LECTURE 3

MICROECONOMIC THEORY

CONSUMER THEORY

Classical Demand Theory

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Introduction and definitions

- In this chapter we will assume that demand is based on the maximization of rational preferences.
- Remember:
  - I. Rationality. A preference relation $\succsim$ is rational if it implies a complete and transitive ordering of all consumption bundles within a consumption set X (see lecture 1).
- Background: without rationality of individuals, normative conclusions cannot be based on methodological individualism,
  - • i.e. explaining and understanding broad society-wide developments as the aggregation of decisions by individuals
- In addition to rationality, specific economic problems may suggest the appropriateness (desirability) of additional assumptions (see next slides).

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Introduction and definitions

- Notation of vector inequalities:

$\gg$ Strictly Greater	means	$>$ in all components
$>$ Greater	means	$\geq$ in all components but $>$ in some
$\geq$ Greater or Equal	means	$\geq$ in all components

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} \underset{\text{strictly greater}}{\gg} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \underset{\text{greater}}{>} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \underset{\text{greater or equal}}{\geq} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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Introduction and definitions

- Monotonicity (more is better). The preference relation $\succsim$ is
  - *monotone* if y>>x $\Rightarrow$ y $\succ$ x.
  - *strongly monotone if y ≥ x* $\Rightarrow$ *y* $\succ$ *x.*
    - "bads" (e.g. garbage) violate monotonicity assumption. Trick: redefine commodity as "absence of bads"
    - monotonicity sometimes justified by defining preferences over goods available for consumption – rather than consumption itself – and assuming *free disposal*
- Remember that
  - y>>x means that $y_n > x_n$ for all n = 1, …, N , i.e. each element of the y vector is larger than the corresponding element of the x vector
  - y ≥ x means $y_n \geq x_n$ for all n = 1, …, N

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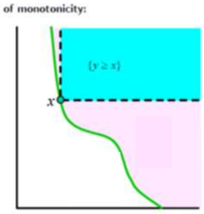
Introduction and definitions

• Illustration of monotonicity:



Monotonicity: More of all goods increases utility.
{the blue dark area not including x or the dotted lines is strictly preferred}

• Strong monotonicity: More of any goods increases utility.
{the blue dark area including the dotted lines but not x is strictly preferred}

NOTE: If a preference relation is monotone, we may have indifference with respect to an increase in the amount of some but not all commodities. In contrast strong monotonicity says that if *y* is larger than *x* for *some* commodity, then *y* is strictly preferred to *x*.

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Introduction and definitions

- Put simply:
  - **Monotonicity :**
    - Increasing the amounts of some commodities (without reducing the amount of any other commodity) cannot hurt

      $y_n \geq x_n$ for all n implies y $\succsim$ x
    - Increasing the amounts of all commodities is strictly preferred

      $y_n > x_n$ for all n implies y $\succ$ x
  - **Strong monotonicity**:

    $y_n \geq x_n$ for all n implies y $\succ$ x

    Even if we increase the amounts of only one of the commodities, we make the consumer better off.

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Monotonicity: Example 1

$$u(x_1, x_2) = \min\{x_1, x_2\}$$

– Monotone, since

$$\min\{x_1 + \delta, x_2 + \delta\} > \min\{x_1, x_2\}$$

for all $\delta > 0$.

– Not strongly monotone, since

$$\min\{x_1 + \delta, x_2\} \ngtr \min\{x_1, x_2\}$$

if $\min\{x_1, x_2\} = x_2$.

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Monotonicity: Example 2

$$u(x_1, x_2) = x_1 + x_2$$

– Monotone, since

$$(x_1 + \delta) + (x_2 + \delta) > x_1 + x_2$$

for all $\delta > 0$.

– Strongly monotone, since

$$(x_1 + \delta) + x_2 > x_1 + x_2$$

- Hence, strong monotonicity implies monotonicity, but the converse is not necessarily true.

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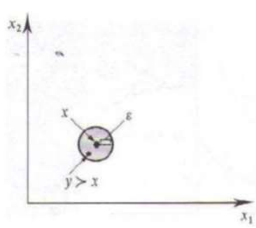
Introduction and definitions

- Local nonsatiation. (you can always increase utility by making a small change in your consumption bundle)

  The preference relation $\succsim$ is *nonsatiated* if for every x and every $\varepsilon > 0$, there is y such that $||y - x|| \leq \varepsilon$ and $y \succ x$

  measure of distance



In words: for any consumption bundle *x* and an arbitrarily small distance away from *x*, denoted by ε>0, there is another bundle *y*, within this distance from *x* that is preferred to *x*.

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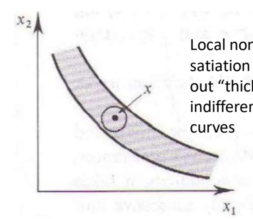
Introduction and definitions

- Implications of local non-satiation.



Local non-satiation rules out "thick" indifference curves

That all goods are bads
– If all goods were bads, zero consumption would be a satiation point. But then all "neighboring" bundles would be worse, conflicting with local non-satiation

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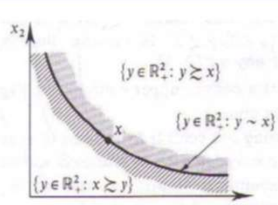
Introduction and definitions



- Given the preference relation $\succsim$, three related sets of consumption bundles can be defined w.r.t. a given bundle x
  - indifference set: $\{y \in X: y \sim x\}$
  - upper contour set: $\{y \in X: y \succsim x\}$
  - lower contour set: $\{y \in X: y \precsim x\}$

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Introduction and definitions

- Convexity

  Recall that a set of points, *X*, is **convex** if for any two points in the set the (straight) line segment between them is also in the set.

