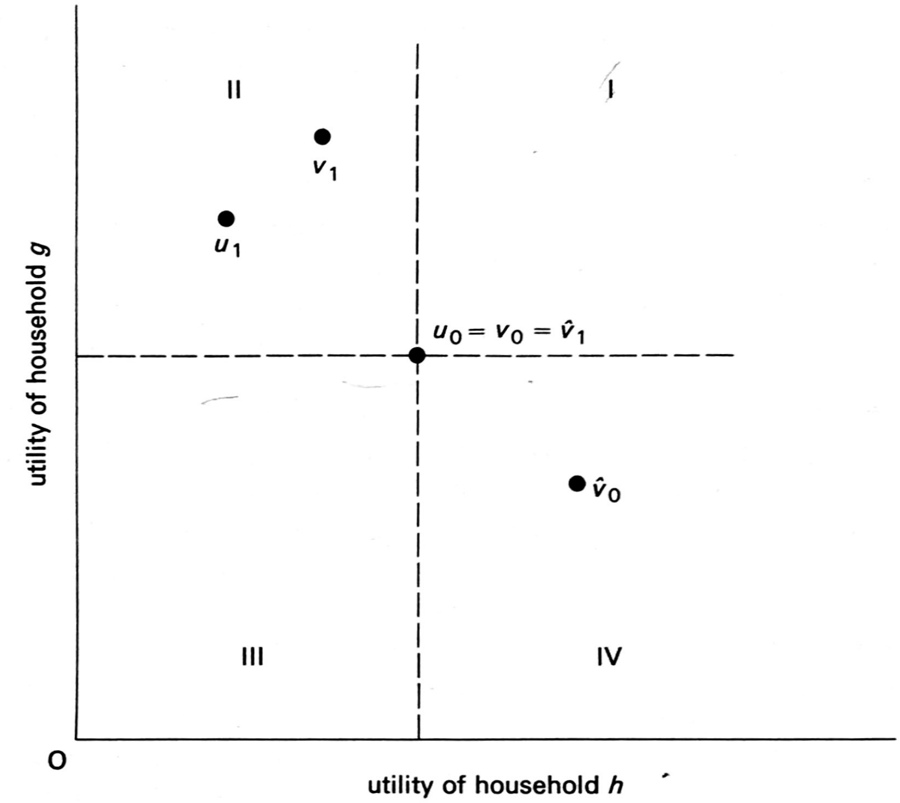

FIGURE 5.4

u_0. By the weak Pareto principle, all points in I must be ranked higher than u_0, whereas u_0 must be ranked higher than all points in quadrant III. The problem is to rank points in quadrants II and IV relative to u_0.

Consider now the informational invariance requirement OS–NC used by Arrow. Formally this assumption means that the social ordering of social states (and by welfarism, the social ordering of utility points) must remain unchanged when the H vector of utility representations is transformed by $\psi = [\psi^1(\ ), \ldots, \psi^H(\ )]$. OS implies that each household transformation $\psi^h(\ )$ is monotonically increasing and NC implies that a different transformation can apply to each household's utility function. This means that any household's indifference curves can be renumbered in any manner which preserves the rankings of its indifference curves, and that different renumberings can be applied to the indifference curve maps of different households.

With the OS–NC assumption we can now show that all points in quadrant II must be ranked against u_0 in the same way. Consider point u_1 in quadrant II, where $u_1^h < u_0^h$ and $u_1^g > u_0^g$. By completeness of the SWO, either u_1 must be ranked above u_0, or u_0 ranked above u_1, or u_1 and u_0 ranked as equivalent. Suppose, without loss of generality, that u_1 is ranked above u_0 according to an SWO. This ranking must be preserved when we apply increasing monotonic transformations to u^g and u^h where, by NC, we can apply different transformations u^g and u^h. Consider applying the

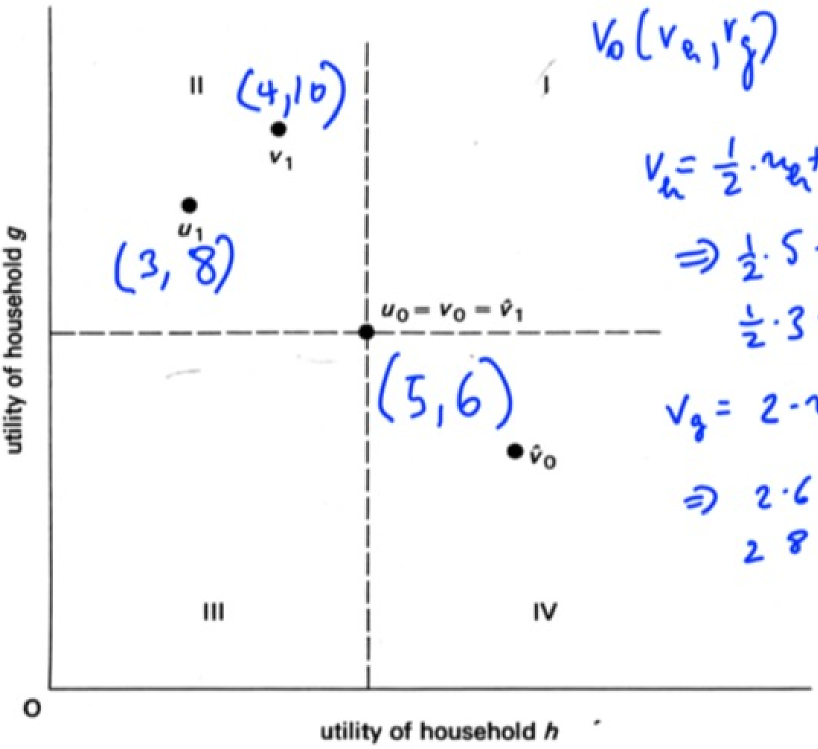

FIGURE 5.4

$V_0 (v_h, v_g)$

$v_h = \frac{1}{2} \cdot u_h + 2{,}5$

$\Rightarrow \frac{1}{2} \cdot 5 + 2{,}5 = 5$

$\frac{1}{2} \cdot 3 + 2{,}5 = 4$

$v_g = 2 \cdot u_g - 6$

$\Rightarrow 2 \cdot 6 - 6 = 6$

$2 \cdot 8 - 6 = 10$

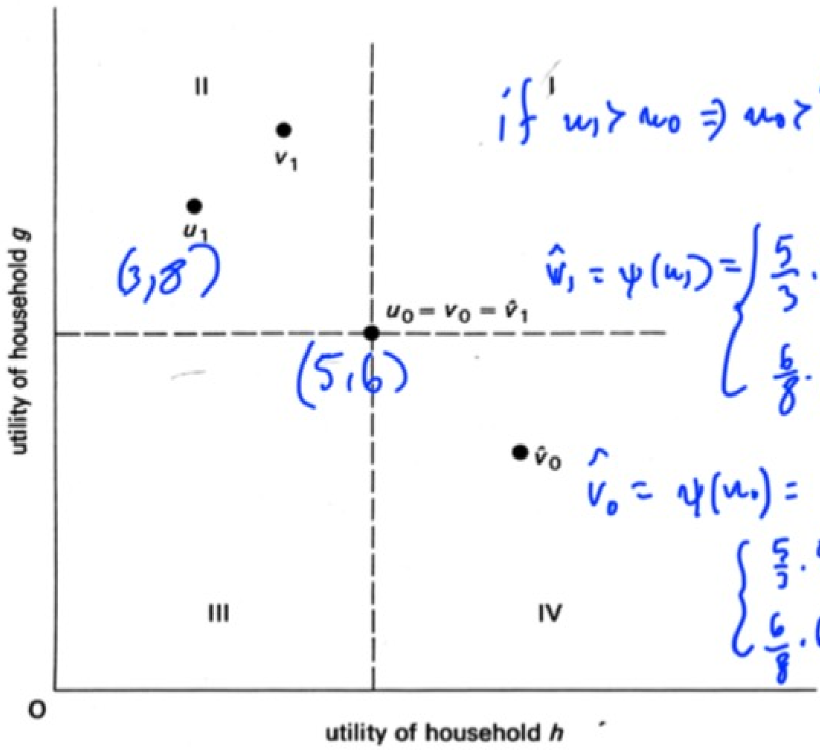

FIGURE 5.4

if $u_1 > u_0 \Rightarrow u_0 > \hat{v}_0$

$\hat{v}_1 = \psi(u_1) = \begin{cases} \frac{5}{3} \cdot 3 \\ \frac{6}{8} \cdot 8 \end{cases}$

$\hat{v}_0 = \psi(u_0) = \begin{cases} \frac{5}{3} \cdot 5 \\ \frac{6}{8} \cdot 6 \end{cases}$

transformation $v^g = \psi^g(u^g)$, $v^h = \psi^h(u^h)$ such that $v_0^g = u_0^g$ and $v_0^h = u_0^h$; that is, point u_0 is mapped back to itself. But, by the choice of $\psi(\ )$, point $[\psi^g(u_1^g), \psi^h(u_1^h)]$ can be mapped anywhere into quadrant II. All that must be retained is $v_1^g > v_0^g$ and $v_1^h < v_0^h$. Thus all points in II must be ranked the same with respect to u_0.

