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THE PRODUCTION FUNCTION IN ALLOCATION AND GROWTH: A SYNTHESIS

By MARVIN FRANKEL*

Two production functions have over recent decades occupied a prominent place in aggregative economics. One of these is the well-known Cobb-Douglas type of function, $P = aK^\beta L^\alpha$, where P is total output, a is a constant, K and L are the quantities employed of capital and labor, and where the exponents sum to unity. This function has played a central role in efforts to explain the alleged relative stability in the income shares of capital and labor and has inspired an admixture of hope, belief, and skepticism that it summarizes some general and fundamental laws governing production and distribution [4] [6] [11] [12]. The other function is a more elementary one of the form $P = f(K)$ or simply $P = aK$, where a may be interpreted either as the output-capital ratio or, in some contexts, as the reciprocal of the accelerator. This function is commonly found in growth models of the kind made popular by Harrod and Domar.

Each of these functions has virtues which, unfortunately, serve to emphasize the limitations of the other. The Cobb-Douglas type of function makes output dependent on at least two productive factors. It implies diminishing marginal returns to either factor with the other fixed and also diminishing marginal rates of substitution between factors. With both factors variable, it exhibits constant returns to scale, a characteristic which perhaps represents a sound middle ground between the other two alternatives. Not the least of its merits lies in what it asserts about factor rewards. Payment to each factor of its marginal product will exactly exhaust total output.¹ These properties, all funda-

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¹ Of less certain merit is the property of unitary elasticity of substitution whose presence insures that, provided factors are paid at rates corresponding to their marginal products,

mental in economic thinking about resource allocation and income distribution, hold appeal on both theoretical and empirical grounds.

The advantages of the function $P = aK$ lie in a quite different direction. When this function is used in a simple growth model, in which investment (saving) is a constant fraction of output,² results are generated that make rough sense in terms of observed experience. Thus, a rate of net investment of 10 per cent together with an output-capital ratio of one-third will yield in such a model an annual growth rate of about 3 per cent. Economists have found such models attractive because of their relatively simple structure, because of the emphasis they give to capital accumulation as an "engine of growth"—an emphasis with deep roots in economic thought—and because of their pragmatically satisfying results. In consequence they have played a central role during recent years in theories of growth and development.

Unfortunately the production function $P = aK$ has nothing interesting to say about resource allocation or income distribution. Worse than this, as a general statement of the resources required in production, it is positively wrong, as any one-factor production function must be. The limitations of the Cobb-Douglas function, this time in a growth setting, are less transparent but nonetheless weighty. When this function is used in a growth model of the kind just described, with growth in the labor force taken as autonomous, the resulting *long-term* rate of growth in output turns out to equal the rate of growth in the labor force. Growth in output per worker, commonly regarded as the essence of development, is zero, and the rate of investment exerts no effect!³

A second limitation of the Cobb-Douglas function appears when it is fitted to historical data. An improved fit generally is obtainable if the exponents β and α are allowed to vary freely instead of being constrained to equal unity. With some sets of data, the resulting sum of the exponents has differed significantly from unity, an outcome that opens the door to economies and diseconomies of scale and that leads to abandonment of the convenient assumption that factors are paid their marginal products. A reluctance to use the Cobb-Douglas function in its pure form results also from findings like those of M. Abramovitz that for the

factor shares will remain stable. The use of a constant elasticity function, a broader class of linear homogeneous production functions in which the Cobb-Douglas function becomes a special case, opens up interesting possibilities for economic analysis. See [2].

Share stability is not an issue of consequence in the present paper. Although the unitary elasticity property of the Cobb-Douglas function, like others of its properties, affects the particular results obtained at various points, the main line of argument is not dependent on it. For the purpose at hand, moreover, there are advantages in the function's simplicity and familiarity.

² See below pp. 1009-10 for an example of such a model. Substitute $P = aK$ for (5) and neglect (5d) and (5e).

³ See below, p. 1010. Also [14].

United States for the interval from 1869–78 to 1944–53, “The source of the great increase in net product per head was not mainly an increase in labor input per head, not even an increase in capital per head. . . . Its source must be sought principally in the complex of little understood forces which caused productivity . . . to rise” [1].

These considerations have led some economists to try an alternative approach in which the constraint on $\beta + \alpha$ is retained and a time trend added to the production function so that it takes the form $P = ae^{gt}K^\beta L^\alpha$, where g is some constant rate of growth per year [3] [16]. In this form the function may also be used in a growth model to give a long-term growth rate in output per worker of $\frac{g}{\alpha}$.⁴ But this approach leads to the unhappy consequence that growth in productivity takes place independently of growth in the capital stock. Output per worker rises steadily because of the action of forces—variously referred to as technical change, improvements in organization, improvements in the human factor—outside the system and independently of forces within it. This adaptation of the Cobb-Douglas function to a growth setting entails, in a sense, the sacrifice of a satisfactory explanation of growth itself.

There is a great deal of intellectual discomfort in this dilemma. The function $P = aK$, being a one-factor function, is clearly not salvageable as a helpful tool in the allocation-distribution sphere. Moreover, it can be viewed as a special and limiting case of the function $P = aK^\beta L^\alpha$ where $\beta = 1$ and $\alpha = 0$. But one feels intuitively that the Cobb-Douglas function, with its many theoretical virtues, possesses an underlying logic and ought to serve much better than it does in the realm of growth.

In the discussion that follows, a method is advanced for reconciling the two production functions so that the desirable properties of each, but none of the limitations, are retained. It is shown, in effect, that each function is but a special case of a more general function and of a more general way of viewing the economic process in which recognition is given to the relation between the production function for the enterprise and that for the economy. The approach recognizes the indirect as well as the direct effects of changes in resource inputs and ties both sets of effects to those changes. A main conclusion is that the Cobb-Douglas type of function can hold fully in the allocation realm while the Harrod-Domar type of function can simultaneously hold fully for growth. An ancillary conclusion is that there is no need for or necessary virtue in an aggregate production function that possesses some desired set of allocation-distribution properties.

Some implications of the approach for growth models also are explored and the results of preliminary efforts to apply the approach to historical data are presented. As a by-product of the main discussion,

⁴ A derivation is given in the appendix.

the question of the secular stability of the rate of return to capital also is briefly discussed.

I. *Relation Between the Enterprise and the Aggregate Production Functions*

Let us begin by considering an economy made up of a large number of enterprises, each with a production function of the form:

$$(1) \quad P_i = aHK_i^\beta L_i^\alpha$$

where the subscript i denotes the i th enterprise, and P , K , L , and a are defined as before. As before also $(\beta + \alpha) = 1$. The symbol H , to be referred to as the development modifier, is intended to denote the level of development of the economy in which the enterprise operates and is, for the enterprise, a parameter. Its logic is simple. Enterprises in relatively developed or advanced economies are able to produce more with given inputs of capital and labor than enterprises in relatively underdeveloped economies. This is the essence of economic development. If the number of enterprises in the economy is large, as we assume it to be, then no single one of them can, through its own actions, affect the value of H . The choice of a specific measure or index of H will, for the moment, be left in abeyance.

Suppose that the i th enterprise, a "typical" enterprise, produces some fraction, $\frac{1}{n}$ th, of aggregate output. Then aggregate output can be written:

$$(1a) \quad nP_i = naHK_i^\beta L_i^\alpha$$

Given that all enterprises have production functions identical with the typical enterprise's, and since that function is linear and homogeneous, it follows that production of $\frac{1}{n}$ th of aggregate output, P , requires the

employment of $\frac{1}{n}$ th of the totals of each resource, K and L , so that:

$$(1b) \quad K_i = \frac{K}{n}, \quad L_i = \frac{L}{n}$$

and

$$(1c) \quad nP_i = naH \left(\frac{K}{n}\right)^\beta \left(\frac{L}{n}\right)^\alpha$$

which gives for the aggregate production function:

$$(1d) \quad P = aHK^\beta L^\alpha$$

This aggregation procedure implies an economy in which individual enterprises may vary in size (scale) but employ K and L in the same ratio, which is equal to the economy-wide ratio, and produce output in

the same proportion as they employ factors. The assumptions are restrictive but are not, it may be contended, out of keeping with the traditions of economic theory or unreasonable considering the orientation of the discussion.⁵ Similar results can be obtained under much less restrictive assumptions, but not without encumbering the exposition. (See the appendix, Part A.)