  Formally, a set X is convex if for any points x and x′ in X, every point *z* on the line joining them,

  z = *tx* + (1-*t*) *x′* for some *t* in [0,1], is also in *X*.
- Before we move on, let's do a thought experiment.
- Consider two possible commodity bundles, x and x′. Relative to the extreme bundles x and x′, how do you think a typical consumer feels about an average bundle, z = tx + (1 − t) x′, t in (0, 1)?
- Although not always true, in general, people tend to prefer bundles with medium amounts of many goods to bundles with a lot of some things and very little of others (examples?). Since real people tend to behave this way, and we are interested in modeling how real people behave, we often want to impose this idea on our model of preferences

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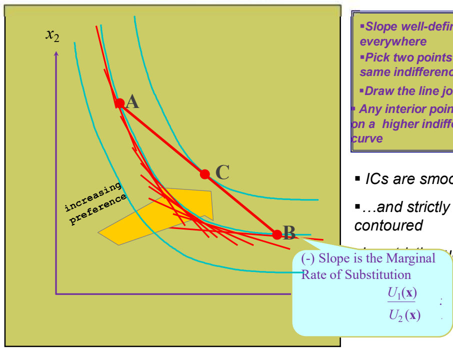
CONVENTIONALLY SHAPED INDIFFERENCE CURVES



- Slope well-defined everywhere
- Pick two points on the same indifference curve.
- Draw the line joining them.
- Any interior point must line on a higher indifference curve

- ICs are smooth
- …and strictly concaved-contoured
- …asiconcave

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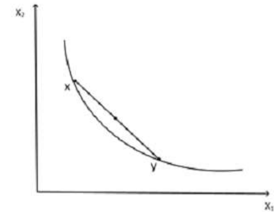
Interpretation of convexity

1) *Taste for diversification*:
- An individual with convex preferences prefers the convex combination of bundles x and y, than either of those bundles alone.



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Interpretation of convexity

2) *Diminishing marginal rate of substitution:*

$$MRS_{1,2} \equiv \frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$

- *MRS* describes the additional amount of good 1 that the consumer needs to receive in order to keep her utility level unaffected.
- *A diminishing MRS* implies that the consumer needs to receive increasingly larger amounts of good 1 in order to accept further reductions of good 2.

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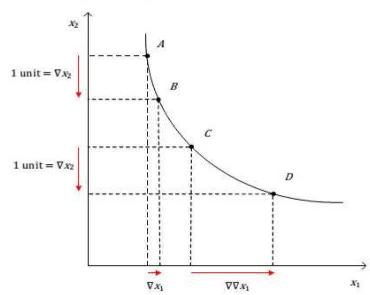
Diminishing marginal rate of substitution



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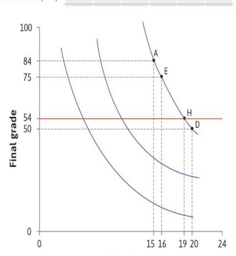
Final grade – free time trade-off

	A	E	F	G	H	D
Hours of free time	15	16	17	18	19	20
Final grade	84	75	67	60	54	50
Marginal rate of substitution between grade and free time (MRS)	9	8	7	6	4	



How many hours should I study?

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Quasiconcavity

- A utility function $u(\cdot)$ is ***quasiconcave*** if, for every bundle $y \in X$, the set of all bundles for which the consumer experiences a higher utility, i.e., the $UCS(x) = \{y \in X \mid u(y) \gtrsim u(x)\}$ is convex.
- The following three properties are equivalent:

Convexity of preferences $\Leftrightarrow$
$UCS(x)$ is convex $\Leftrightarrow$
$u(\cdot)$ is quasiconcave

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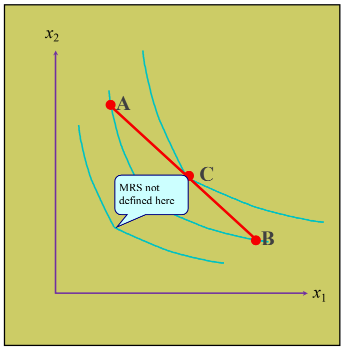
OTHER TYPES OF IC: KINKS



- *Strictly quasiconcave*
- *But not everywhere smooth*

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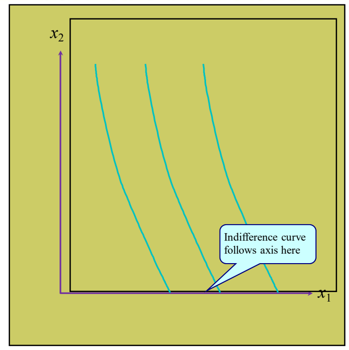
OTHER TYPES OF IC: NOT STRICTLY QUASICONCAVE



- *Slope well-defined everywhere*
- *Indifference curves with flat sections make sense*
- *But may be a little harder to work with...*

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Introduction and definitions

- justification of convexity assumption
  - *diminishing marginal rates of substitution*: starting at $x \in \mathbb{R}^2$, it takes increasingly larger amounts of one commodity to compensate for losses of the other
  - inclination for diversification, esp. for situations with uncertainty
- nevertheless, convexity is a debatable assumption
  - e.g. you may prefer milk or orange juice to a mixture of both
  - sometimes, convexity can be obtained by appropriate aggregation, e.g. milk and orange juice over a week

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Preference and Utility

- The previous analysis about preferences is not extremely useful because you have to do it one bundle at a time.
- If we could somehow describe preferences using mathematical formulas, we could use math techniques to analyze consumer behaviour.
- The tool we use is the utility function (already introduced in lecture 1).
- A utility function assigns a number to every consumption bundle *x* in *X*. According to its definition, the utility function assigns a number to *x* that is at least as large as the number it assigns to *y* if and only if *x* is at least as good as *y*.