We can now rule out the case that all points in II are ranked equivalent to u_0. Suppose this to be the case, and consider a transformation that maps u_0 back to u_0 and u_1 to v_1, where $v_1^g > u_1^g$, $v_1^h > u_1^h$. By PW, v_1 must be ranked above u_1. However, we have already supposed that v_1 and u_1 are both indifferent to u_0. This violates transitivity. Thus either all points in II are ranked above u_0 or u_0 is ranked above all points in II. They all cannot be equivalent with u_0.

By the same line of reasoning we can prove that all points in quadrant IV must be ranked above u_0 or u_0 ranked above all points in IV. It can be further established that if u_0 is ranked above all points in II (or vice versa), all points in IV must be ranked above u_0 (or vice versa). This follows because the relationship of u_0 to points in II is the same as that of points in IV to u_0. That is, if u_1 is preferred to u_0 then we can transform the utility scales so that $\hat{v}_1 = \psi(u_1) = u_0$ and $\hat{v}_0 = \psi(u_0)$ lies in quadrant IV. Thus if u_1 is ranked above u_0 then $\hat{v}_1$ $(= u_0)$ is ranked above $\hat{v}_0$.

Finally, it is obvious that if two quadrants are ranked the same way with respect to u_0 then points on the boundary between the two quadrants are ranked in the same way. Therefore, what we have established so far is that either quadrants I and II (and their common boundary) are preferred to u_0 and u_0 is preferred to III and IV, or quadrants I and IV are preferred to u_0 and u_0 is preferred to II and III. In the former case, we still have not ranked the points along the horizontal line through u_0, whereas in the latter we have not ranked the points along the vertical line through u_0. For illustration, let us concentrate on the former case. There are two possibilities here:

Strong dictator The first possibility is that all points along the horizontal line through u_0 are socially indifferent. In other words, this line is a social welfare indifference curve. This implies that household g is a strong dictator, since if it is indifferent between two states, the states are ranked indifferent socially. The entire preference map would consist of a series of horizontal lines and the SWF would correspond with household h's own ordinal utility function. Of course, if h were the dictator, the SWO would be represented by a set of vertical lines. This result generalizes readily to the case of more than two persons. The SWF would simply be represented by the dictator's utility function.

Lexicographic dictatorship The assumptions we have made do not require that all points along the horizontal line through u_0 be socially indifferent as they would be under the strong dictator. It is also possible that u_0 is preferred to any point to its left but not preferred to any point to its right. Since u_0 was arbitrarily chosen, any point on the horizontal line is preferred to any point to its left. In other words, the ranking of points on the horizontal line increases as one moves right. Such a social ordering corresponds to a *lexicographic dictatorship*, analogous to the lexicographical ordering of bundles by a household familiar from consumer theory.[11] In this case, there is some arbitrary ordering such that if, as in this example, household g is the prior dictator but is indifferent between two social states, then the mantle of dictatorship falls on household h providing h strictly prefers one state to the other. If not, the next household becomes the dictator, and so on. As with household preferences, when the social ordering is lexicographic over utilities it is not continuous; that is, there is no possibility of indifference between social states. The SWO cannot, in this case, be represented by a SWF.[12]

So far we have talked about possibility results. We will obtain an *impossibility result* (i.e. the set of SWO possibilities is empty) by imposing a non-dictatorship requirement directly (in addition to welfarism and weak Pareto), or by imposing a requirement such as anonymity which rules out dictatorship by implication. This is why the Arrow result is often referred to as the *Arrow impossibility theorem.*

Suppose we substitute the strong Pareto principle (PS) for the weak one. This is sufficient to rule out the strong dictator as a possibility, since now no one person can dictate social indifference. The strong Pareto principle states that if someone is made better off and no one is made worse off in a state x as compared with state y, then x must be preferred to y even if the dictator is indifferent. In the two-person case above, point u_0 must be preferred to any point to its left by the strong Pareto principle. More generally, if there are more than two persons, one can always imagine there being a set of household preferences such that for two states x and y between which the dictator is indifferent, x will be preferred to y by at least one other household and not nonpreferred by any. If so, letting the dictator dictate social indifference would violate the strong Pareto principle (but not the weak). Thus, when the Pareto principle is strengthened from the PW to PS, the strong dictator is ruled out and we are left with the lexicographical dictatorship. The ordering of households is still done arbitrarily, so many different lexicographical dictatorships are possible.

It is fair to say that the Arrow theorem generated a lot of controversy. Statements such as 'Arrow's theorem implies that, in general, a non-dictatorial SWO is impossible' were not uncommon. Various ways of getting around the dictatorial result have since been sought. All of these necessarily relax Arrow's assumptions. One solution is to relax the invariance requirements and admit more information to the planner. Arrow's theorem can be interpreted as saying that the OS–NC invariance requirement, when combined with welfarism, is simply too restrictive to

[11] This possibility was noted by Gevers (1979).

[12] The strong dictatorship and the lexicographic dictatorship are not the only possible ways to rank points along the horizontal (or vertical) lines, and thus are not the only possible SWOs. Any way of arbitrarily ranking points along the horizontal line which is consistent with welfarism and PW is permissible (e.g. flipping a coin).

permit any meaningful SWO possibilities. An alternative procedure is to relax some of the requirements that the social ordering must satisfy. This is equivalent to relaxing the assumption of welfarism. The reader is referred to Sen (1970) for a discussion of this. We shall restrict our discussion to relaxing the informational restrictions on the planner which are really very strict in the Arrow framework.

We shall see that by relaxing the invariance requirements in certain ways, additional SWO possibilities will be available. Before proceeding, however, it is useful to point out that relaxing the measurability assumption, *ceteris paribus*, does not necessarily allow us to escape Arrow's dictatorship. In particular, the dictatorship (strong and lexicographic) results derived above hold with equal force if we assume cardinal non-comparability (CS–NC). That is, 'cardinalizing' household utility by permitting positive linear affine transformations $v^h = a^h + b^h u^h$ while maintaining non-comparability across households leaves the SWO possibilities unchanged. This result was proven by Sen (1970).

In terms of the diagrammatic framework, it is easily seen that the logic of the 'proof' is unchanged by allowing positive affine transformations (as in Blackorby, Donaldson and Weymark, 1983). All of the transformations utilized to prove dictatorship can be accomplished with CS measurability. This is shown for household h in figure 5.5 where the