It is convenient at this point to introduce a more specific definition of the development modifier, H . A number of expedients might be used to measure it, like birth or death rates, literacy rates, nutritional levels, levels of per capita income, or levels of capital per worker. Without intending to deprecate any of the other alternatives, let us provisionally choose the last of these, which ranks among the more familiar indexes of development. Explicitly, let $H = \left(\frac{K}{L}\right)^\gamma$, where the exponent γ is a

parameter and gives the expression a more general form.⁶

For the economy as a whole H must be treated as a variable rather than a parameter. Like the other variables in the aggregate function, K and L , it reflects the outcome of the actions of all enterprises. When a single enterprise adds, say, to its capital, the level of development is not significantly affected. But when all enterprises do so, the modifier changes. The aggregate production function may now be written

$$(1e) \quad \begin{aligned} P &= a \left(\frac{K}{L}\right)^\gamma K^\beta L^\alpha \\ &= a K^{\beta+\gamma} L^{\alpha-\gamma}. \end{aligned}$$

Only one more step is needed to complete our synthesis. Suppose—and the empirical validity of the supposition is considered below (see Section IV)—that $\gamma = \alpha$. Then the aggregate function reduces to

$$(1f) \quad P = aK.$$

The nature of this synthesis can be summarized as follows. Production in the typical enterprise is governed by a function of the form (1). Under these conditions the properties of the Cobb-Douglas function hold fully for the enterprise. As the enterprise varies its factor inputs, say accumulating capital in response to market and other opportunities,

⁵ It may be noted that any linear homogeneous production function lends itself to the method of aggregation employed here. The result is always an aggregate function that mirrors the enterprise function.

⁶ It will be noted that the sum of the coefficients, $\beta + \alpha + \gamma - \gamma$, remains equal to unity. This constraint could be eliminated by putting the modifier into the yet more general form $\frac{K^\gamma}{L^{\gamma'}}$, and this procedure is followed in Section IV and in the appendix. Meanwhile, use of the simpler version in this and the subsequent two sections does not affect the conclusions of the discussion in any essential way.

the modifier shifts. The shifts are exogeneous for the enterprise in question and reflect the collective impact of the actions of all enterprises as they respond in similar fashion to economic opportunities. The typical enterprise thus remains "in step" with the economy, continuing to employ factors in the same proportion as the (changing) national average. The upshot is that instead of moving along its production function (1), $P_i = aHK_i^\beta L_i^\alpha$, which is an *ex ante* function, the enterprise moves along a realized function which mirrors (1f), namely $P_i = aK_i$.

Figure 1 depicts this situation. The enterprise's movement is resolvable into two components, one to the right along the *ex ante* function and the other an upward displacement to the realized function.

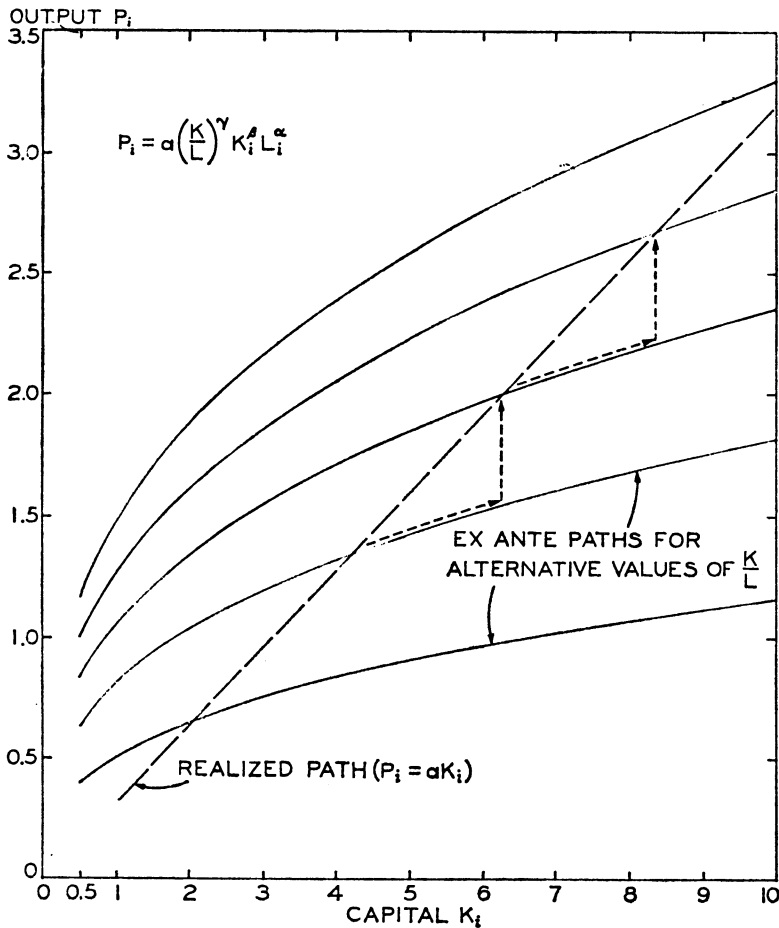


FIGURE 1. *Ex Ante* AND REALIZED PRODUCTION PATHS FOR THE ENTERPRISE WHEN $\gamma = \alpha$ (L_i constant)

It is characteristic of the aggregate function, in which the modifier is a variable, that it internalizes all of the effects on development that are collectively generated by enterprises. Here the term development may be construed broadly to include the several indirect, as distinct from the direct, effects of resource changes. Among these indirect effects are improvements in organization and the quality of labor, technical change, external economies of scale, and better social overhead facilities in the form of transport and communication networks.⁷ As enterprises expand (say) their K_i 's, there is a double impact on the aggregate function: aggregate output rises as a direct result of an increase in one of the factor inputs, and it rises also because the numerator of the modifier has grown. The aggregate function is, in effect, a summation of the realized paths of all enterprises. Provided that $\gamma = \alpha$, the net outcome is as in (1f).⁸

There is no reason to require that an aggregate production function possess properties which accord with our beliefs about the allocation-distribution realm. For it is in the enterprise that factors are employed

⁷ This interpretation is rough. Treatment of H as a parameter for the enterprise implies ignorance by it of all the factors placed in the indirect category. Enterprises will, however, be aware to some extent of these factors and this awareness may play a role in their decisions. This point is treated further in Section III below. An alternative would be to label all factors recognized by the enterprise as direct and all others as indirect. But this treatment would sacrifice any clear distinction among the forces affecting growth and, in mixing the effects from purely quantitative changes in resource inputs with others, would weaken the rationale for using a linear homogeneous production function for the enterprise.

Are the indirect effects neutral or nonneutral in their impact? The answer is that for the *ex ante* enterprise functions they are neutral, for they leave unchanged the relation between the relative marginal productivities of factors and relative factor quantities. But for the aggregate function they are nonneutral.

⁸ The more general point is that any significant departure of γ from zero will result in an *ex ante* relation between inputs and output that is significantly different at the aggregate and the enterprise levels.

The sense in which this approach achieves greater generality differs from that in which the approach of Arrow *et al.* [2] achieve it. The latter drop the assumption of unitary elasticity of substitution, replacing the Cobb-Douglas function by one whose elasticity is constant. This function is then applied at the industry and economy-wide levels in an effort to explain a range of observed data. The present paper proceeds by developing the relation between the micro- and macro-functions. (Although "micro" as here used refers to enterprises, in principle it may denote higher levels of aggregation provided resource allocation decisions at those levels do not by themselves significantly affect the modifier.) The Cobb-Douglas function is retained but is used as a micro-function with one of its parameters, H , dependent on aggregate behavior. The macro-function, which turns out to be different in form from the micro-function, is then developed via the aggregation process. Thus there are two functions, one relevant to the allocation-distribution sphere and the other to the domain of growth.