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Preference and Utility

- Question: Under what circumstances can the preference relation $\succsim$ on $X = \mathbb{R}^L_+$ be represented by a utility function?

As it turns out rationality is not sufficient.
For example, define on $X = R^2_+$ as follows:
$x \succsim y$ if either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$.
– i.e. good 1 has highest priority, as the first letter in dictionary

These *lexicographic preferences* cannot be represented by a utility function.
– intuition: no two distinct bundles are indifferent so that indifference sets are singletons

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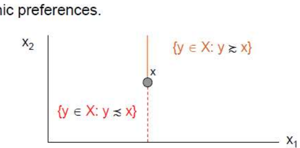
Preference and Utility

- lexicographic preferences.



- upper contour set: all points to the right of vertical line or on its solid part
- lower contour set: all points to the left of vertical line or on its dashed part
  - no point is indifferent to x;
  - hence, since x has been chosen arbitrarily, all indifference sets are singletons

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Preference and Utility

An additional property is needed.

Continuity. The preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is continuous if it is preserved under limits. That is, for any sequence of pairs

$$\{(x^n, y^n)\}_{n=1}^{\infty} \text{ with } x^n \succsim y^n \text{ for all } n,$$

$$x = \lim_{n\to\infty} x^n, \text{ and } y = \lim_{n\to\infty} y^n, \text{ we have } x \succsim y$$

- continuity rules out „jumps" in the preferences
  - e.g. that a consumer prefers each element in the sequence $\{x^n\}$ to the corresponding element in the sequence $\{y^n\}$, but suddenly reverses her preferences to $y > x$

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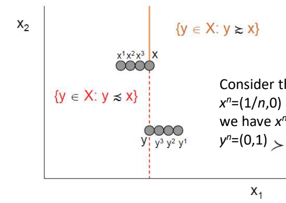
Preference and Utility

- continuity rules out lexicographic preferences.



Consider the sequence of bundles $x^n=(1/n,0)$ and $y^n=(0,1)$. For every n, we have $x^n \succ y^n$. But at the $\lim_{n\to\infty} y^n=(0,1) \succ (0,0) = \lim_{n\to\infty} x^n$.

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Preference and Utility

- Proposition:

  If $\succeq$ is rational and continuous then we can always have a continuous utility function to represent these preferences

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Preference and Utility

Axiom	Implication
1. Completeness	No gaps in the commodity space. Any two bundles can be compared.
2. Transitivity	Orders bundles in terms of preferences.
3. Nonsatiation	A household can always do a little bit better.
4. Diminishing Marginal Rate of Substitution (Strict Convexity)	Averages are preferred to extremes.

Rational Household

Necessary for Utility Maximization

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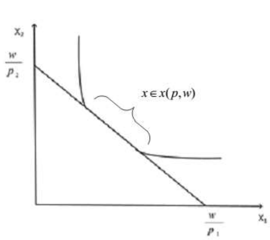

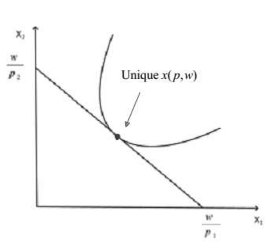

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Preference and Utility

- Utility is an ordinal concept, therefore any strictly increasing transformation of a utility function $u(.)$ that represents the preference relation $\succeq$ also represents $\succeq$.
  - Suppose f strictly increasing. Suppose that u is a utility function representing a preference relation. If $x \succ y$, then $u(\underline{x}) > u(\underline{y})$. With f strictly increasing, $f(u(\underline{x})) > f(u(\underline{y}))$. Therefore $f(u(.))$ is also a utility function representing the same preference relation.
  - The difference between the utility of two bundles doesn't mean anything. This makes it hard to compare things such as the impact of two different tax programs by looking at changes in utility.

- Common assumptions w.r.t. the utility function
  - Continuity
  - Differentiability
    - but: some preferences cannot be represented by a differentiable utility function,
      – e.g. Leontief preferences $u(x) = \min(x_1, x_2)$

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IRRELEVANCE OF CARDINALISATION

- $U(x_1, x_2,..., x_n)$
- $\log(U(x_1, x_2,..., x_n))$
- $\exp(U(x_1, x_2,..., x_n))$
- $\sqrt{}(U(x_1, x_2,..., x_n))$
- $\varphi(U(x_1, x_2,..., x_n))$

- *So take any utility function...*
- *This transformation represents the same preferences...*
- *...and so do both of these*
- *And, for any monotone increasing φ, this represents the same preferences.*

- *U is defined up to a monotonic transformation*
- *Each of these forms will generate the same contours.*
- *Let's view this graphically.*

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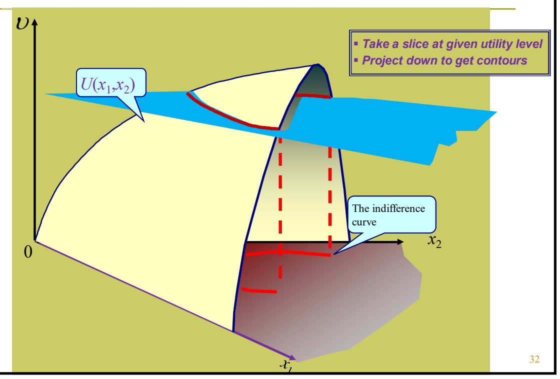
A UTILITY FUNCTION