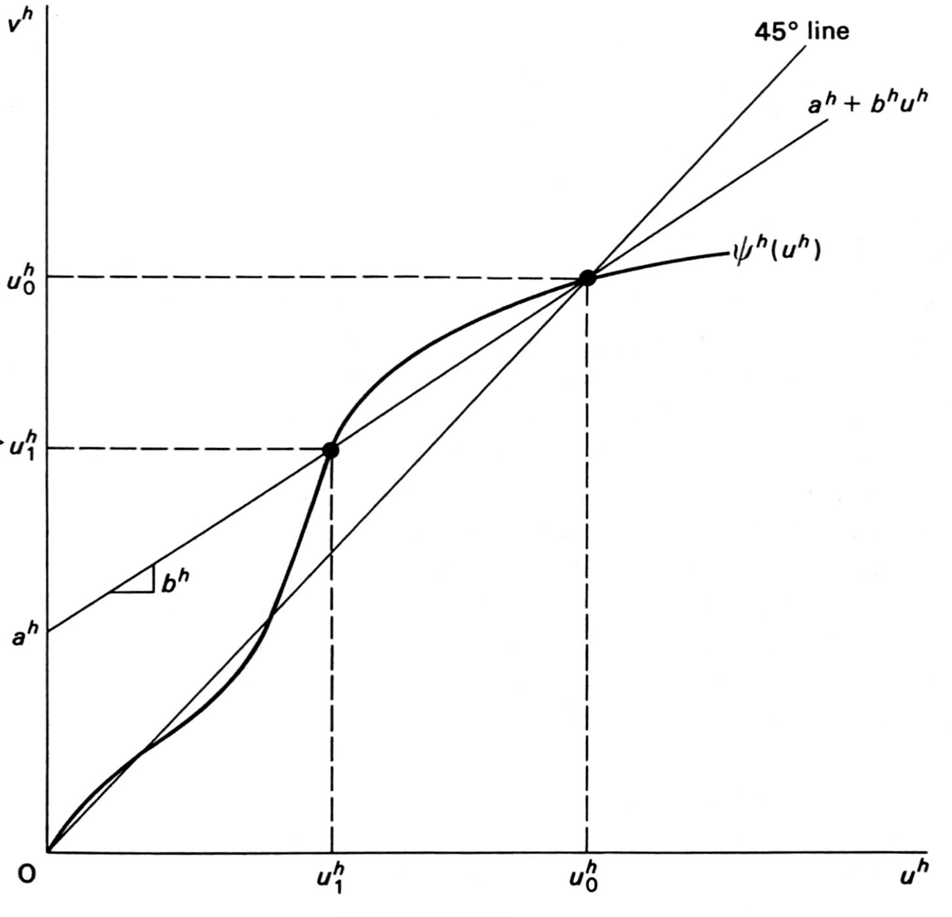

FIGURE 5.5

monotonic transformation that maps u_0^h back to itself and plots u_1^h to v_1^h is labelled $\psi^h(u^h)$. Exactly the same transformation can be accomplished by the positive affine transformation labelled $a^h + b^h u^h$. Thus, with NC, whether individual household utilities are cardinally or ordinally measurable is irrelevant to the question of SWO possibilities. Dictatorship (either of the strong sort or lexicographic) is the only possibility in either case.

Finally, note that more restrictive measurability assumptions cannot be combined with non-comparability; hence they cannot generate the Arrow result. RS measurability implies that proportional unit changes in utility must be comparable across households, since the household-specific transformation b^h cancels out when $\Delta v^h/v^h = \Delta u^h/u^h$ is calculated. Therefore, proportionate utility changes between two states are uniquely defined and comparisons of them can be made across households. As mentioned, AS measurability for every household implies full comparability across households.

5 SWO Possibilities with Full Comparability

Under FC the admissible transformations that can be applied to each household's utility function are the same. This means that the information available in making utility comparisons for the individual household is also available for utility comparisons across households. In contrast to the NC case, increasing the measurability of household utility significantly expands the SWO possibilities set under FC.

5.1 *Ordinal scale measurability* (SO)

Under this measurability assumption only utility levels can be compared by the individual household; that is, statements such as 'this *increment* in utility is larger (smaller) than that increment' have no meaning. Under FC, utility levels can also be compared across households whereas increments cannot be so compared. The combination of OS and FC means that any monotonic transformation can be applied to households' utility functions as long as the *same* transformation is applied to the utility function of every household; that is, $v^h = \psi(u^h)$ for all h. Formally, this means that $v^g(x) \gtreqless v^h(y)$ as $u^g(x) \gtreqless u^h(y)$ for any two households g and h and any two social states x and y. Thus Alice with x is better (worse) off than Bob with y both before and after the transformation, so such information on rankings is preserved and can be utilized by the social planner. Conversely, we can say that if the planner is only able to compare utility levels across and within households, the information available to the planner is OS–FC.

The fact that utility levels are comparable across households means that households can be ranked by utility position for any social state. This now permits SWO possibilities based on the utility positions of the households. Such possibilities were obviously excluded under the NC assumptions of Arrow and Sen.

If the requirements of welfarism and the weak Pareto principle are added to OS–FC, the ability to compare utility levels across households opens the SWO possibility of *positional dictatorships* in addition to the Arrow case of strong and lexicographic dictatorships. In this case, the SWO is dictated not by a particular household but by the preferences of the household occupying a particular utility position. A common example is the Rawlsian maximin case, where the SWO is dictated by the preferences of the household in the lowest utility position. If the worst-off household in state x is better off than the worst-off household in state y, then state x is preferred to state y in the SWO. Note that which household happens to be worst off can differ in the two states. Also note that the maximin case is an example of a positional dictatorship but not the only one possible under assumptions W, PW and OS–FC. For example, a maximax social welfare ordering would be possible, or a dictatorship by the nth well-off person. Only by adding an equity axiom of some type does one narrow the positional dictatorship to the maximin (Rawlsian) form.

Also possible under W, PW and OS–FC is the *positional lexicographic* SWO. In this case there is a hierarchy of households ranked according to utility level (first household, second household etc., not necessarily going from the worst-off to the best-off household or vice versa) such that the SWO is dictated by the first household in the hierarchy providing it has strict preference; if not, the strict preferences of the second household dictate the SWO etc. If one adopts the strong Pareto principle instead of the weak, the positional dictatorship is not possible. This is because allowing a household in a particular position in the ranking of utility levels to dictate indifference can violate PS, since it would be possible for the dictating household to be indifferent between states x and y whereas some other household prefers x and none prefers y. Thus, under PS, positional lexicographic SWOs (and lexicographic dictatorships) are possible but not positional (or strong) dictatorships.

The SWO possibilities are narrowed further by adding other restrictions. Adding anonymity rules out all of the dictatorship forms. If the further assumption of separability is made then the positional lexicographic forms are narrowed to the so-called leximin and leximax forms. The leximin is a positional lexicographic SWO where the positional hierarchy runs from this worst-off to the best-off position. For the leximax case the hierarchy runs in the opposite direction.

This result, which was proved by Hammond (1976) and Strasnick (1975), can be illustrated in figure 5.6 again adapted from Blackorby, Donaldson and Weymark (1983). By separability we can analyse the case of two households, g and h, independently of other households. We begin, as before, with an arbitrary reference point u_0. By the Pareto principle, points in the positive orthant (north-east of u_0) are ranked above u_0 whereas u_0 is ranked above points in the negative orthant (south-west

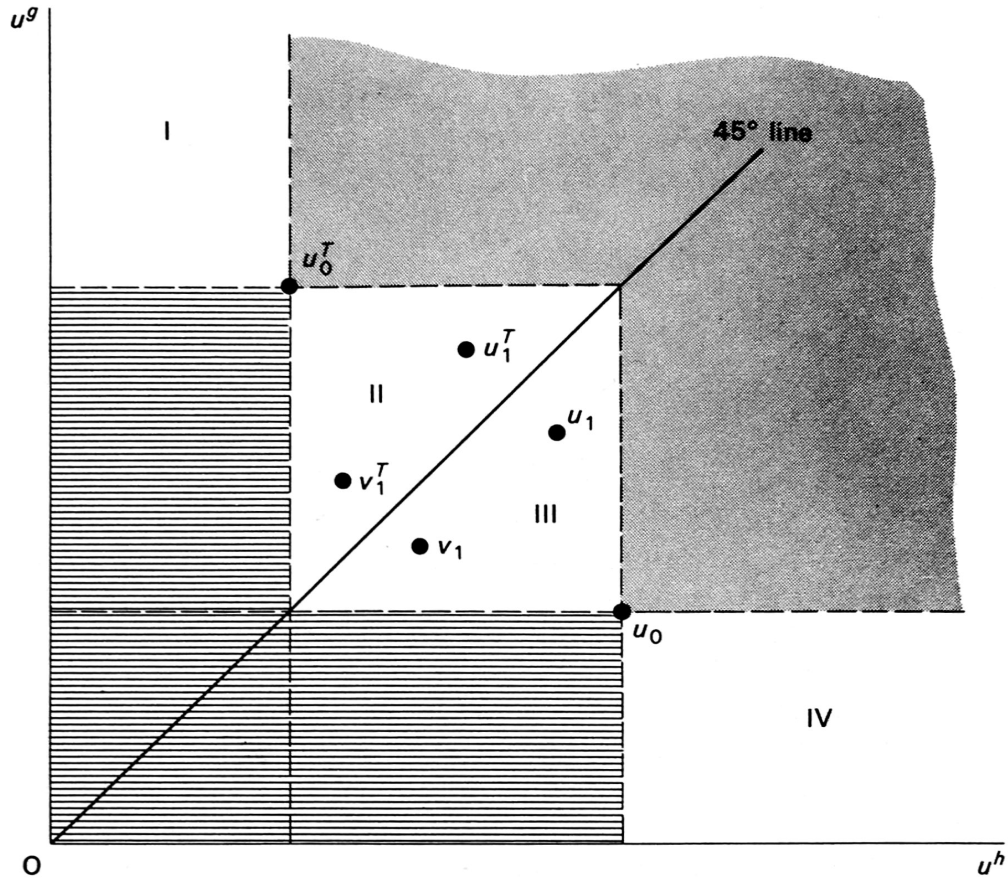

FIGURE 5.6

of u_0). By anonymity, the transposed point u_0^T, where the utility levels of h and g are interchanged, must be ranked equivalently with u_0. The positive orthant of u_0^T must be preferred to u_0^T (and u_0), whereas u_0^T (and u_0) are preferred to the negative orthant. In figure 5.6 the combined preferred area is shaded and the combined non-preferred area is cross-hatched. This leaves four areas to consider, labelled I to IV.