A synthesis of the enterprise function with the aggregate function $P = aK$ might also be achieved by using, in place of the Cobb-Douglas function, a constant elasticity of substitution function. In general form it may be written $P = (bK^u + cL^u)^{1/u}$ where $u \leq 1$. (See Arrow, *et al.* [2, p. 230], and R. Solow [14, p. 77]. In this case one form of the required modifier, H , would be

$$a \left[b + \left(\frac{K}{L} \right)^{-u} \right]^{-1/u}$$

and rewards distributed, and here a different, though related, function governs.⁹ By the same token, there is no reason to expect the enterprise function, (1), to give us an accurate account of aggregate growth, since that function accords no recognition to the developmental or indirect effects that enterprises collectively generate. A corollary for growth models is that the production functions employed in them need not possess any particular allocation properties. The use, for example, of a one-factor production function like $P = aK$ may be entirely appropriate and in itself carry no necessary implications, long- or short-run, about the scope for factor substitution and, in so far as it may be related to factor substitution, the equilibrium of the system. Similarly, the use of an aggregate function with nonconstant returns to scale need not carry any implication that factors don't get their marginal products.¹⁰

II. *Some Allocation Implications*

Consider the marginal products of capital and labor in the enterprise. These take the following form:

$$(2) \quad \frac{\partial P_i}{\partial K_i} = aH\beta K_i^{\beta-1} L_i^\alpha$$

$$(2a) \quad \frac{\partial P_i}{\partial L_i} = aHK_i^\beta \alpha L_i^{\alpha-1}.$$

Let us call these *ex ante* marginal products to distinguish them from a variant to be defined in a moment. As with the Cobb-Douglas function in its more customary form, (2) and (2a) state that the marginal product of either factor varies inversely with its quantity relative to the quantity of the other factor.

Following through on an assumption made earlier that the typical enterprise employs $\frac{1}{n}$ th of each resource so that $\frac{K}{n} = K_i$ and $\frac{L}{n} = L_i$

⁹ It was long ago suggested in connection with the fitting of a production function to cross-sectional data for the manufacturing sector that payments to factors in accord with the marginal products of this function—the interfirm marginal product—did not necessarily imply payments to factors in accord with the marginal products in individual enterprises—intrafirm marginal products. M. W. Reder [12, pp. 262–63]. The reverse proposition in the present article, though its basis is very different, is in principle similar: Payment to factors by enterprises of a wage equal to their marginal products carries with it no necessary implication that factors receive their marginal products as determined from the (historical) aggregate production function.

¹⁰ This is clear if we alter the modifier so that

$$H = \frac{K^\gamma}{L^{\gamma'}} \quad (\text{where } \gamma \neq \gamma') \quad \text{and} \quad P = aK^{\beta+\gamma}L^{\alpha-\gamma'}.$$

The resulting aggregate function may have nonconstant returns to scale even though derived from *ex ante* enterprise functions possessing constant returns to scale.

$\left(\text{and } \frac{K}{L} = \frac{K_i}{L_i}\right)$, then:

$$(2c) \quad \frac{\partial P_i}{\partial K_i} = a \left(\frac{K}{L}\right)^\gamma \beta \left(\frac{K}{n}\right)^{\beta-1} \left(\frac{L}{n}\right)^\alpha = a\beta K^{\beta+\gamma-1} L^{\alpha-\gamma}$$

$$(2d) \quad \frac{\partial P_i}{\partial L_i} = a \left(\frac{K}{L}\right)^\gamma \left(\frac{K}{n}\right)^\beta \alpha \left(\frac{L}{n}\right)^{\alpha-1} = aK^{\beta+\gamma} \alpha L^{\alpha-\gamma-1}.$$

Call these realized marginal products, or perhaps better realized payment functions, to distinguish them from the previous set. The *ex ante* marginal product functions, like the *ex ante* production function, describe the results expected by the enterprise if, with all else unchanged, it varies the quantity of either factor. They relate to what the enterprise “sees” when it contemplates a change in factor proportions. They also indicate what will result if in fact all else remains unchanged. By contrast, the realized payment functions indicate the outcome when the enterprise makes only such changes in factor proportions as keep it “in step” with the economy as a whole—that is with the national average.

This is implicit in the substitution of $\frac{K_i}{L_i}$ for $\frac{K}{L}$. Stated differently, the realized payment functions describe the outcome for the typical enterprise when *all* enterprises seek to move along their *ex ante* marginal product functions. No one enterprise can, through its own actions, affect the development modifier, H , and it is on this supposition that each enterprise (rightly) makes, say, its investment decisions. But when all enterprises accumulate capital, the modifier will change and the realized and *ex ante* outcomes will differ. In this way enterprise production and marginal product functions shift or are transformed, as the economy accumulates capital and alters the proportions in which it employs factors.

Taking again the case where $\gamma = \alpha$, our realized payment functions reduce to

$$(2e) \quad \frac{\partial P_i}{\partial K_i} = a\beta$$

$$(2f) \quad \frac{\partial P_i}{\partial L_i} = aK\alpha L^{-1} \left(\text{or alternatively, } \frac{\partial P_i}{\partial L_i} = aK_i\alpha L_i^{-1}\right).$$

Here labor's return rises more sharply in response to a relative increase in capital than is true of the *ex ante* function. Should capital increase more rapidly than labor, as historically it has, the resulting rise in the real wage rate would be greater than the *ex ante* function indicates. The case

of capital is perhaps more interesting, for it is seen to be constant and independent of relative factor quantities. If one enterprise alone were to add to its capital, it would encounter diminishing returns to that factor. But when all do so, all are beneficiaries of compensatory shifts in the modifier. Figure 2 describes this situation.

There is in this outcome more than a suggestion of observed experience. During the past three-quarters of a century, perhaps much longer, real wage rates have persistently forged upward in the United States as well as some other Western countries. By contrast profit rates, while undergoing wide oscillations, have not exhibited a marked secular

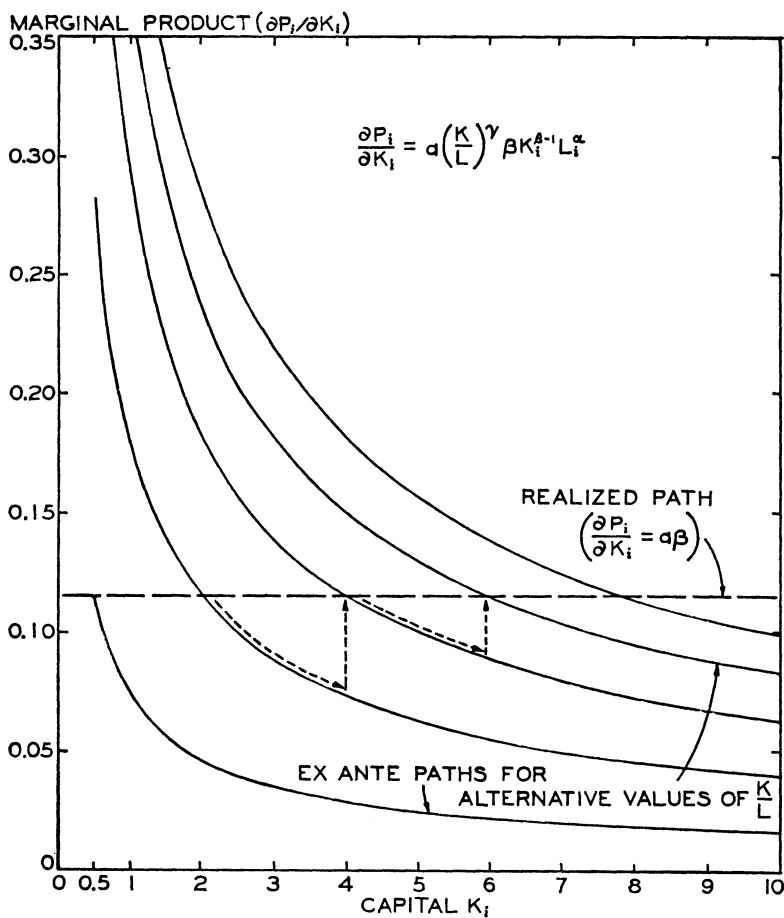


FIGURE 2. *Ex Ante* and REALIZED MARGINAL PRODUCT PATHS FOR THE ENTERPRISE WHEN $\gamma = \alpha$ (L_i constant)

trend.¹¹ It may be worth noting further that the magnitude of the realized marginal product of capital, as implied by (2e) is within reasonable limits. From (1f) a may be interpreted as the output-capital ratio, a plausible value for which might be .33. Data on capital's share in the national product suggest for the United States a figure for β of perhaps .35. Hence $a\beta$ would equal about 11.5 per cent.