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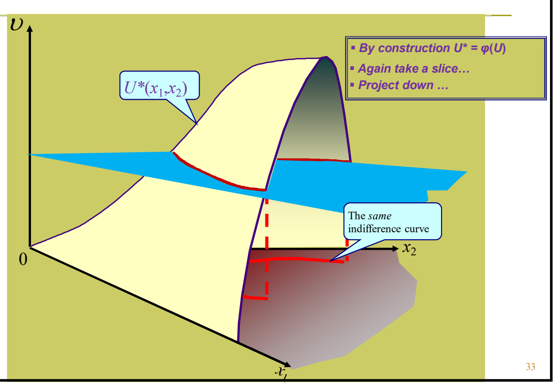
ANOTHER UTILITY FUNCTION



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Preference and Utility

Assumptions about the preference relation translate into implications for the utility function.

- Monotonicity of the preferences imply that the utility function is increasing: u(x) > u(y) if x>>y.
- Convex preferences lead to quasiconcave utility, i.e.
  - for convex preferences

    $u(\alpha x + (1-\alpha)y) \geq \text{Min}\{u(x), u(y)\}$ for any x,y and all $\alpha \in [0,1]$,

    which is the definition of a quasiconcave function.

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The utility maximization problem

- We compute the maximal level of utility than can be obtained at given prices and wealth.
- Difference with choice-based approach:
  - In choice-based approach we never said anything about why consumers make the choices they do.
  - Now we say that the consumer acts to maximise utility with certain properties.

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The utility maximization problem

- In order to ensure that the problem is "well-behaved", we assume that:
  - Preferences are rational, continuous, convex and non-satiated.
  - Therefore, the utility function u(x) is continuous and the consumer's choices will satisfy Walras' law.
  - We further assume that u(x) is differentiable in each of its arguments, so that we can use calculus techniques (the indifference curves have no kinks).

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The utility maximization problem

- Consumer utility maximization problem (UMP)

$$\max_{x \geq 0} u(x) \ s.t. \ p \cdot x \leq w$$

- Proposition (MWG 3.D.1): If p>>0 and u(.) is continuous, then the utility maximization problem has a solution.
- If the optimal set x(p,w) is single valued, we call it the Walrasian (or ordinary or market) demand function

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The utility maximization problem

- *Properties of Walrasian demand (*assuming that u(.) is continuous and represents a locally nonsatiated preference relation)

i. Homogeneity of degree zero in p and w: x(p,w) = x(αp, αw), for any p,w and scalar α > 0.

ii. Walras law: p·x = w for any x in the optimal set x(p,w).

iii. Convexity/uniqueness: if $\succeq$ is convex, so that u(.) is quasiconcave, then x(p,w) is a convex set. Moreover, if $\succeq$ is strictly convex so that u(.) is concave, then x(p,w) consists of a single element.

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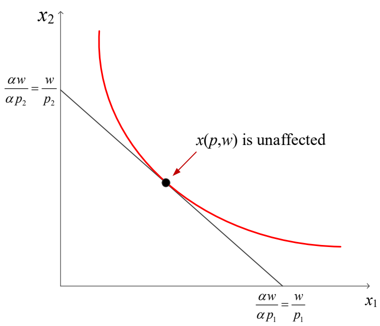
Properties of Walrasian Demand
Homogeneity of degree zero in p and w



- Note that the preference relation can be linear, and homog(0) would still hold.

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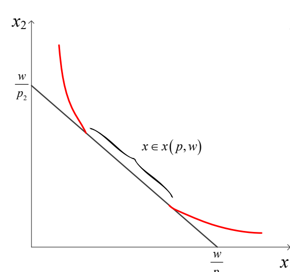
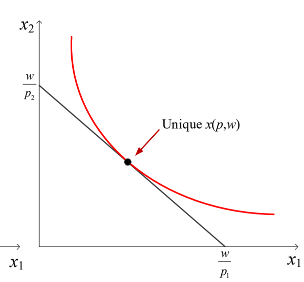
Properties of Walrasian Demand
Convexity and Uniqueness

Convex preferences



Strictly convex preferences



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The utility maximization problem

- Maximise

Lagrange multiplier

$$U(\mathbf{x}) + \lambda\left[w - \sum_{i=1}^{n} p_i x_i\right]$$

- If U is strictly quasiconcave we have an interior solution.

- A set of $n+1$ First-Order Conditions

$$\left.\begin{aligned} U_1(\mathbf{x}^*) &= \lambda^* p_1 \\ U_2(\mathbf{x}^*) &= \lambda^* p_2 \\ \dots \quad &\dots \quad \dots \\ U_n(\mathbf{x}^*) &= \lambda^* p_n \end{aligned}\right\}\ \text{one for each good}$$

budget constraint

$$w = \sum_{i=1}^{n} p_i x_i^*$$

- Use the objective function
- ...and budget constraint
- ...to build the Lagrangean
- Differentiate w.r.t. $x_1, \dots, x_n$ and set equal to 0.
- ... and w.r.t λ
- Denote utility maximising values with a $*$.

Interpretation

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From the FOC

- If both goods i and j are purchased and MRS is defined then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} = \frac{p_i}{p_j}$$

- (same as before)

- MRS = price ratio

- "implicit" price = market price

- If good i could be zero then...

$$\frac{U_i(\mathbf{x}^*)}{U_j(\mathbf{x}^*)} \leq \frac{p_i}{p_j}$$

- $\text{MRS}_{ji} \leq$ price ratio

- "implicit" price $\leq$ market price

Solution

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The solution...