Consider another point u_1 anywhere in region III which is to be ranked against u_0. By A, u_1^T must be ranked the same way. Since we can take any monotonic transformation of both households' utility we can map $v_0 = \psi(u_0)$ back to u_0 (and v_0^T to u_0^T) and u_1 (u_1^T) to any point v_1 (v_1^T) in region III (II). Note that the 45° line cannot be crossed because household h must remain better off than household g under OS–FC. Thus all points in II (and by anonymity, III) must be ranked the same way against u_0 and u_0^T.

By the logic followed in section 4, regions II and III must be strictly preferred or strictly not preferred to u_0 and points in areas I and IV must be ranked in the opposite way. This leaves two possibilities: II and III preferred and I and IV not preferred (figure 5.7(a)) or II and III not preferred and I and IV preferred (figure 5.7(b)). The former is a leximin result between the two households g and h, whereas the latter is the leximax. By SE, we can perform the same analysis for any two house-

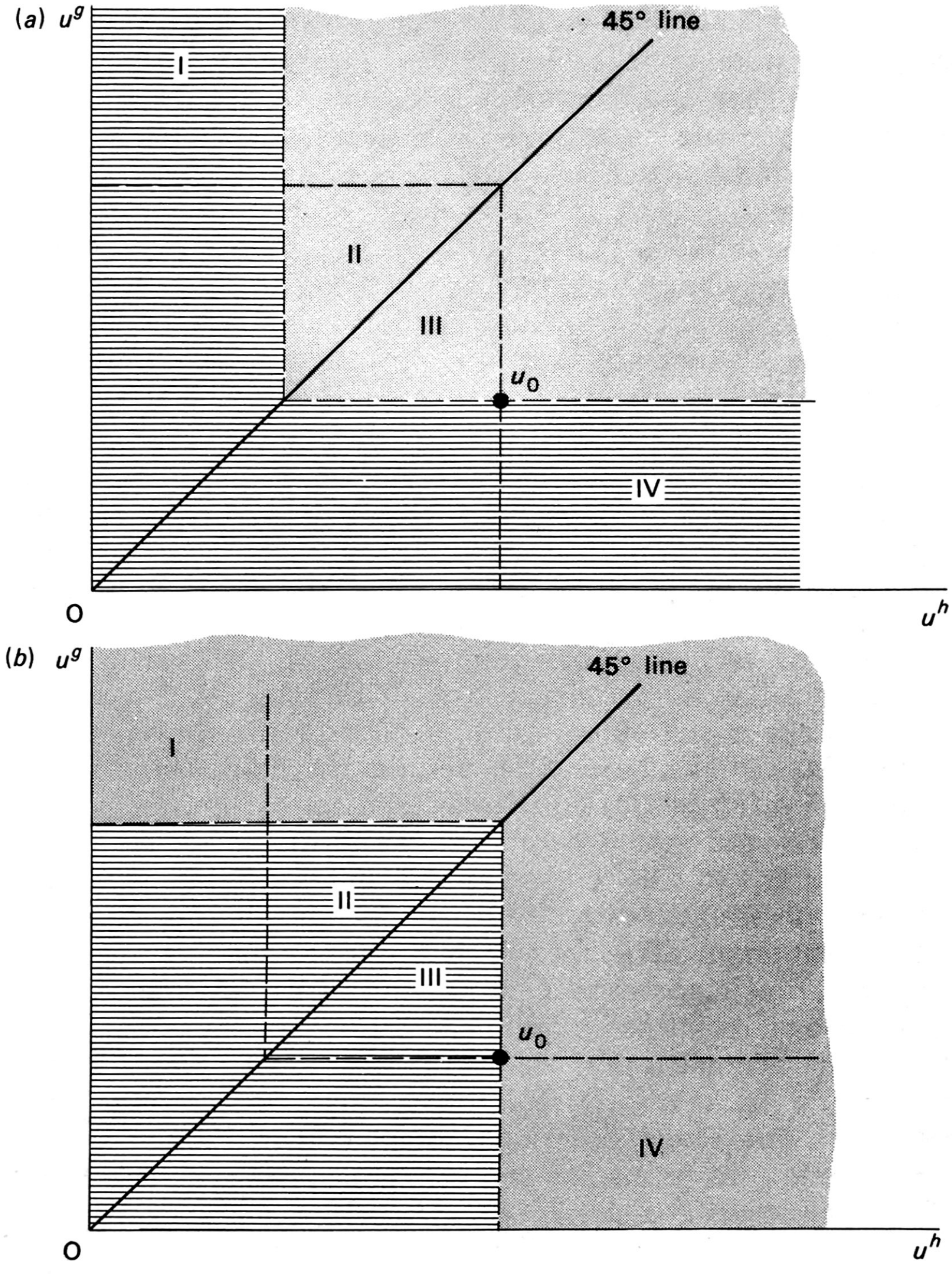

FIGURE 5.7

holds, so the two-household leximin–leximax results chain together to get the H household result.

Finally we can narrow the possibilities to the leximin case along by making the minimal equity assumption (EM). This rules out the leximax case by excluding priority to the preferences of the best-off household.

5.2 *Cardinal scale measurability* (CS)

Under CS measurability levels of utility and increments in utility can both be meaningfully compared for the individual household. By FC, such comparisons can also be made across households. In addition to statements such as, 'Alice is better off (worse off) in x than Bob is in y', we can also make statements such as, 'The increment in Alice's utility is greater (smaller) than the increment in Bob's utility'. These sorts of comparisons can be made because by CS measurability each household's utility function can be transformed by any positive affine transformation $v^h = a^h + b^h u^h$ and by FC, $a^h = a^g = a$ and $b^h = b^g = b$ for all h and g. It is then easily established that $v^g(x) \geqslant v^h(y)$ only as $u^g(x) \geqslant u^h(y)$ and $v^g(x) - v^g(y) \geqslant v^h(y) - v^h(z)$ only as $u^g(x) - u^g(y) \geqslant u^h(y) - u^h(z)$. In words, both levels and first differences in utility are comparable across households. The planner now has more information and this increases the range of SWOs possible.

Since levels of utility are still comparable, all of the positional forms of SWO obtained under OS are permissible as are the dictatorship forms of the non-comparable case. But since increments in (or 'units' of) utility are now meaningful for utility comparisons across households, additional SWO possibilities are admitted; specifically, those relying on cross-household comparisons of changes in utility. The additional SWO possibilities include SWF of the utilitarian and generalized utilitarian forms. The former is a social welfare function (recall that an SWF is a continuous SWO) that ranks social states on the basis of the unweighted sum of household utilities. The latter SWF permits the household utilities to be 'weighted' with different but positive weights for each household.

Consider first the case where only welfarism and the weak Pareto principle are added to CS–FC. The simple and positional dictatorship and lexicographic possibilities are still open, of course, and in addition the generalized utilitarian SWF (of which utilitarianism is a special case) is possible. Also possible is some combination of the generalized utilitarian and the positional dictatorship SWF.

To see this geometrically, assume that the SWF is a differentiable function $W(u^1(\ ), \ldots, u^H(\ ))$ and that $u^h(\ )$ depends only on its own income m^h.[13] The social ordering can be depicted by a set of social indifference contours in income space. The absolute value of the slope of one of these contours at a given point m^g, m^h space is given by

$$\frac{\partial W/\partial m^g}{\partial W/\partial m^h} = \frac{\partial W/\partial u^g \;\; \partial u^g/\partial m^g}{\partial W/\partial u^h \;\; \partial u^h/\partial m^h} \tag{5.6}$$

These contours must be unchanged when the households' utility functions are submitted to allowable transforms, since the ordering of social states must be unchanged. Therefore, the left-hand side must be unchanged when the households' utility functions are transformed by identical

[13] This 'selfishness' assumption involves no loss in generality. Specifically, one can let m^h be a money metric utility measure where actual utility is derived from the allocation vector in a manner which can include empathy, jealousy etc.

positive affine transformations. Suppose $v^h = a + bu^h$. Then

$$\frac{\partial v^g/\partial m^g}{\partial v^h/\partial m^h} = \frac{b\partial u^g/\partial m^g}{b\partial u^h/\partial m^h}$$

is unchanged by all such transformations. Therefore, for the left-hand side of (5.6) to be unchanged, we also require that $(\partial W/\partial u^g)/(\partial W/\partial u^h)$ be unchanged by the transformation.