For the aggregate function, (1f) $P = aK$, the marginal products are very different:

$$(2g) \quad \frac{\partial P}{\partial K} = a$$

$$(2h) \quad \frac{\partial P}{\partial L} = 0.$$

These marginal products, like the aggregate function itself, have no meaning for enterprises. Perforce each enterprise makes its decisions in terms of its own *ex ante* function. To a central planner, however, the aggregate function and its derivatives would appear as the *ex ante* functions.¹² Giving recognition to the developmental or indirect effects flowing from changes in factor inputs by enterprises, he would observe a much higher marginal product for capital and a much lower one—zero—for labor. He might rightly decide that the national product would benefit far more from policies stressing higher investment than those emphasizing higher population growth or labor-force participation rates. To him the output-capital ratio would be the true measure of capital's contribution, even though the excess of this figure over $a\beta$, the amount paid to capital by enterprises, accrued wholly to labor.¹³ Similar considerations might, other things equal, lead him to an autarkic position that favored domestic over foreign investment. For investment abroad by domestic enterprises would yield them only the direct return on capital, while investment at home would cause both the direct and indirect benefits to accrue to domestic factors. More simply, investment

¹¹ For the United States, see I. Kravis [9, Table 10, Cols. 8 and 9] who gives estimates of the yield on reproducible assets and total wealth. For Great Britain the yield on consols, for Germany the return on bonds, and for France the yield on securities might be taken as indicative. See [10, charts 53-56] [17, p. 46].

¹² They also would appear to the planner as realized functions, descriptive of the actual outcome when inputs are varied.

¹³ It may be well at this point to note an implication of a modifier of the form K/L . It implies that the level of development, as we have called it, is reversible and will decline if growth in the labor force exceeds that in capital. There may be some truth in this for some underdeveloped economies. But it is probably not a valid assumption for industrialized economies. Having reached a relatively advanced level of development it seems unlikely that such economies would fall away from it under the conditions noted. Accordingly, one might wish to constrain K/L by specifying its movement as monotonic.

abroad raises the level of development abroad, whereas investment at home raises it at home.

III. A Qualification and Extension

The approach we have taken depicts the typical firm as being pushed—from behind, as it were—to successively higher production functions. The firm never anticipates any of the forces that are, in effect, summarized by parametric changes in the modifier, and hence it never foresees the path along which it will actually move. This view is an oversimplification. Firms will at times, in conjunction, say, with changes in their resource inputs and the associated adoption of new technologies, see themselves as moving upward to new production functions and anticipate, to one or another extent, their realized paths of movement. Their experience is a blend of the foreseen and the unforeseen. Hence the question arises whether the approach we have taken is consistent with the possession by firms of partial or complete foreknowledge of their realized production paths.

Suppose the typical firm which, as before, keeps in step with the economy, anticipates its realized path fully. Then its *ex ante* and realized paths will be the same, and it will see its production function not as:

$$(1) \quad P_i = aHK_i^\beta L_i^\alpha$$

but as:

$$(3) \quad P_i = aK_i^{\beta+\gamma} L_i^{\alpha-\gamma}$$

where $K = nK_i$ and $L = nL_i$. When as in Section I we let $\gamma = \alpha$, this becomes simply $P_i = aK_i$.

Its *ex ante* marginal product functions will be the derivatives of (3). Thus for capital the function will be:

$$(3a) \quad \frac{\partial P_i}{\partial K_i} = a(\beta + \gamma)K_i^{\beta+\gamma-1} L_i^{\alpha-\gamma}$$

which may be rewritten approximately:

$$(3b) \quad \frac{\partial P_i}{\partial K_i} = aH(\beta + \gamma)K_i^{\beta-1} L_i^\alpha$$

This last expression differs from the corresponding one for firms without foresight, as given in (2), by the inclusion of γ within the parentheses on the right-hand side.

The production function of the firm with foresight, which is both an *ex ante* and a realized function, gives us no basis by itself for inferring the realized payment functions. There is no necessary connection between the firm's foreseeing its realized production path and foreseeing

its realized payment paths. It may, for example, be aware of and give due recognition to technological changes in its investment decisions and it may even take notice of certain concurrent environmental changes, such as the construction of an improved and cheaper transport network that will be available for its use. But its foresight in these respects does not insure foresight with respect to the realized payment functions—that is, with respect to the outcomes in economy-wide or region-wide factor markets; nor need it affect in a consequential way the outcomes in those markets.

An alternative approach to the problem is to suppose that the enterprise, besides anticipating the indirect effects, anticipates also that their contribution to higher profits will be lost through the forces of competition. This approach permits results identical to those obtained when foresight is assumed to be absent.

Accordingly, the case of foresight does not lead logically to or call for realized payment functions different from those applicable in the case without foresight, namely expressions (2c) and (2d). Payment in accord with these expressions is consistent with both cases and, for both, causes aggregate factor payments exactly to exhaust total product.

To summarize, the line of argument developed in Sections I and II and based upon a typical firm without foresight may be extended to cover the alternative of a typical firm with foresight. The approach taken toward the firm and its production function does not require that managers be completely blind to the forces denoted by the modifier. Possession of some foresight by them is consistent with the same set of realized outcomes presented earlier. Movement by the typical firm along the realized production function, whether the firm foresees the movement or not, may be thought of as the outcome of movement along a linear homogeneous production function that is undergoing transformation; and movement by such a firm along the realized payment functions may be thought of as the outcome of a movement along the derivatives of the same linear homogeneous production function, with the derivative functions undergoing an identical transformation.

IV. *Some Implications for Growth*

A. *The Aggregate Production Function*

The implications for growth of an aggregate production function that incorporates a modifier differ in important ways from those of a similar function in which the modifier is omitted or one in which, instead of the modifier, a time trend is employed. This much is true whether one considers the production function in isolation or as an ingredient in a growth model. Consider first a function of the form $P = ae^{\sigma t} K^\beta L^\alpha$, where the term $e^{\sigma t}$ describes a time trend that causes output to grow at the rate

g per year. A function of this form has been fitted by Solow [16] to U. S. data for the years 1900–1949, with the results, $g = .015$, $\beta = .35$, $\alpha = .65$. Differentiating with respect to t , dividing by P and substituting these numerical values gives:

$$(4) \quad \frac{\dot{P}}{P} = .015 + .35 \frac{\dot{K}}{K} + .65 \frac{\dot{L}}{L}.$$

If the quantities of capital and labor employed are constant, output will grow at 1.5 per cent per year. An annual increase of 1 per cent in the stock of capital will raise the rate of growth of output by an additional .35 percentage points, while a 1 per cent increase in the employed labor force will raise it by .65 of a point.

One may be led by this kind of analysis to the conclusion that increases in capital, and hence the rate of investment, do not exercise much influence on the rate of growth in output.¹⁴ The finding that a 1 per cent increase in capital causes output to grow by only about one-third of one per cent stands in sharp contrast to the rather popular view, conditioned and supported by growth models of the Harrod-Domar variety, that output will grow at about the same rate as the growth in capital.

This depressing conclusion about the efficacy, or lack thereof, of investment is sometimes reached by a different but nonetheless related route. Data for the U. S. (private domestic) economy, over the period 1889 to 1957, indicate about a ninefold rise in real output. In the same interval tangible capital rose a little over four times and hours of labor about one and a half times, while the weighed total input of labor and capital combined increased only a little over two times.¹⁵ The figures suggest that increases in capital and labor account for only a fraction of the increase in output and that the contribution of capital alone, when allowance is made for the fact that it represents but a third by weight of value added, has been almost negligible. This approach is, in an essential way, analogous to that which makes output dependent on a Cobb-Douglas function adjusted by a time trend or which fits such a function to historical data. Any increase in output not matched by an equivalent increase in the weighed inputs combined tends to be ascribed to the influence of outside forces unrelated to the resource inputs. The assumptions, of course, constrain the conclusions.