- Solving the FOC, you get a utility-maximising value for each good...

$x_i^* = D^i(\mathbf{p}, w)$

- ...for the Lagrange multiplier

$\lambda^* = \lambda^*(\mathbf{p}, w)$

- ...and for the maximised value of utility itself.

Remark: In general the Largrange multiplier is the shadow value of the constraint, meaning that it is the increase in the value of the objective function resulting from a small relaxation of the constraint. The Lagrange multiplier is the marginal utility of wealth or income (mathematical property of the Lagrange multiplier).

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Interpreting the Lagrangian Multiplier

$$\lambda = \frac{\partial U/\partial x_1}{p_1} = \frac{\partial U/\partial x_2}{p_2} = \dots = \frac{\partial U/\partial x_n}{p_n}$$

$$\lambda = \frac{MU_{x_1}}{p_1} = \frac{MU_{x_2}}{p_2} = \dots = \frac{MU_{x_n}}{p_n}$$

- At the optimal allocation, each good purchased yields the same marginal utility per € spent on that good
- So, each good must have identical marginal benefit (MU) to price ratio
- If different goods have different marginal benefit/price ratio, you could reallocate consumption among goods and increase utility. Hence, you would not be maximizing utility.

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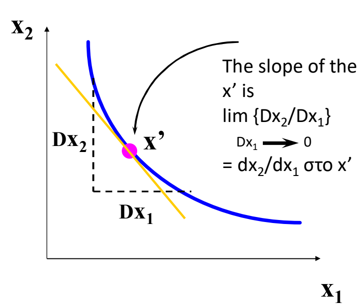
A two-goods example



The slope of the indifference curve at x' is

lim $\{Dx_2/Dx_1\}$

$Dx_1 \longrightarrow 0$

= dx_2/dx_1 στο x'

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A two-goods example

- The general form for an indifference curve is

  $U(x_1,x_2) \equiv k$, a constant.

  Taking the total derivative:

$$\frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 = 0$$

Or $\frac{\partial U}{\partial x_2}dx_2 = -\frac{\partial U}{\partial x_1}dx_1$ or $\frac{dx_2}{dx_1} = -\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{MU_1}{MU_2}$.

We call this the Marginal Rate of Substitution

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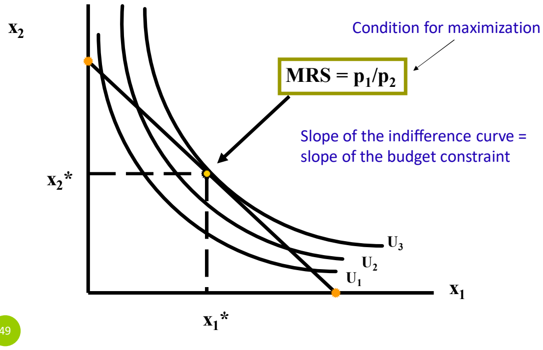
A two-goods example



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A Numerical Illustration

- Assume that the individual's *MRS* = 1
  - willing to trade one unit of *x* for one unit of *y*
- Suppose the price of *x* = \$2 and the price of *y* = \$1
- The individual can be made better off
  - trade 1 unit of *x* for 2 units of *y* in the marketplace
- So, it cannot be an optimal bundle if MRS is different from the ratio of prices

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The indirect utility function

Solving the FOC, you get a utility-maximising value for each good, for the Lagrange multiplier and for the maximised value of utility itself.

The *indirect utility function* is defined as

$$V(\mathbf{p}, w) := \max_{\{\Sigma p_i x_i \leq w\}} U(\mathbf{x})$$

It gives me the max utility I can attain given **p** and w

vector of goods prices

money income or wealth

I call it *indirect* because while utility is a function of the commodity bundle consumed, *x*, the indirect utility function *V*(**p**,*w*) is a function of **p** and w.

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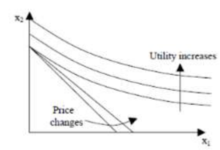
The Indirect Utility Function has some properties...

(All of these can be established using the known properties of the Walrasian demand function)

- Non-increasing in every price. Decreasing in at least one price



- Increasing in wealth *w*.

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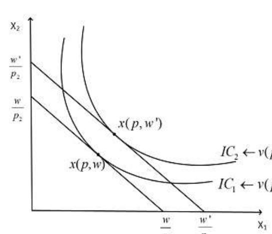
The Indirect Utility Function has some properties...

(All of these can be established using the known properties of the Walrasian demand function)

- Increasing in wealth *w*.

$v(p, w') > v(p, w)$ for $w' > w$.



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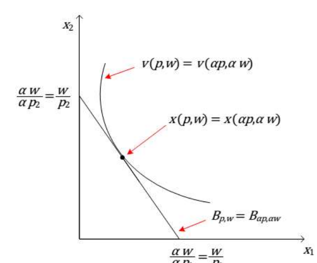
The Indirect Utility Function has some properties...

- Homogeneous of degree zero in (**p**, *w*) (since the bundle you consume does not change when you scale all prices and wealth by the same amount, neither does the utility you earn).



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The Indirect Utility Function has some properties...

- Roy's Identity

But what's this...?