The implication of all this is shown in figure 5.8, which depicts social welfare contours in utility space. At any arbitrary reference point u_0 the slope of the SWF indifference curve (i.e. $-(\partial W/\partial u^h)/(\partial W/\partial u^g)$) is given by the slope of the line segment through u_0. This slope must remain unchanged when we transform u^g and u^h by the same positive affine transformation. Such a transformation can relocate u_0 to any v_0 point below the 45° line by some combination of a movement along a ray through the origin (multiplying each household's utility by the same positive scalar) to bu_0 plus a movement along a 45° line through bu_0 (adding a common intercept term to each household's utility). By inspection it can be seen that v_0 can be placed anywhere below the 45° line by a positive affine transformation. Therefore, the SWF indifference curves must have the

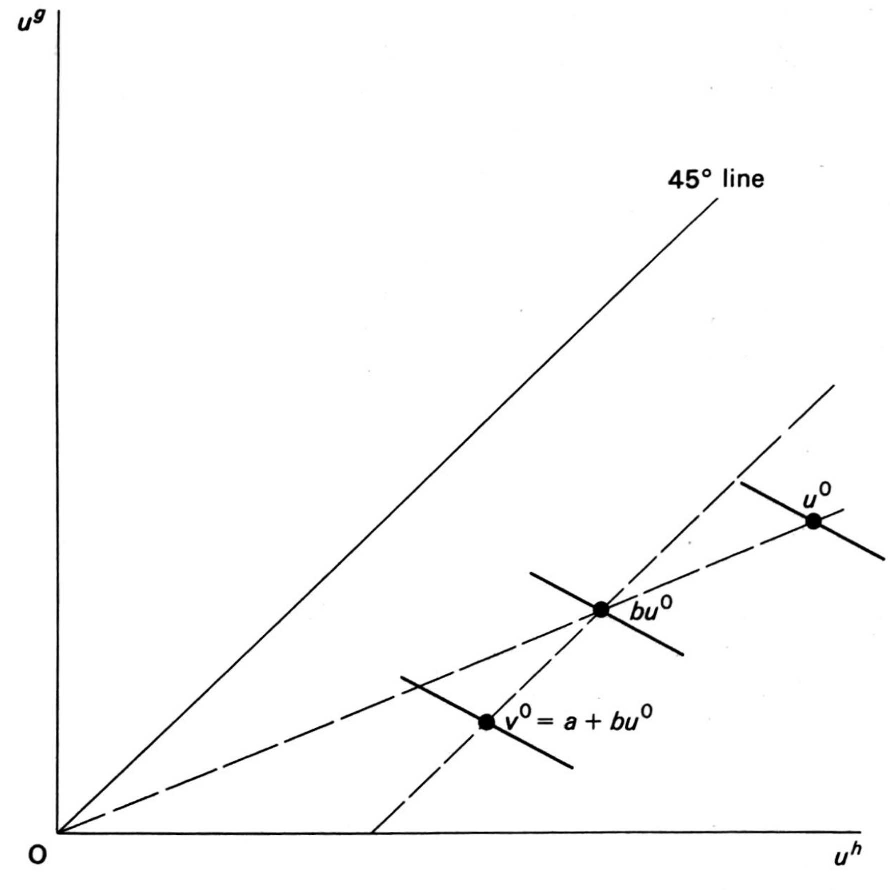

FIGURE 5.8

same slope as that at u^0 throughout that part of the quadrant. By the same logic the SWF indifference curves must also have a constant slope at all points above the 45° line (though not necessarily the same slope as below the line).

The types of SWF indifference curves admitted are shown in figure 5.9(*a*)–(*c*). In figure 5.9(*b*) the SWF indifference curves happen to have the same slope (not necessarily -1). This is the generalized utilitarian case (utilitarian if the slope is -1). In figure 5.9(*a*) the SWF is a linear combination of the (generalized) utilitarian and the maximax positional dictatorship. In figure 5.9(*c*), the utilitarian is combined with maximin. More generally we have

$$W = W^{\mathrm{u}} + \alpha(W^{\mathrm{d}} - W^{\mathrm{u}}) \tag{5.7}$$

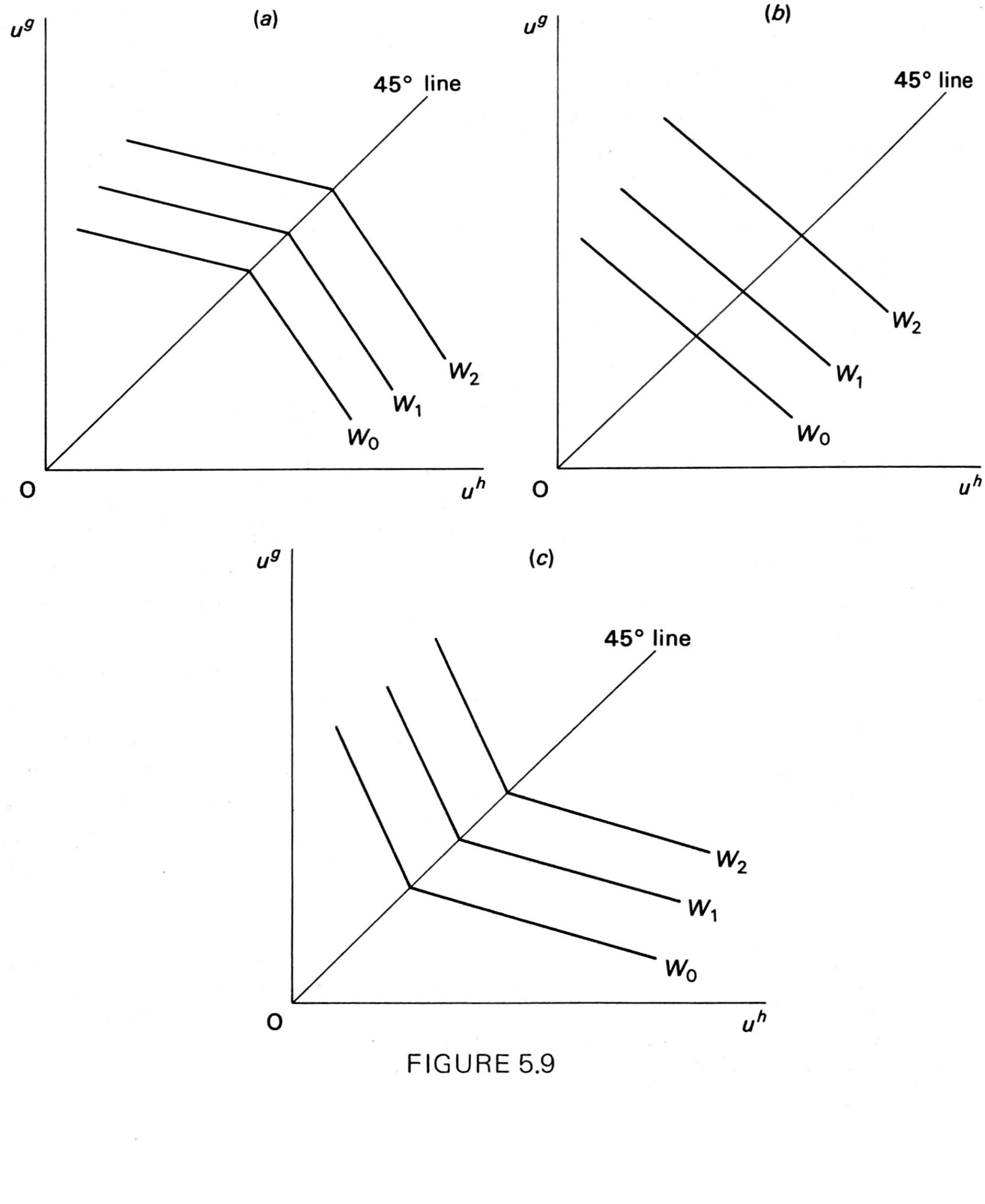

FIGURE 5.9

where W^{u} is the generalized utilitarian form, W^{d} is a positional dictatorship form such as the maximin and α is a scalar between zero and one (for details see Roberts, 1980).

If the strong Pareto principle is invoked, the strong and positional dictatorship forms of the SWO are excluded but lexicographic forms remain possible. Adding anonymity precludes the lexicographic dictatorship and the generalized utilitarian forms, leaving the possibilities of the positional lexicographic and simple utilitarian forms. With the separability of indifferent individuals' requirements (SE), the lexicographic forms are narrowed to the leximin and leximax forms. Adding the minimal equity requirement (EM) leaves available the leximin and utilitarian forms (Deschamps and Gevers, 1978). Adding a continuity requirement leaves available only the utilitarian form (Maskin, 1978) while a strong equity requirement leaves only the leximin possibility.