¹⁴ "It seems that the rate of growth which can be attained in a modern industrial economy is not strongly influenced by the investment policy which is applied. Whether investments are high or low, within reasonable limits, national product will increase by 2–3 percent a year, if the volume of employment remains constant. This is mainly because the human factor alone (*i.e.*, *time trend*) is sufficient to ensure a growth of 1.5 percent a year" [3, p. 49]. Aukrust's statistical application was to Norwegian data. His statement assumes an average (and constant) capital-output ratio of 3.5 and a rate of net investment ranging from about 5 to 20 per cent.

¹⁵ Solomon Fabricant [5, Table A]. The data are John W. Kendrick's.

By contrast, consider the aggregate function

$$(1e) \quad P = a \frac{K^\gamma}{L^{\gamma'}} K^\beta L^\alpha$$

where the modifier is given a more general form than in earlier discussion. This formulation does not in itself explain the indirect effects any more than does a time trend. But it differs in two related respects from the latter approach. First it ties the indirect to the direct effects. The indirect effects are implemented through resource changes rather than materializing independently of them. Second, it carries quite different implications about the effects on growth of an increase in a resource like capital.

Differentiating (1e) with respect to time and dividing by P gives:

$$(4a) \quad \frac{P}{P} = \gamma \frac{\dot{K}}{K} + \beta \frac{\dot{K}}{K} + \alpha \frac{\dot{L}}{L} - \gamma' \frac{\dot{L}}{L}.$$

A change in a resource input here has two effects, one direct and the other indirect. Depending on the sign and magnitude of γ and γ' , the indirect effects may either amplify or diminish the direct effects. In the special case, for example, in which $\gamma = \alpha$, a one per cent increase in capital will result in a one per cent increase in output. In this case, and it is a meaningful one, for it conforms rather well to U. S. data for the period 1889 to 1933 (see below, p.1012), investment remains a strategic element in the growth picture.

B. Growth Models

The use in a Harrod-Domar type of growth model of a production function with a modifier opens up a number of interesting possibilities some of which have empirical relevance. Let us limit our attention to the following model which differs from one of general familiarity only in the production function that is used:

$$(5) \quad P = aHK^\beta L^\alpha$$

$$(5a) \quad P = C + I$$

$$(5b) \quad I = \frac{dK}{dt} = K$$

$$(5c) \quad C = (1 - s)P$$

$$(5d) \quad L = L_0 e^{\lambda t}$$

$$(5e) \quad H = \frac{K^\gamma}{L^{\gamma'}}.$$

The variables P , K and L have the same meanings as before. The variable I should be understood as denoting net investment, since assets are assumed to be perpetual. The variable C denotes consumption in real terms, the parameter s the net savings ratio, the parameter L_0 the labor force in the initial year, and the parameter λ some constant growth rate of the labor force. The modifier, $H = \frac{K^\gamma}{L^{\gamma'}}$, can be multiplied through in (5) to give for that equation:

$$P = aK^\theta L^\phi, \quad \text{where } \theta = \beta + \gamma \quad \text{and} \quad \phi = \alpha - \gamma'.$$

The system of equations can be solved to get the time paths of the variables and their rates of growth. (Part B of the appendix sets forth the procedure.) For the growth rate of output we have

$$(5f) \quad \frac{P_t}{P_t} = \frac{\theta\lambda\phi as L_0^\phi e^{\lambda\phi t}}{\lambda\phi K_0^{1-\theta} - (1-\theta)asL_0^\phi + (1-\theta)asL_0^\phi e^{\lambda\phi t}} + \lambda\phi$$

where $(\theta+\phi) \geq 1$ and $\theta \neq 1$. For $\theta=1$ the result is

$$(5g) \quad \frac{P_t}{P_t} = asL_0^\phi e^{\lambda\phi t} + \lambda\phi.$$

In the special case in which the modifier raises the exponent of K to unity and reduces that of L to zero, so that $\theta=1$, $\phi=0$ and $P=aK$, we get for (5g):

$$(5h) \quad \frac{P_t}{P_t} = as.$$

The growth rate is constant and proportionate, being the product of the output-capital ratio and the savings ratio.

Two other cases, more general than the preceding special case, are of interest. Though ordinarily disregarded, they become relevant, as does the special case, when the influence of the modifier is taken into account. Consider first the case where $(\theta+\phi)=1$ and where θ is relatively large so that ϕ must be relatively small. Then in the very long run the first two terms in the denominator of (5f) become negligibly small and the expression reduces to

$$(5i) \quad \frac{P_t}{P_t} = \frac{\theta\lambda\phi}{1-\theta} + \lambda\phi = \lambda.$$

The long-run growth rate of output is equal to the growth rate of the labor force, which is a familiar result. Now, however, with θ relatively

large, it is meaningful to ask, How long is the long run? Clearly if θ and ϕ are in the neighborhoods of unity and zero respectively, the outcome ought, at least for a time, to approximate that of the special case (5h), and the savings-investment ratio should be influential during this period. Figure 3, based on equation (5f), illustrates some results for alternative values of θ and ϕ . Growth in the labor force, λ , is taken to be .015 per year. Alternate values of .20 and .10 are assumed for the rate

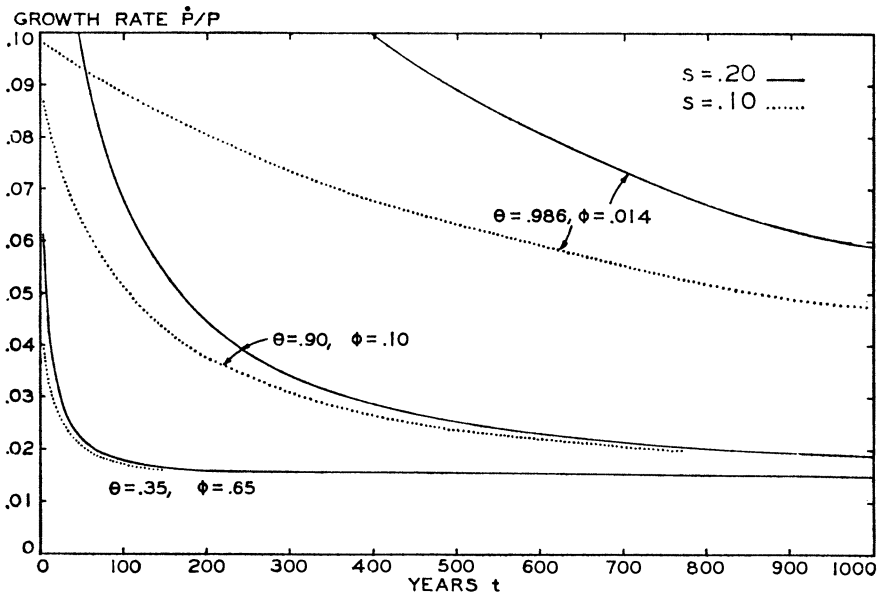


FIGURE 3. TIME PATHS OF THE GROWTH RATE OF OUTPUT,
GIVEN ALTERNATIVE VALUES FOR θ , ϕ AND s
(a , K_0 and L_0 equal 1)

of saving (equal to the rate of investment), s , while the value of unity is arbitrarily assigned to a , K_0 and L_0 .

In all cases the long-run growth rate in output is the same, .015. However, the movement toward this position occurs at disparate rates. When θ and ϕ take respective values of .35 and .65—which might be thought of as the case in which there is no modifier—the growth rate falls very rapidly toward the .015 level, for all practical purposes reaching it within 100 years. Variation in the rate of investment is significant in the early stages, but this influence rapidly diminishes, all but disappearing within 40–50 years.

As θ rises and ϕ falls, the .015 level is approached more slowly and the rate of investment remains influential for a longer time. With $\theta = .95$, $\phi = .05$ (not shown) the growth rate is still above the .015 level even after

1,000 years, and variation in the rate of investment retains significance for growth for perhaps 200 to 300 years. The results are yet more dramatic for higher values of θ .