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The indirect utility function

- The definition of the indirect utility function implies that the following identity is true:

$V(\mathbf{p},w) \equiv u(x(\mathbf{p},w))$

Differentiating both sides w.r.t. p_l: $\frac{\partial V}{\partial p_l} = \sum_{i=1}^{L} \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial p_l}$

Using that $\partial u/\partial x_i = \lambda p_i$ and that $\lambda = \partial V/\partial w$, after some manipulations we get:

$$x_l(\mathbf{p},w) = -\frac{\frac{\partial V}{\partial p_l}}{\frac{\partial V}{\partial w}}$$

Roy's identity: allows us to derive the demand function from the indirect utility function

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Roy's identity

Powerful result, since in many cases it is easier to compute the derivatives of $v(p,w)$ than solving the UMP with the system of FOCs.

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The expenditure minimization problem

- The expenditure minimization problem asks the question "if prices were **p**, what is the minimum amount the consumer would have to spend to achieve utility level u?"
- Officially:

$$\min_{x\geq 0} \mathrm{p}\cdot\mathrm{x} \quad \text{s.t. } u(\mathrm{x}) \geq u$$

In other words, the EMP computes the minimal level of wealth required to reach utility level u.

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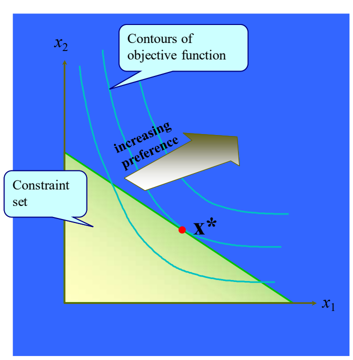
The primal problem (Utility Maximization Problem)



- *The consumer aims to maximise utility...*
- *Subject to budget constraint*
- *Defines the primal problem.*
- *Solution to primal problem*

max $U(\mathbf{x})$ subject to

$\sum_{l=1}^{L} p_l x_l \leq w$

- *But there's another way at looking at this*

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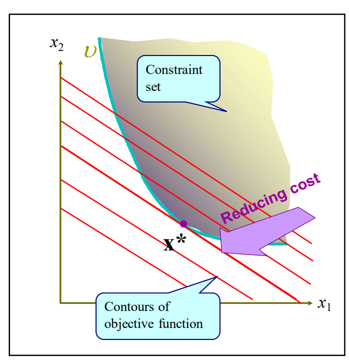
The dual problem (Expenditure Minimization Problem)



- *Alternatively the consumer could aim to minimise cost...*
- *Subject to utility constraint*
- *Defines the dual problem.*
- *Solution to the problem*

minimise

$\sum_{l=1}^{L} p_l x_l$

subject to $U(\mathbf{x}) \geq \upsilon$

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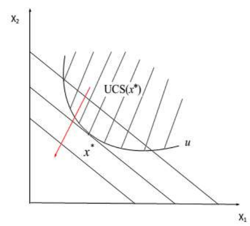
The dual problem (Expenditure Minimization Problem)

- Consumer seeks a utility level associated with a particular indifference curve, while spending as little as possible.
- Bundles strictly above x^* cannot be a solution to the EMP:
  - They reach the utility level u
  - But, they do not minimize total expenditure
- Bundles on the budget line strictly below x^* cannot be the solution to the EMP problem:
  - They are cheaper than x^*
  - But, they do not reach the utility level u



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The Primal and the Dual…

- There's an attractive symmetry about the two approaches to the problem
- In both cases the *p*s are given and you choose the *x*s. But…
- …constraint in the primal becomes objective in the dual…
- …and vice versa.

$$\sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})]$$

$$U(\mathbf{x}) + \mu\Big[\, w - \sum_{i=1}^{n} p_i x_i \,\Big]$$

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The expenditure minimization problem

EMP: $\min_{x \geq 0} \; p \cdot x \quad$ s.t. $u(x) \geq u$

$L_{EMP} = p \cdot x - \lambda\,(u(x) - u)$

FOC: $p_l - \lambda\, u_l(x) = 0 \quad$ for $l = 1, \ldots, L$

$\lambda\,(u(x) - u) = 0$

- The Hicksian demand function (or "compensated demand function") is the solution **h(p,u)** of the above problem

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The expenditure minimization problem

- Solving the FOC, you get a cost-minimising value for each good…

  $\mathbf{x}_i^* = h^i(\mathbf{p}, u)$

- …for the Lagrange multiplier

  $\lambda^* = \lambda^*(\mathbf{p}, u)$

- …and for the minimised value of expenditure itself.
- The *consumer's cost function* or *expenditure function* is defined as

  $e(\mathbf{p}, u) \;:=\; \min_{\{U(\mathbf{x}) \geq u\}} \Sigma\, p_i\, h^i(\mathbf{p}, u)$

vector of goods prices

Specified utility level

It is equal to the minimum cost of achieving utility u, for any given p and u

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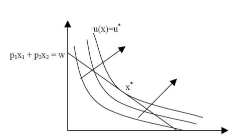
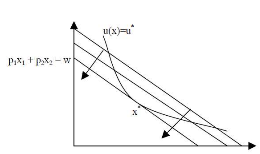
Duality properties

- The UMP picks out the point that max utility given the budget constraint.



- The EMP picks the point that achieves certain utility at min cost.



- The two points are the same!

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Duality properties

- If x* solves the UMP when prices are **p** and wealth is *w*, then x* solves the EMP when prices are **p** and the target utility level is u(x*).
- Further, maximal utility in the UMP is u(x*) and minimum expenditure in the EMP is *w*.
- This result is called the "duality" of the EMP and the UMP.