5.3 *Ratio scale measurability* (RS)

When utility is measurable using a ratio scale, still further SWOs are admitted. With RS measurability, proportional changes in utility can be compared by the individual household and, under FC, can also be compared across households. Thus statements such as 'The proportional change in Alice's utility is greater (smaller) than that of Bob', are meaningful. Under RS, transformations of the type $v^h = b^h u^h$ are admitted, whereas FC implies that $b^h = b$ for all h. Then

$$\frac{v^g(y)}{v^g(x)} \gtreqless \frac{v^h(y)}{v^h(z)} \quad \text{as} \quad \frac{u^g(y)}{u^g(x)} \gtreqless \frac{u^h(y)}{u^h(x)}$$

Note that $v^g(y)/v^g(x)$ can also be written as $((v^g(y)-v^g(x))/v^g(x))+1$; thus proportional changes in utility are comparable. The reader can ascertain that such comparability is not possible with CS measurability. Levels and increments of utility still remain comparable across households. Hence, the information available to the planner is again increased and further SWO possibilities are admitted.[14]

In figure 5.10 we have a reference point u^0 and a line segment the slope of which is equal to $-(\partial W/\partial u^h)/(\partial W/\partial u^g)$, the slope of the SWF indifference curve through u_0. As before, this slope must be unchanged when utilities are transformed according to the linear transformation $v^h = bu^h$ for all h. This means that the slope of the SWF indifference curve must be the same along a ray from the origin through point u_0. As point u_0 is chosen arbitrarily, this condition must hold along any ray

[14] In the discussion of ratio scale measurability we restrict the range of individual utility functions to the positive real line. This is done in order that the addition of a positive proportion of the utility level to itself increases utility; that is, $(1+f)u \geqslant u$ if $f \geqslant 0$. This involves no loss of generality because we could have left the range of the utility functions as the entire real line and considered ratio scale measurability in terms of the ratio to the absolute value of utility.

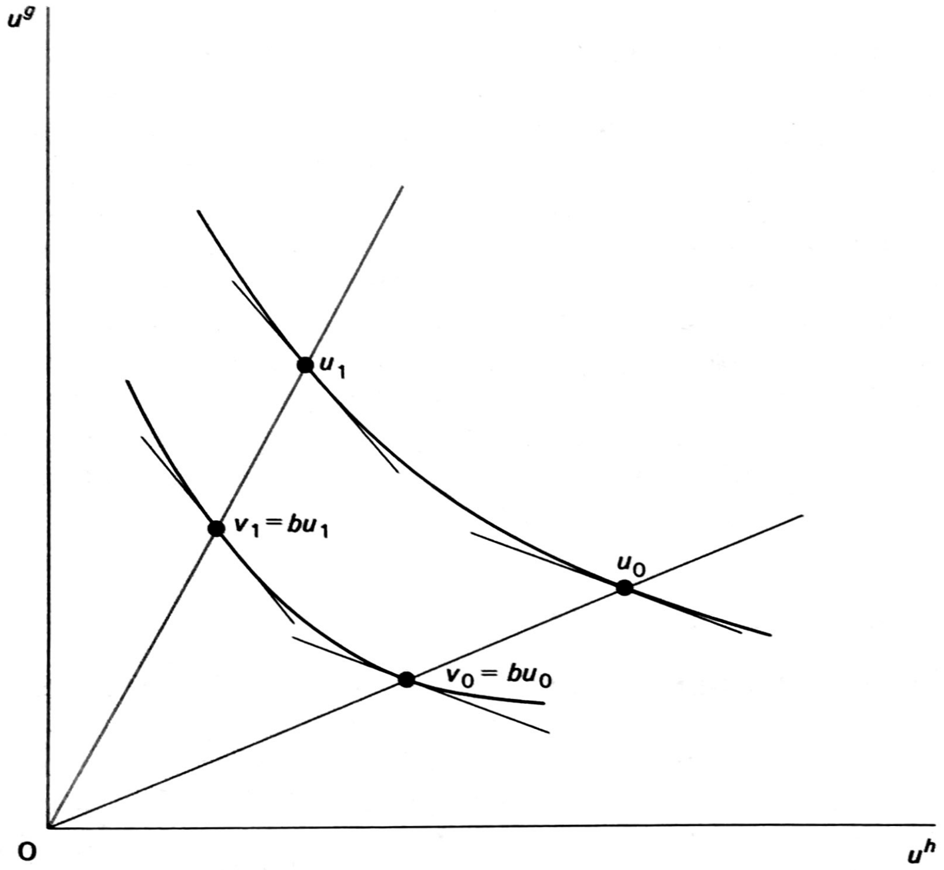

FIGURE 5.10

(e.g. the ray passing through u_1). Any homothetic SWF satisfies this property; but since the SWF indifference curves can be numbered in any increasing manner, we can restrict our attention to the linearly homogeneous SWF form. Thus the linearly homogeneous SWF possibility is added to the possibilities open under RS measurability. Adding A requires that the linearly homogeneous form be symmetric. Finally, if SE and A are assumed, the linearly homogeneous SWF must be of the constant elasticity of substitution form

$$W = \sum_{h=1}^{H} \frac{(u^h)^{1-\rho}}{1-\rho} \tag{5.5'}$$

where $1/\rho$ is the elasticity of substitution between any two households' utilities. As mentioned above, this SWF is very useful because ρ can be taken as an equity parameter. When $\rho = 0$, W is utilitarian. The limiting case as $\rho \to 1$ is the Bernoulli–Nash (Cobb–Douglas) case and the limiting case as $\rho \to \infty$ $(-\infty)$ is the maximin (maximax) form. Note that the latter two are limiting cases since A precludes a positional dictatorship. In other words, as ρ increases, more weight is given to the equality of utilities *per se* and the SWF indifference curves become more convex.

5.4 *Absolute scale measurability* (AS)

When utility is measurable to an absolute scale and full comparability is assumed, the SWO possibilities are the widest possible. With AS, the only transformation permitted is the identity transform $v^h = u^h$ for all h. In this case the invariance requirement is trivial. In terms of figure 5.10, the only possible transformation of reference point u_0 is one which maps it back to itself; thus the slope of the SWF indifference curve can be different at every point in utility space. In other words, AS measurability of utility permits the general Bergson–Samuelson form of the SWF. The Pareto principle (strong) makes the SWF indifference curves negatively sloped, A makes the SWF symmetric, and SE makes the SWF additively separable, i.e. can be expressed in the form

$$W(x) = \sum_{h=1}^{H} g[u^h(x)]$$

An equity requirement is necessary to make the SWF indifference curves convex.

The results of sections 4 and 5 are summarized in table 5.1. It shows the sorts of SWOs that are possible under various informational assumptions. It shows that comparability is the *sine qua non* for non-dictatorial SWOs. With FC, the SWO possibilities are widened by greater measurability (less restrictive invariance requirements) of individual household utilities. The SWO possibilities are narrowed by the addition of requirements such as A, SE, EM or ES and CO. +

6 SWO Possibilities with Partial Comparability

If some of the information implied by the measurability of the individual household's utility function is not available for comparisons across households, then comparability is said to be partial. In this case, certain utility comparisons can be made by the individual household which cannot be used for making comparisons across households.

6.1 *Cardinal scale measurability with unit comparability* (CS–UC)

In this case households can make comparisons both of levels and of increments in their own utility, but only increments can be compared across households. Formally, the utility functions of the households can be transformed by $v^h = a^h + b^h u^h$, where $b^h = b$ for all h but a^h can differ across households. Thus level comparisons across households are precluded by the transformation but increment comparisons are possible.

It is easily seen that CS–UC when combined with welfarism and the Pareto principle permits only the generalized utilitarian SWF (in addition to dictatorship). In figure 5.11, the reference utility point u_0 and a line

TABLE 5.1 SWF possibilities under non-comparability and full comparability*

Informational requirement \ *Ethical requirement*	$W + PW$	$W + PS$	$W + PS + A$	$W + PS + A + SE$	$W + PS + A + SE + EM$	$W + PS + A + SE + CO + ES$
NC, OS or CS	DS, DL (Arrow (1951a), Sen (1970))	DL	none	none	none	none
FC						
(a) OS	above and DP, LP	above and LP	LP	LXN, LXX (Hammond (1976), Strasnick (1975))	LXN	none
(b) CS	above and UG UG–DP (Roberts (1980))	above and UG	above and U	above and U	above and U (Deschamps & Gevers (1978))	U (Maskin 1978))
(c) RS	above and H	above and H	above and HS	above and CE	above and CE	CEC
(d) AS	above and B	above and B	above and BS	above and BSS	above and BSS	above and BSSC

Note: * The above entries are not necessarily exclusive except where 'none' is indicated.