The values of .986 for θ and .014 for ϕ are included because (together with a value of a of .321) they represent one set of results for regression analysis as applied to U. S. data for the period 1889–1933.¹⁶ A more extended period running into the fifties might have been selected, but the data do not appear homogeneous for so long a span.¹⁷ The correlation coefficient was .977. If we assign to β and α respective values of .35 and .65 to accord approximately with the distributive shares of capital and labor, then in the context of the approach developed in this paper the regression results imply for the aggregate production function:¹⁸

$$(5j) \quad P_t = .321 \left(\frac{K_t}{L_t} \right)^{.636} K_t^{.35} L_t^{.65} .$$

This case is explored further in Figure 4, but this time, instead of unity, the observed values are used for K_0 and L_0 and the regression-determined value is used for a . A rate of saving of 4.7 per cent is just sufficient to sustain a steady rate of growth at the long-term level of 1.5 per cent. Higher rates of saving generate initial growth rates that

¹⁶ The data are taken from Kendrick [8] Appendix A, Tables III, X, XV relating respectively to gross domestic product, man-hours (civilian) and capital stock (structures, equipment and inventories). The data were first smoothed by means of a moving average. A justification for this or some similar procedure is that the adjustment of these variables to one another is slow, that the relations among them make themselves felt only over the intermediate and longer term, and that the influence of short-term random and cyclical movements that tend to obscure the relations sought should be minimized. A second regression using unadjusted data did not, however, produce markedly different results.

¹⁷ The line of relation between P and K/L moves in a comparatively smooth and steady way until the early thirties and then evidences a definite break. Subsequent movement is along a different path that rises more sharply than in earlier years. Lack of homogeneity as between the pre- and post-'33 periods is reflected also in the behavior of the ratio K/L . It rises continuously through 1933 but then reverses direction and continues downward through 1943. Thereafter it rises steadily, though it does not regain its 1933 level until 1954.

¹⁸ Taking $\gamma = \gamma'$, the function may first be written

$$P_t = a \left(\frac{K_t}{L_t} \right)^\gamma K_t^\beta L_t^\alpha = a K_t^{\beta+\gamma} L_t^{\alpha-\gamma} .$$

Dividing through by L_t gives

$$\frac{P_t}{L_t} = a \left(\frac{K_t}{L_t} \right)^\theta \quad \text{where } \theta = \beta + \gamma .$$

This function was fitted to the series on output, capital and employment referred to in footnote 16. One would expect the fitting of a production function to cross-sectional data to yield different results from those given here, since in that case the level of development and hence the modifier would be held constant. In so far as an aggregate production function has meaning, one might argue, contrary to the customary belief, that it is the historical rather than cross-sectional function that is valid. For unlike the enterprise, the economy cannot alter its resource inputs without generating indirect effects and hence changing the modifier.

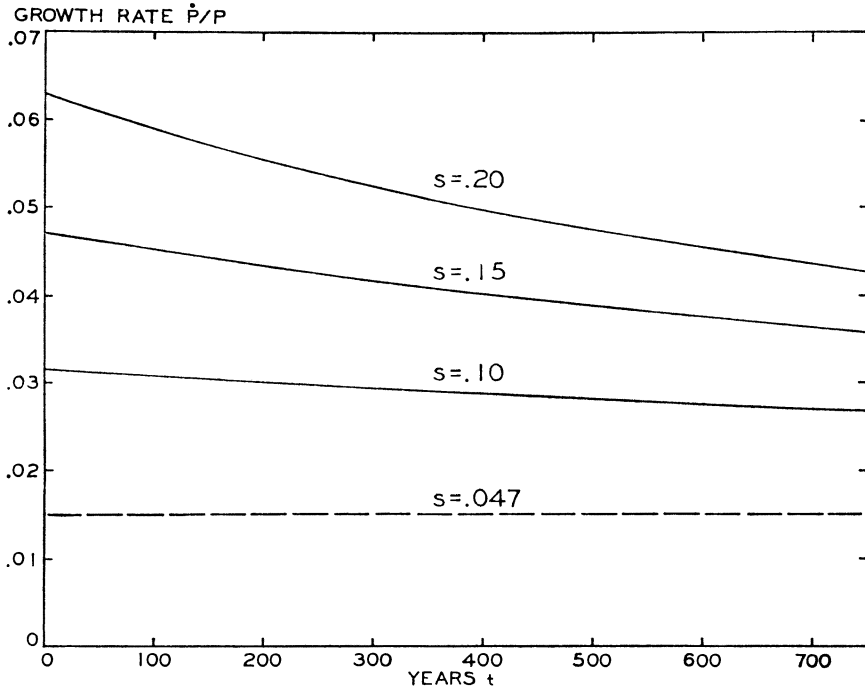


FIGURE 4. TIME PATHS OF THE GROWTH RATE OF OUTPUT, GIVEN THE PRODUCTION FUNCTION $P = .322K^{.986}L^{.014}$ AND ALTERNATIVE VALUES OF THE SAVINGS RATIO (s)

are higher and that decline with almost imperceptible slowness toward the long-term rate. Lower rates (not shown) result in initial growth rates below the long-term rate that rise with equal slowness toward it. The growth rate is, for all practical purposes, as sensitive to the rate of investment as in the case where $P = aK$.

Regression analysis covering the later years 1944–54¹⁹ resulted in a somewhat lower value for θ of .89 and a higher value for ϕ of .11. (The value for the a coefficient was .91). The period covered is unfortunately quite short. An idea of the growth implications of this outcome can be gleaned from the approximating case in Figure 3 for $\theta = .90$, $\phi = .10$.

A second case of interest is that where $\theta + \phi$ is not constrained. The long-run growth rate of output is:

$$(5k) \quad \frac{\dot{P}}{P} = \frac{\theta\lambda\phi}{1 - \theta} + \lambda\phi$$

¹⁹ Selection of any earlier initial year for the post-1933 era would have involved the inclusion of years in which the ratio K/L was moving downward. Such years were excluded on grounds that the influence of the modifier is not reversible, at least not in any simple fashion. Once any given level of development has been achieved, it seems unlikely that it would be reduced by a slowing in the growth of the capital stock below that in the labor force.

which now will be greater than λ except when $\theta > 1$.²⁰ Regression analysis for the 1889–1933 period, applied this time without imposing any constraints on $(\theta + \phi)$, gave $\theta = .76$, $\phi = .55$, $a = .09$, with a correlation coefficient of .998. Assigning values to β and α that are approximately equal to the shares of capital and labor in the national product, our implied production function is:

$$(5l) \quad P_t = .09 \frac{K_t^{.41}}{L_t^{.10}} K_t^{.35} L_t^{.65}.$$

For the years 1944–54 regression analysis gave for the parameters:

$$\theta = .92, \phi = .29, a = .19.$$

Limiting attention to the 1889–1933 interval and using the regression-determined values for θ and ϕ in the production function of our growth model, we obtain from (5k) a long-run growth rate of 3.3 per cent, which is more than double the growth rate of the labor force. Figure 5 illustrates the case, once more using observed values for K_0 and L_0 and the regression-determined values for the other parameters. A rate of saving of about 11 per cent yields a growth rate from the outset just equal to the long-term rate. At higher rates of saving the growth rate approaches the long-term rate from above, while at lower rates the long-term rate is approached from below. The rate of saving retains significance for a much shorter period than in Figure 4. Still, it remains as a force to be reckoned with for perhaps 150 to 200 years, a period which is long relative to the span of about 65 years covered by the data.

Given the kind of setting portrayed in Figures 4 and 5, with all of their underlying assumptions, one's judgment on the current importance of the rate of investment for growth depends on where on the time scale one places the present U. S. economy. There is no necessary implication in this kind of analysis that, whatever the economy's present position, it must eventually be dominated by the long-term outcome. Structural changes of one or another sort, discontinuities in the flow of technological change or some other circumstance may from time to time shift the values of one or more parameters, thereby restoring the *status quo ante* and with it the influence of investment. In statistical terms this is to say that the subperiods comprising any extended period may not all be homogeneous.²¹

²⁰ When $\theta > 1$ the first term on the right becomes negative. The long-run rate of growth will then be lower than λ and may be negative.

²¹ It is not claimed of the regression findings offered here that they are either the right ones or the best ones. The data can and have been approached in different ways by different observers, with a variety of results. Moreover, the hazards of correlating time series are well known and the results inherently suspect. Interest in the present findings derives less from any worth they may have as statistics than from their value as a supplement to the main ideas offered in this paper.