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Duality properties

- $\underline{x}(\underline{p},w) = \underline{h}(\underline{p}, v(\underline{p},w))$ i.e. the commodity bundle that maximizes your utility when prices are p and wealth is w, is the same bundle that minimizes the cost of achieving the maximum utility you can achieve when prices are p and wealth is w.

solution to the EMP (minimum expenditure)

- $\underline{h}(\underline{p},u) = \underline{x}(\underline{p}, \underline{p}\cdot \underline{h}(\underline{p},u)) = \underline{x}(\underline{p}, e(\underline{p},u))$ i.e. the commodity bundle that minimizes the cost of achieving utility u when prices are p, is the same bundle that maximizes utility when prices are p and wealth is equal to the minimum amount of wealth needed to achieve utility u at those prices.

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A USEFUL CONNECTION

- The indirect utility function maps prices and budget into maximal utility
  $u = v(\mathbf{p}, w)$

The indirect utility function works like an "inverse" to the cost function

- The cost function maps prices and utility into minimal budget
  $w = e(\mathbf{p}, u)$

The two solution functions have to be consistent with each other. Two sides of the same coin

- Therefore we have:
  $u = v(\mathbf{p}, e(\mathbf{p},u))$
  $w = e(\mathbf{p}, v(\mathbf{p}, w))$

Odd-looking identities like these can be useful

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68

Duality properties

- e ($\underline{p}$, v($\underline{p}$,w)) = w

  $e(p, v(p,w)) = w$, i.e., the minimal expenditure needed in order to reach a utility level equal to the maximal utility that the individual reaches at his UMP, $u = v(p,w)$, must be w.

- v ($\underline{p}$, e($\underline{p}$,u)) = u

  $v(p, e(p,u)) = u$, i.e., the indirect utility that can be reached when the consumer is endowed with a wealth level w equal to the minimal expenditure he optimally uses in the EMP, i.e., $w = e(p,u)$, is exactly u.

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Relationship between Expenditure function and Hicksian demand function

- Start from: $e(p,\bar{u}) \equiv p \cdot h(p,\bar{u})$
- Differentiating w.r.t. p_i: $\dfrac{\partial e}{\partial p_i} \equiv h_i(p,\bar{u}) + \sum_j p_j \dfrac{\partial h_j}{\partial p_i}.$
- Substituting the FOC, $p_j = \lambda\, u_j$

$$\frac{\partial e}{\partial p_i} \equiv h_i(p,\bar{u}) + \lambda \sum_j u_j \frac{\partial h_j}{\partial p_i}. \qquad (1)$$

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Relationship between Expenditure function and Hicksian demand function

- The constraint is binding at any optimum of the EMP,

$$u(h(p,\bar{u})) \equiv \bar{u}$$

- Differentiate w.r.t. p_i: $\sum_j u_j \dfrac{\partial h_j}{\partial p_i} = 0$
- Substituting into (1): $\dfrac{\partial e}{\partial p_j} \equiv h_j(p,\bar{u})\,.$

I.e. the derivative of the expenditure function w.r.t. p_j is just the Hicksian demand for commodity j.

Importance: we can derive the Hicksian demand function from the expenditure function.

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The Hicksian demand function

Hicksian compensation

We have:

$$h(p,u) = x(p, \underbrace{e(p,u)}_{w})$$

When prices vary, $h(p,u)$ indicates how the Marshallian demand would adjust if wealth was modified to ensure that the consumer still obtains utility u (i.e. adjusting the consumer's wealth so that the new wealth exactly enables him to buy a quantity that will yield the utility level u when spent efficiently).

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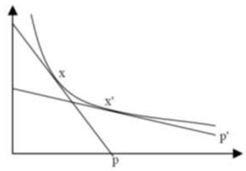
The Hicksian compensation



The Hicksian demand curve is also known as the compensated demand curve. The reason for this is that implicit in the definition of the Hicksian demand curve is the idea that following a price change, you will be given enough wealth to maintain the same utility level you did before the price change (since demand is calculated for given p and *u*). When prices change from p to p', the consumer is compensated by changing wealth from w to w' so that he is exactly as well off in utility terms after the price change as he was before. E.g. if prices increase, (p',*u*) would imply some kind of wealth compensation.

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Hicksian Compensation

Definition (Hicksian compensation)

Hicksian compensation is the variation in wealth Δw following a variation in price ($p \to p'$) such that the utility-maximizing consumer keeps the same initial utility $v(p, w)$.

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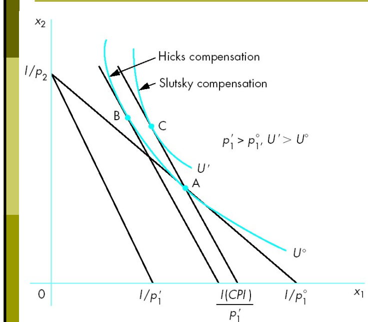
Hicks and Slutsky compensation



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Other properties of the Hicksian demand function

- Recall $\frac{\partial e}{\partial p_j} \equiv h_j(p, \bar{u})$ (1)
- How does the compensated demand of commodity *i* change when the price of commodity *j* changes? Take first derivative of (1) w.r.t. p_j:

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j}$$

But this is exactly the *ij*th element of the Slutsky substitution matrix!