Abbreviations:

A	anonymity
AS	absolute scale measurability
B	Bergson–Samuelson SWF
BS	Bergson–Samuelson SWF (symmetric)
BSS	Bergson–Samuelson SWF (symmetric and separable)
BSSC	Bergson–Samuelson SWF (symmetric, separable and quasi-concave)
CE	constant elasticity SWF
CEC	constant elasticity SWF (concave, $\rho \geqslant 0$)
CO	continuity
CS	cardinal scale measurability
CSSB	quasi-concave separable symmetric Bergson–Samuelson SWF
DL	dictatorship (lexicographic)
DP	dictatorship (positional)
DS	dictatorship (strong)
EM	equity (minimal)
ES	equity (strong)
FC	full comparability
H	homogeneous SWF
HS	homogeneous (symmetric) SWF
LP	lexicographic by positions
LXN	leximin
LXX	leximax
NC	non-comparable
OS	ordinal scale measurability
PS	Pareto principle (strong)
PW	Pareto principle (weak)
RS	ratio scale measurability
SE	separability of indifferent households
U	utilitarian
UG	utilitarian (generalized)
UG–DP	linear combination of UG and DP

Permitted transformations

OS–NC	$v^h = \psi^h(u^h)$	$\psi^h(\ )$ any monotonic
CS–NC	$v^h = a^h + b^h u^h$	
OS–FC	$v^h = \psi(u^h)$	$\psi(\ )$ any monotonic
CS–FC	$v^h = a + bu^h$	
RS–FC	$v^h = bu^h$	
AS–FC	$v^h = u^h$	

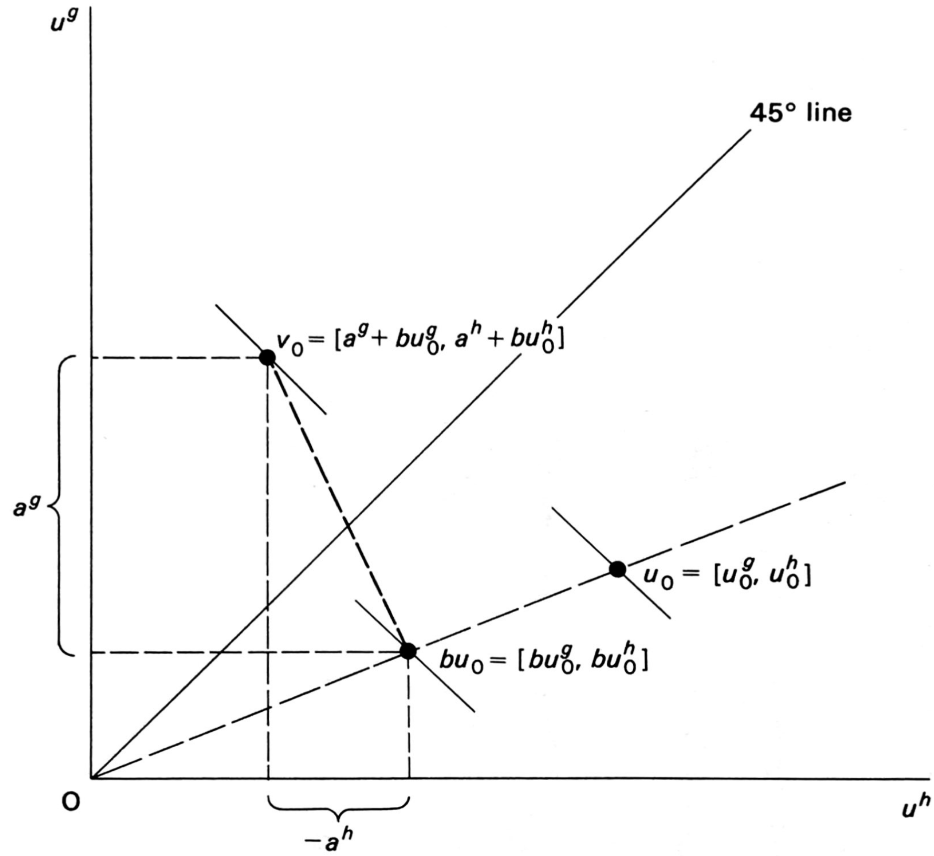

FIGURE 5.11

segment having a slope equal to the slope of the SWF indifference curve are shown. As shown, the transform $a^h + bu^h$ permits u_0 to be mapped to v_0 anywhere in the utility space, so the SWF indifference curves must have the same slope everywhere in the utility space. The slope need not be equal to -1, so the SWF is a generalized utilitarian form. Adding A precludes the dictatorship possibility and leaves available the simple (unweighted) utilitarian SWF (a version of this result was proved by D'Aspremont and Gevers (1977)).

6.2 *Ratio scale measurability* (RS)

If utility is measurable by a ratio scale, then unit comparability implies level comparability. However, it is possible for proportional comparisons of utility (which are possible for the individual household under RS) to be comparable across households even though units and levels are not. In fact, this must be the case: non-comparability under RS measurability is not possible.

Consider the case where the permissible transformations are $v^h = b^h u^h$ for all h and b^h can differ across households. Note that this transformation leaves $u^h(x)/u^h(y)$, and therefore $(u^h(x) - u^h(y))/u^h(y)$ unchanged for every household. Thus

$$\frac{v^g(x) - v^g(y)}{v^g(y)} \gtrless \frac{v^h(x) - v^h(y)}{v^h(y)}$$

as

$$\frac{u^g(x) - u^g(y)}{u^g(y)} \gtrless \frac{u^h(x) - u^h(y)}{u^h(y)}$$

In other words, comparisons such as, 'Household Alice's increment in utility as a proportion of her utility level is greater (less) than that of Bob's', can still be made.

It can be shown that this admits the possibility that the SWF be of the Bernoulli–Nash (Cobb–Douglas) form. That is,

$$W = \prod_{h=1}^{H} (u^h)^{a_h} \tag{5.3'}$$

To see this, recall that we require

$$\frac{\partial W/\partial m^g}{\partial W/\partial m^h} = \frac{\partial W/\partial u^g}{\partial W/\partial u^h}\,\frac{\partial u^g/\partial m^g}{\partial u^h/\partial m^h} \tag{5.6}$$

to be unchanged when the permissible linear transformations of utility functions are undertaken. At first this seems impossible because $(\partial u^g/\partial m^g)/(\partial u^h/\partial m^h)$ will depend on the ratio b^g/b^h, which is arbitrary. However, multiplying and dividing the right-hand side of (5.6) by u^g/u^h we get

$$\frac{\partial W/\partial m^g}{\partial W/\partial m^h} = \frac{(\partial W/\partial u^g)u^g}{(\partial W/\partial u^h)u^h}\,\frac{(\partial u^g/\partial m^g)/u^g}{(\partial u^h/\partial m^h)/u^h} \tag{5.6'}$$

The last term is unchanged by the linear transformations even if $b^g = b^h$, since b^g cancels out of the numerator and b^h cancels out of the denominator. Thus $(\partial W/\partial m^g)/(\partial W/\partial m^h)$ will be unchanged for an SWF that satisfies

$$\frac{\partial W/\partial u^g}{\partial W/\partial u^h} = \beta\frac{u^g}{u^h} \qquad \text{for constant } \beta > 0 \tag{5.8}$$

In figure 5.12 we have reference point u_0 where the slope of the SWF indifference curve $(\partial W/\partial u^h)/(\partial W/\partial u^g)$ is equal to the (absolute) slope of the line segment through u_0. Expression (5.8) requires that the slope of the SWF indifference curve be inversely proportional to the slope of