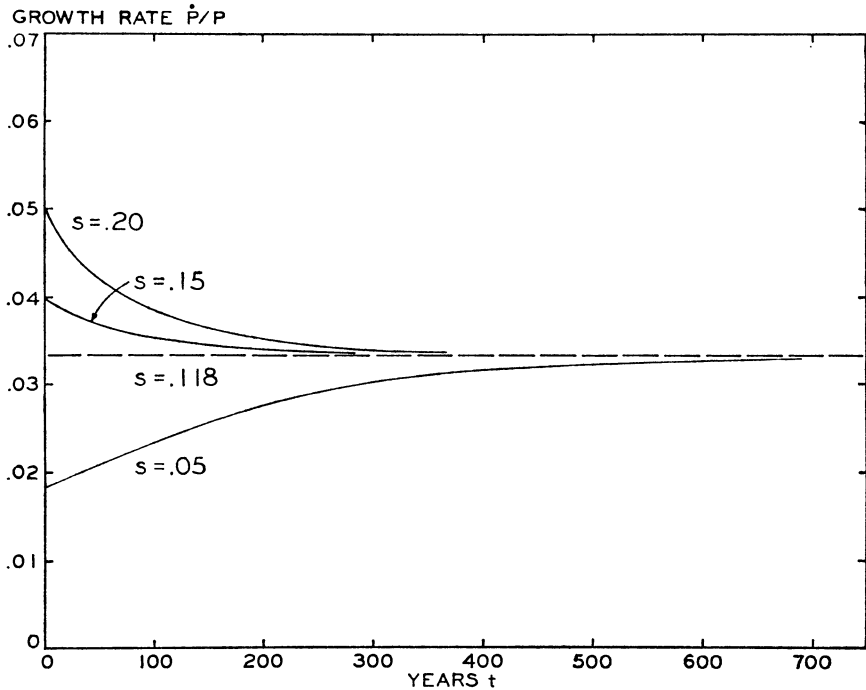


FIGURE 5. TIME PATHS OF THE GROWTH RATE OF OUTPUT, GIVEN THE PRODUCTION FUNCTION $P = .089K^{.78}L^{.66}$ AND ALTERNATIVE VALUES OF THE SAVINGS RATIO (s)

In discussions of economic growth the question occasionally arises whether, as an economy matures, there is any tendency for the growth rate to decline. In so far as a long-run terminal rate has any practical meaning, Figure 5 suggests that the approach to it may be either from above or from below, depending on the economy's point of departure. For some economies, therefore, the passage of time may bring a rising growth rate while for others it may bring a decline. In yet other cases it may vary little or not at all.

V. *Some Qualifications and Caveats*

In the writer's view production functions, when used with reference to the typical enterprise or the whole economy, are at best useful fictions. They may suggest underlying reality but can hardly be expected to replicate it. The particular production function and modifier used in the preceding pages as the basis for formal analysis must be understood in this light. Their choice, moreover, is provisional and not necessarily optimal and is not intended to exclude the consideration of other possibilities.

Functions other than the Cobb-Douglas function may be better

suiting to the needs both of theory and practice. Two properties of the Cobb-Douglas function are perhaps unduly limiting. One of these is an elasticity of substitution equal to unity. This property has been central to explanations, via an aggregate production function, of an apparent stability in the United States and other countries of labor's share in the national product. The results of recent studies of the data differ as to whether stability in fact has prevailed (cf. [9 *passim*] [13, pp. 196-99]). In any event, it is a contention of this paper that answers to questions concerning income distribution, if they are to be sought in production functions, are better sought at lower levels of aggregation. When one turns to the enterprise and considers the wide diversity among enterprises, both cross-sectionally and over time, in their ability to substitute productive factors for one another, the unitary elasticity assumption seems rather restrictive. One would prefer the greater flexibility of a function whose elasticity might vary from zero upwards, even though this required looking elsewhere for any needed explanation of share stability.²²

A second restrictive property of the Cobb-Douglas function, less consequential than the preceding one, is the requirement that both capital and labor be used in production. Here too greater flexibility would permit accommodation to a wider range of experience. The extremes of the fully automated plant and the use in production of labor alone ought not to be ruled out.

A function of the form $P = (bK^u + cL^u)^{1/u}$, (where $u \leq 1$), is one possible answer [14, p. 77] [2]. This function preserves the features of linearity and homogeneity which, like the other restrictive features of the Cobb-Douglas function, also represent an oversimplification. But on balance they are probably desirable features, for they help to fulfill some theoretical needs and to minimize mathematical complexity. A second possible answer may lie in the use not of a single function but of a set of functions each of which is intended to describe a class of enterprises sharing common characteristics. The mathematics of such an approach are likely to be difficult. But if analytical techniques did not suffice, something might be done with Monte Carlo methods.

Improvement in the modifier, as in the production function proper, also is a distinct possibility. A modifier of the form $\frac{K}{L}$ has the virtue of being simple and of lending itself to an easy economic interpretation. These are useful qualities but they do not alone ensure analytical and empirical adequacy. In principle there is no reason why a term that de-

²² In Appendix A, however, less restrictive assumptions are used in aggregation than are used in Section I. As a result, returns to scale can vary among enterprises and individual elasticities of substitution need not equal unity.

scribes how the production function is transformed need be simple or susceptible of ready economic interpretation. The modifier purports to summarize the indirect effects of quantitative changes in resource inputs. Such effects are varied and include technological change, changes in organization, changes in the contribution of social overhead capital, changes in the quality of labor inputs, and external economies of scale. It is an obvious limitation on the modifier used in this paper that it does not distinguish among these effects. It is also an obvious and related limitation that some of these indirect effects may in fact make themselves felt at times without being accompanied by changes in factor inputs,²³ though adjustments in the concepts of an measuring rods for capital and labor might help to remedy this neglect.

Other types of modifiers might well be developed that would be stronger in these respects and that would meet more efficiently the tests of theory and data. It is worth emphasizing that with modifiers, perhaps more so than with production functions, no single form may suffice to fit all the facts. What works well for one economy over a given period of time may not serve well for a different period or a different economy. A deeper understanding of differences, temporal and regional, in economic development may perhaps come from a fuller knowledge of the forms which modifiers may reasonably be expected to take.

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²³ A way of overcoming this problem is by an approach that introduces the indirect effects through a replacement function. Technological change has been handled in this way by Leif Johansen [7] and by R. M. Solow [15]. In the present case, our model in Section V might be expanded to include a replacement function, and the modifier might then be made to depend in some fashion on both the level of the capital stock and its age structure as determined by the rate of replacement.

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APPENDIX

A. *The Relation Between Enterprise and Aggregate Production Functions*

For illustration, let the economy consist of three enterprises to be designated by the subscripts j , k , and l . Neglect the modifier for the present. Let the production function of the j th enterprise be

$$(A1) \quad P_j = a_j K_j^{\beta_j} L_j^{\alpha_j}$$

where $P_j = jP$, $K_j = j'K$ and $L_j = j''L$, P , K , and L are the corresponding macro-variables, $j \geq j' \geq j''$ and $0 < (\beta_j + \alpha_j) \leq 1$. Represent the production functions of the k th and l th enterprises in parallel fashion, with

$$j + k + l = j' + k' + l' = j'' + k'' + l'' = 1.$$

Thus a given enterprise may employ different fractions of the totals of each resource and produce a yet different fraction of total output, and it may have increasing, constant or decreasing returns to scale. Substituting the macro- for the micro-variables in each function and expressing each in logarithmic form and summing over all enterprises gives:

$$\begin{aligned} 3 \log P + \log j + \log k + \log l &= \log a_j + \log a_k + \log a_l + \beta_j \log j' \\ &\quad + \beta_k \log k' + \beta_l \log l' + \alpha_j \log j'' \\ &\quad + \alpha_k \log K'' + \alpha_l \log L'' \\ &\quad + (\beta_j + \beta_k + \beta_l) \log K \\ &\quad + (\alpha_j + \alpha_k + \alpha_l) \log L. \end{aligned}$$

Transposing, taking the antilog and rearranging terms gives

$$(A2) P = \left(\frac{a_j a_k a_l (j')^{\beta_j} (k')^{\beta_k} (l')^{\beta_l} (j'')^{\alpha_j} (k'')^{\alpha_k} (l'')^{\alpha_l}}{jkl} \right)^{1/3} K^{(\beta_j + \beta_k + \beta_l)/3} L^{(\alpha_j + \alpha_k + \alpha_l)/3}.$$

Let the first term on the right, which is some constant term, equal a . Let the arithmetic averages of the exponents for capital and labor equal, respectively, β and α . Assume that $\beta + \alpha = 1$, which is less restrictive than assuming the equivalent for each enterprise. Then the resulting linear homogeneous production function is

$$(A3) \quad P = aK^\beta L^\alpha.$$

This procedure can be extended from an economy of three enterprises to one that covers any number of enterprises.