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The Slutsky substitution matrix

- The L x L matrix of partials $s_{ij} = \partial h_i/\partial p_j$ is called Slutsky substitution matrix:

$$S(p,w) = D_p h(p,u) = \begin{bmatrix} s_{11}(p,w)\ldots s_{1L}(p,w) \\ . \quad . \quad . \\ . \quad . \quad . \\ s_{L1}(p,w)\ldots s_{LL}(p,w) \end{bmatrix}$$

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The Slutsky substitution matrix

- Properties:
  - It is symmetric, i.e. cross-price effects are the same, the effect of increasing p_j on h_i is the same as the effect of increasing p_i on h_j. (The order in which we take derivatives does not make a difference). (In choice approach not necessarily symmetric unless L =2)
  - It is negative semidefinite, since it is the matrix of second derivatives (Hessian) of a concave function (exp.function). Therefore $\partial h_i/\partial p_i \leq 0$, diagonal elements are non-positive. (Also true in Choice approach)

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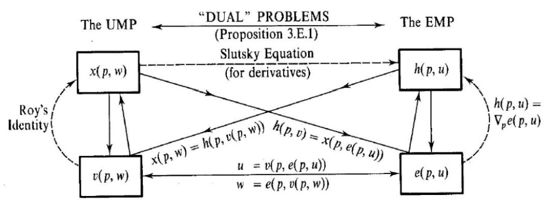
Duality summarized!

Fig.: Relationship between UMP and EMP (Duality)



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Duality summarized in words

- Start with UMP:

$$\max u(x)$$

$$s.t \;\; : \;\; p \cdot x \leq w.$$

- The solution to this problem is $x(p,w)$, the Walrasian demand functions.

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Duality summarized in words

- Substituting $x(p,w)$ into u(x) gives the indirect utility function $v(p,w) \equiv u(x(p,w))$.
- By differentiating $v(p,w)$ w.r.t. p_i and w, we get Roy's identity,

$$x_i\,(p,w) \equiv -v_{p_i} / \, v_w$$

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Duality summarized in words

- Solve the EMP

$$\min p \cdot x$$

$$s.t. \;\; : \;\; u(x) \geq u.$$

- The solution to this problem is $h(p,u)$, the Hicksian demand functions.

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Duality summarized in words

- The expenditure function is defined as

$$e(p,u) \equiv p \cdot h(p,u)$$

- Differentiating the expenditure function w.r.t. p_j gets you back to the Hicksian demand

$$h_j\,(p,u) \equiv \frac{\partial e(p,u)}{\partial p_j}$$

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Utility and expenditure

- Utility maximisation
- ...and expenditure-minimisation by the consumer
- ...are effectively two aspects of the same problem.
- So their solution and response functions are closely connected:

	Primal	***Dual***
Problem:	$\max_{\mathbf{x}} U(\mathbf{x}) + \mu\left[w - \sum_{i=1}^{n} p_i x_i\right]$	$\min_{\mathbf{x}} \sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})]$
Solution function:	$V(\mathbf{p}, w)$	$e(\mathbf{p}, \upsilon)$
Response function:	$x_i^* = D^i(\mathbf{p}, w)$	$x_i^* = h^i(\mathbf{p}, \upsilon)$

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Duality summarized in words

- The connections between the two problems are provided by the duality results. Since the same bundle that solves the UMP when prices are *p* and wealth is *w* solves the EMP when prices are *p* and the target utility level is *u*(*x*(*p*,*w*)) (=*v*(*p*,*w*)), we have that

$$x(p,w) \equiv h(p, v(p,w))$$

$$h(p,u) \equiv x(p, e(p,u))$$

- Applying these to the expenditure and indirect utility functions

$$v(p, e(p,u)) \equiv u$$

$$e(p, v(p,w)) \equiv w$$

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Duality summarized in words

- Finally, we can also prove the Slutsky equation:

$$\frac{\partial h_i(p,u)}{\partial p_k} = \frac{\partial x_i(p,w)}{\partial p_k} + \frac{\partial x_i(p,w)}{\partial w} x_k(p,w) \quad \text{for all } i, k.$$

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Example using duality properties

Consider the following utility function,

$u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$

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Start from the utility function and derive the Marshallian demand for x_1 $x_1 = y/2p_1$	Minimise expenditures s.t. u to find the Hicksian demand function $x_1^h = u\,(p_2/p_1)^{1/2}$
Plug in the respective demand functions to get the indirect utility function $v = y/(4p_1p_2)^{1/2}$	expenditure function $e = u(4p_1p_2)^{1/2}$
Substitute the expenditure function into the Marshallian demand function to derive the Hicksian demand function $x_1 = (u(4p_1p_2)^{1/2})/2p_1 = u(p_2/p_1)^{1/2}$	Substituting the indirect utility function into the Hicksian demand function to derive the Marshallian demand function $x_1^h = (p_2/p_1)^{1/2} y/(4p_1p_2)^{1/2} = y/2p_1$
Invert v and replace y by u to get the expenditure function $v^{-1} = u(4p_1p_2)^{1/2}$	Invert e and replace u by v to get the indirect utility function $e^{-1} = y(4p_1p_2)^{-1/2}$
Check Roy's identity $-\frac{\partial v/\partial p_1}{\partial v/\partial y} = \frac{2y(p_1p_2)^{1/2}}{4(p_1^3p_2)^{1/2}} = y/2p_1$	Check Shephard's lemma $\frac{\partial e}{\partial p_1} = \frac{u4p_2}{2(4p_1p_2)^{1/2}} = u(p_2/p_1)^{1/2}$
Establish the Slutsky equation $\frac{\partial x_1}{\partial p_2} = \frac{u}{2(p_1p_2)^{1/2}} - \frac{y}{2p_2} \cdot \frac{1}{2p_1}$ substitute $u = v(\mathbf{p}, y)$ into the substitution effect $\frac{\partial x_1}{\partial p_2} = \frac{y}{4p_1p_2} - \frac{y}{4p_1p_2} = 0$	

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