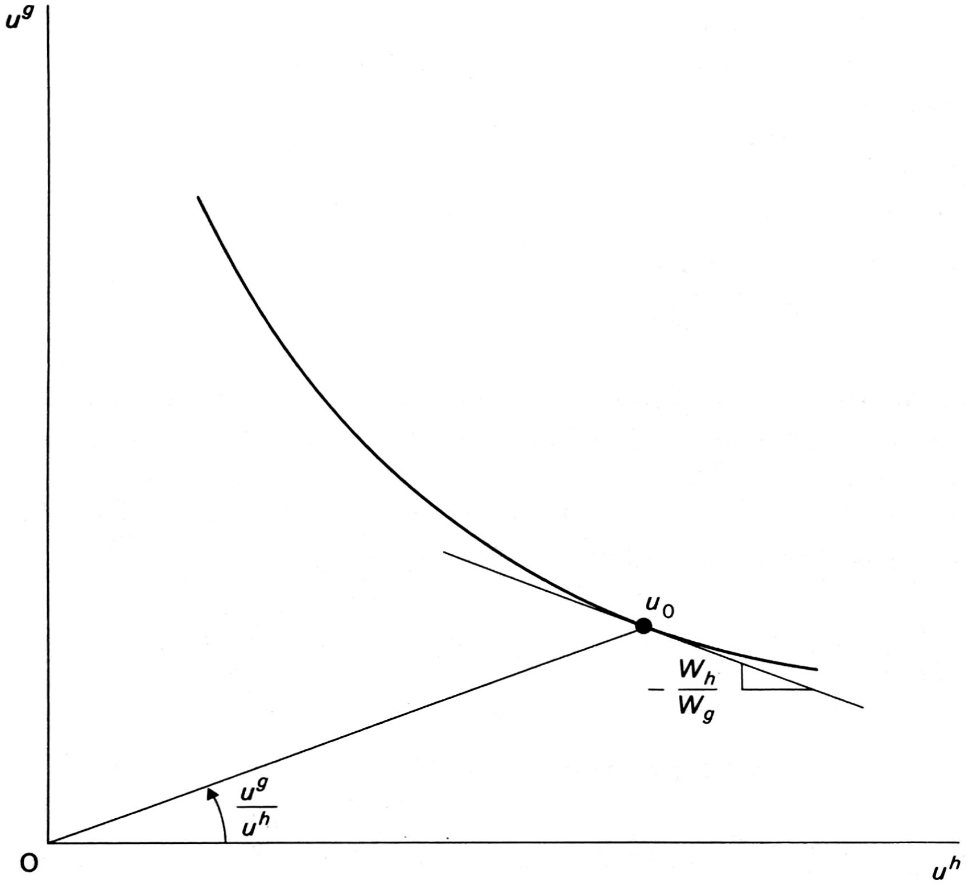

FIGURE 5.12

the ray from the origin through u_0. It immediately follows that all of the SWF indifference curves have the same slope along the ray, implying that the SWF is homothetic which, since we can number the social welfare indifference curves in any increasing way, is equivalent to a linearly homogeneous SWF form. However, (5.8) also implies that the slope of the SWF indifference curve must change in inverse proportion to the slope of the ray u^g/u^h. This requires that every SWF indifference curve must have an elasticity of substitution of unity at all points. The only SWF satisfying this property is the Bernoulli–Nash (Cobb–Douglas) form.

Adding anonymity makes the SWF symmetric; that is, $a^h = a$ for all h in (5.3′). It also precludes the dictatorship possibility leaving the symmetric Bernoulli–Nash as the only SWF possibility under RS–PC, W, P and A.

This exhausts the partial comparability cases since full comparability is implied by AS measurability whereas only FC or NC is possible under OS measurability. The results are summarized in Table 5.2.

7 Summary and interpretation

This chapter has presented what might be referred to as the *informational approach* to social welfare orderings. The informational approach builds

TABLE 5.2 SWF possibilities under partial comparability

Informational restrictions \ *Ethical restrictions*	$W + PS$	$W + PS + A$
CS–UC	DS or DL UG	U (D'Aspremont and Gevers (1977))
RS–PC	DS or DL BN	BNS

Abbreviations

A	anonymity
BN	Bernoulli–Nash form
BNS	Bernoulli–Nash form (symmetric)
CS	cardinal scale
DL	dictatorship (lexicographic)
DS	dictatorship (strong)
PC	proportion comparability
PS	Pareto principle (strong)
PW	Pareto principle (weak)
U	utilitarian form
UG	utilitarian form (generalized)
W	welfarism

Permitted transformation

CS–UC	$v^h = a^h + bu^h$
RS–PC	$v^h = b^h u^h$

upon Arrow's (1951a) crucially important possibility theorem. According to that theorem, if we wish the social ordering to satisfy certain plausible axioms or value judgments (the Pareto principle, the independence of irrelevant alternatives, and unrestricted domain), and to be a complete and transitive ordering, and if we restrict the planner to knowing only the preference orderings of all households in the economy, then the only possible ordering is of a dictatorship form (either the dictatorship of a particular person or a lexicographical dictatorship of persons ordered in some particular way). The informational approach investigates how the set of possible SWOs expands as more 'information' is made available to the planner. This information can take the form of increasing degrees of measurability of household utilities and increasing degrees of interpersonal comparability of utilities. The latter is the *sine qua non* of meaningful SWOs. The more information that is available to the planner, the greater the range of possible SWO forms that are compatible with the value judgments being made. In the limit, full measurability of individual utilities and full comparability in conjunction with the axioms we have adopted permit the general Bergson–Samuelson form. On the other hand, the set of SWO possibilities is narrowed by allowing only partial comparability or measurability, or by imposing additional properties such as anonymity or separability.

It would, of course, have been possible to relax welfarism to obtain a different set of possible SWOs. We have chosen not to pursue that route here. (Interested readers may consult Sen, 1970 or Sen, 1977.) Instead, we have restricted ourselves to a similar set of axioms to those used by

Arrow. The only difference with Arrow's axioms is in our use of Pareto principle. Arrow required only the weak version of the Pareto principle, whereas we have also investigated the consequences of admitting Pareto indifference and the strong Pareto principle. As we have seen, the use of Pareto indifference together with the independence and unrestricted domain axioms implies that the SWO will be welfaristic; that is, the SWO depends only on utility outcomes of the social states. In addition to making the analysis more tractable, this seems to be a fairly reasonable requirement for choosing among alternative resource allocations.

The addition of measurability and comparability information, as in this chapter, complements the results of the preceding chapters. It will be recalled that if the Pareto and individualism are the only value judgments made and if household preference orderings are the only source of information, then social states cannot be completely ordered. Only those which are Pareto comparable can be ordered. This chapter has investigated the sorts of complete social orderings which are *possible* given the different kinds of information available to the planner. Except in a few special cases, the informational approach does not leave us with a unique SWO (or SWF if the ordering is continuous). To select a unique method of ordering social states from the various possible SWOs requires further ethical judgments. Ethical arguments for certain SWO forms which exist in the literature will be discussed in the next chapter.

Before considering these ethical arguments it is worth considering exactly how one might interpret the informational approach to social orderings. What does it mean to say that the planner has available information on the measurability and comparability of utilities? Is this to be taken as information obtained in a scientific or empirical fashion or is it information which represents some person's subjective evaluation of individual utility levels? It seems to us that there are at least two ways that one may interpret the informational approach, each of which leads to a slightly different view of the role of the planner.

First, one may take the view that the measurement of utility is, in principle, an objective matter. Once utility levels are empirically determined, they can then naturally be compared among individuals. This seems to have been the view taken by the classical utilitarians and their followers (e.g. Bentham, Mill, Edgeworth), but also appears to be held today by some (e.g. Ng, 1979). The planner then takes this information and chooses among the SWOs which the information permits. The choice itself involves an ethical judgment as to how to trade one person's utility off against another's, but the information used is treated as objective. Of course, as above, the information may involve only partial measurability or comparability, in which case the possible SWOs are restricted accordingly.

The theory developed in this chapter is perfectly compatible with this view; the objections to it may be both ethical and empirical. One may take the view that the measurement of utility and, even more, its comparability among persons involves a fundamental value judgment. Alternatively one may object that, even if one thought that utility were in principle measurable, there exists no agreed method for obtaining more than ordinal measurement or for comparing utility levels. This being the case, the objective information available to the planner as revealed by the behaviour of households is what we have called ordinal non-comparable utilities. If this is the only information allowed, we are back to the Arrow possibility theorem.

A second and more fruitful possibility is to view the information not as being given to the planner from an outside source but as reflecting the planner's own ethical judgment of the measurability and comparability of utility. Thus, OS–FC means the planner is ethically prepared to measure utility ordinally and to compare utility levels fully among persons but not utility increments. This is fundamentally different from the first view outlined above in that it is recognized that the information itself reflects an ethical judgment of the planner (or someone else) and does not comprise some objectively determined data. In a sense, the use of the term 'information' in the literature to convey the measurability and comparability of utilities is unfortunate, since it almost connotes empirical data.

If this is to be the interpretation placed on the information used by the planner, some further questions are raised. We have already seen that under most combinations of measurability and comparability, no unique SWO emerges. The planner has a set of possible SWOs from which one must be chosen. This choice requires a further ethical judgment involving how the measured utilities are to be traded off. It seems rather artificial to separate these two ethical judgments in the analysis. Furthermore, if the measurability and comparability assumptions reflect the planner's judgment, why should the planner restrict himself to partial rather than full measurability and comparability, especially since these restrict the set of SWOs from which he may choose? In other words, why not simply let him choose the Bergson–Samuelson SWO that represents his ethical preferences?

In any case, it is clear that the informational approach to SWOs does not generally leave the planner with a unique method of ordering social states, that is, with a unique SWF. What it does is provide the planner with a set of possible candidates for the SWF, a set which depends upon the information which is assumed to be available. The more information that is available, or the higher the degree of measurability and comparability the planner is faced with or is prepared to assume, the larger the set of SWOs there are to choose from. The choice of a specific form for the SWO then involves a further ethical judgment about how to aggregate the individual utilities.