It is clear that the fractions of resources used and output produced and the extent of economies or diseconomies of scale in each enterprise need not remain constant over time in order that the aggregate function remain stable. All that is required is that the a term and the average of each exponent remain stable, and this stability can prevail if there are compensating changes of various kinds within and among enterprises.

The inclusion of the modifier H in each enterprise function would result in its inclusion also in the aggregate function so that:

$$(A4) \quad P = aHK^\beta L^\alpha.$$

A less restrictive treatment also is possible by providing that enterprises may be affected by the modifier—or experience the indirect effects of resource changes—in differing degrees. Defining H_j as the modifier applicable to the j th enterprise, we may write

$$H_j = j''' \frac{K^{\gamma_j}}{L^{\gamma_j'}}$$

where j''' , γ_j and γ_j' are the relevant constants for this enterprise. Other enterprises may be treated similarly. The effect on the aggregate function (A4) would be to alter a by the multiplicative inclusion in the numerator of the bracketed term of (A2) of $(j''' k''' l''')$. It would result also in defining γ and γ' (in $H = K^\gamma/L^{\gamma'}$) as the respective arithmetic averages of the corresponding parameters for the enterprises.¹

In short, it is possible to develop an aggregate production function of the kind in (A4) by assuming only the form taken by each enterprise's production function. Returns to scale need not be constant for the enterprise, nor need the indirect effects be the same among enterprises.

Let us now define a typical enterprise as that hypothetical i th enterprise

¹ A variant of this approach would be to develop the modifier term directly from each enterprise's micro-variables instead of assuming it initially. This would involve specifying, first, the quantitative relation between an enterprise's micro-variables and the corresponding macro-variables, and second, the relation between the macro-variables and the indirect effects for the enterprise.

which employs $1/n$ th of each resource and produces $1/n$ th of total output, and which has a production function of the same form and parameters identical with those in (A4), so that

$$(A5) \quad P_i = aHK_i^\beta L_i^\alpha.$$

As was shown in Section I, aggregating over an economy of such enterprises yields the aggregate function (A4). Hence the same aggregate function that prevails in an economy of typical enterprises is consistent also with an economy whose enterprises are governed by much less restrictive assumptions.

B. Further Notes

1. Solution for (5f), the growth rate of output or \dot{P}_t/P_t , given the following system of equations:

$$(5) \quad P = aHK^\beta L^\alpha, \quad \text{where } \beta + \alpha = 1$$

$$(5a) \quad P = C + I$$

$$(5b) \quad I = \frac{dK}{dt} = \dot{K}$$

$$(5c) \quad C = (1 - s)P$$

$$(5d) \quad L = L_0 e^{\lambda t}$$

$$(5e) \quad H = \frac{K^\gamma}{L^{\gamma'}}.$$

Letting $\beta + \gamma = \theta$ and $\alpha - \gamma' = \phi$, (5) may be written:

$$(6) \quad P = aK^\theta L^\phi.$$

Inserting (5b) and (5c) in (5a) and the result in (6) we arrive at

$$(7) \quad \frac{dK}{dt} = saK^\theta L^\phi.$$

Inserting (5d) in (7) gives:

$$(8) \quad \frac{dK}{dt} = saK^\theta (L_0 e^{\lambda t})^\phi, \quad \text{and}$$

$$(9) \quad \frac{dK}{K^\theta} = (saL_0^\phi e^{\lambda \phi t}) dt.$$

Now integrate (9):

$$\int \frac{dK}{K^\theta} = saL_0^\phi \int e^{\lambda \phi t} dt$$

which for $\theta \neq 1$ results in:

$$\frac{K^{1-\theta}}{1-\theta} = \frac{saL_0^\phi e^{\lambda\phi t}}{\lambda\phi} + k_1$$

where k_1 is an arbitrary constant of integration. Hence

$$(10) \quad K = \left[\frac{(1-\theta)saL_0^\phi e^{\lambda\phi t}}{\lambda\phi} + k_2 \right]^{1/(1-\theta)}$$

where k_2 is another arbitrary constant. When $t=0$, $K=K_0$ so

$$k_2 = K_0^{1-\theta} - \frac{(1-\theta)saL_0^\phi}{\lambda\phi}.$$

Therefore

$$(11) \quad K_t = \left[K_0^{1-\theta} - \frac{(1-\theta)saL_0^\phi}{\lambda\phi} + \frac{(1-\theta)saL_0^\phi e^{\lambda\phi t}}{\lambda\phi} \right]^{1/(1-\theta)}$$

where K_t denotes the capital stock in year t . Substituting (11) and (5d) in (6) gives:

$$(12) \quad P_t = a \left[K_0^{1-\theta} - \frac{(1-\theta)saL_0^\phi}{\lambda\phi} + \frac{(1-\theta)saL_0^\phi e^{\lambda\phi t}}{\lambda\phi} \right]^{\theta/(1-\theta)} [L_0 e^{\lambda t}]^\phi.$$

Placing (12) in logarithmic form and differentiating with respect to t gives:

$$(13) \text{ and } (5f) \quad \frac{P_t}{P_t} = \frac{\theta\lambda\phi saL_0^\phi e^{\lambda\phi t}}{\lambda\phi K_0^{1-\theta} - (1-\theta)saL_0^\phi + (1-\theta)saL_0^\phi e^{\lambda\phi t}} + \lambda\phi$$

which in the very long run reduces to:

$$(14) \quad \frac{P_t}{P_t} = \frac{\theta\lambda\phi}{1-\theta} + \lambda\phi.$$

In the special case where $\theta=1$, return to (9) and divide through by dt to get:

$$(9a) \quad \frac{\dot{K}_t}{K_t} = saL_0^\phi e^{\lambda\phi t}.$$

Differentiate (5d) with respect to t and divide by L_t to get

$$(9b) \quad \frac{L_t}{L_t} = \lambda.$$

Differentiate (6) with respect to t and divide by P_t to get:

$$(9c) \quad \frac{P_t}{P_t} = \theta \frac{\dot{K}_t}{K_t} + \phi \frac{L_t}{L_t}.$$

Substitute (9a) and (9b) in (9c), so that:

$$(15) \text{ and } (5g) \quad \frac{P_t}{P_t} = saL_0^\phi e^{\lambda\phi t} + \lambda\phi.$$

2. A slight modification in the general solution will give the growth rate in output when a time trend is added to the production function. Rewrite (6) in the form:

$$(6a) \quad P = ae^{\sigma t} K^\theta L^\phi$$

where g is some growth rate per year because of the time trend. Inserting (5b) and (5c) in (5a) and the result in (6a) gives instead of (7)

$$(7a) \quad \frac{dK}{dt} = sae^{\sigma t} K^\theta L^\phi.$$

Inserting (5d) in (7a) then gives instead of (9):

$$(9d) \quad \frac{dK}{K^\theta} = (saL_0^\phi e^{(\sigma+\lambda\phi)t}) dt.$$

Carrying through this change in succeeding steps changes the long-run growth rate (14) to:

$$(14a) \quad \frac{P_t}{P_t} = \frac{\theta(g + \lambda\phi)}{1 - \theta} + \lambda\phi + g.$$

If the modifier is neglected, i.e. equals unity (so that $\theta = \beta$ and $\phi = \alpha$), then (14a) reduces to:

$$(14b) \quad \frac{P_t}{P_t} = \frac{g}{\alpha} + \lambda$$

and output per worker grows at the rate:

$$(14c) \quad \frac{\left(\frac{P}{L}\right)_t}{\left(\frac{P}{L}\right)_t} = \frac{g}{\alpha}.$$