MICROECONOMICS

Principles and Analysis

THE FIRM: OPTIMISATION

OVERVIEW…

Approaches to the firm's optimisation problem

- Firm: Optimisation
  - The setting
  - Stage 1: Cost Minimisation
  - Stage 2: Profit maximisation

THE OPTIMISATION PROBLEM

- We want to set up and solve a standard optimisation problem.
- Let's make a quick list of its components.
- ... and look ahead to the way we will do it for the firm.

THE OPTIMISATION PROBLEM

- Objectives *-Profit maximisation?*
- Constraints ***-Technology; other***
- Method ***- 2-stage optimisation***

CONSTRUCT THE OBJECTIVE FUNCTION

- Use the information on prices…

w_i •price of input *i*

p •price of output

- …and on quantities…

z_i •amount of input *i*

q •amount of output

- …to build the objective function

How it's done

THE FIRM'S OBJECTIVE FUNCTION

- Cost of inputs: $\sum_{i=1}^{m} w_i z_i$

•Summed over all *m* inputs

- Revenue: pq

•Subtract Cost from Revenue to get

- Profits: $pq - \sum_{i=1}^{m} w_i z_i$

OPTIMISATION: THE STANDARD APPROACH

- Choose q and $\mathbf{z}$ to maximise

$$\Pi := pq - \sum_{i=1}^{m} w_i z_i$$

- ...subject to the production constraint...

$$q \le \phi(\mathbf{z})$$

- ..and some obvious constraints:

$$q \ge 0 \qquad \mathbf{z} \ge \mathbf{0}$$

- Could also write this as $\mathbf{z} \in Z(q)$

- You can't have negative output or negative inputs

A STANDARD OPTIMISATION METHOD

- If ϕ is differentiable…
- Set up a Lagrangean to take care of the constraints

$$L(\dots)$$

- Write down the First Order Conditions (FOC) — necessity

$$\frac{\partial}{\partial \mathbf{z}} L(\dots) = 0$$

- Check out second-order conditions — sufficiency

$$\frac{\partial^2}{\partial \mathbf{z}^2} L(\dots)$$

- Use FOC to characterise solution

$$\mathbf{z}^* = \dots$$

USES OF FOC

- First order conditions are crucial
- They are used over and over again in optimisation problems.
- For example:
  - Characterising efficiency.
  - Analysing “Black box” problems.
  - Describing the firm's reactions to its environment.
- More of that in the next presentation
- Right now a word of caution...

A WORD OF WARNING

- We've just argued that using FOC is useful.
  - But sometimes it will yield ambiguous results.
  - Sometimes it is undefined.
  - Depends on the shape of the production function ϕ.
- You have to check whether it's appropriate to apply the Lagrangean method
- You may need to use other ways of finding an optimum.
- Examples coming up…

A WAY FORWARD

- We could just go ahead and solve the maximisation problem
- But it makes sense to break it down into two stages
  - The analysis is a bit easier
  - You see how to apply optimisation techniques
  - It gives some important concepts that we can re-use later
- The first stage is ***"minimise cost for a given output level"***
  - If you have fixed the output level q…
  - …then profit max is equivalent to cost min.
- The second stage is ***"find the output level to maximise profits"***
  - Follows the first stage naturally
  - Uses the results from the first stage.
- We deal with stage each in turn

OVERVIEW…

A fundamental multivariable problem with a brilliant solution

- Firm: Optimisation
  - The setting
  - **Stage 1: Cost Minimisation**
  - Stage 2: Profit maximisation

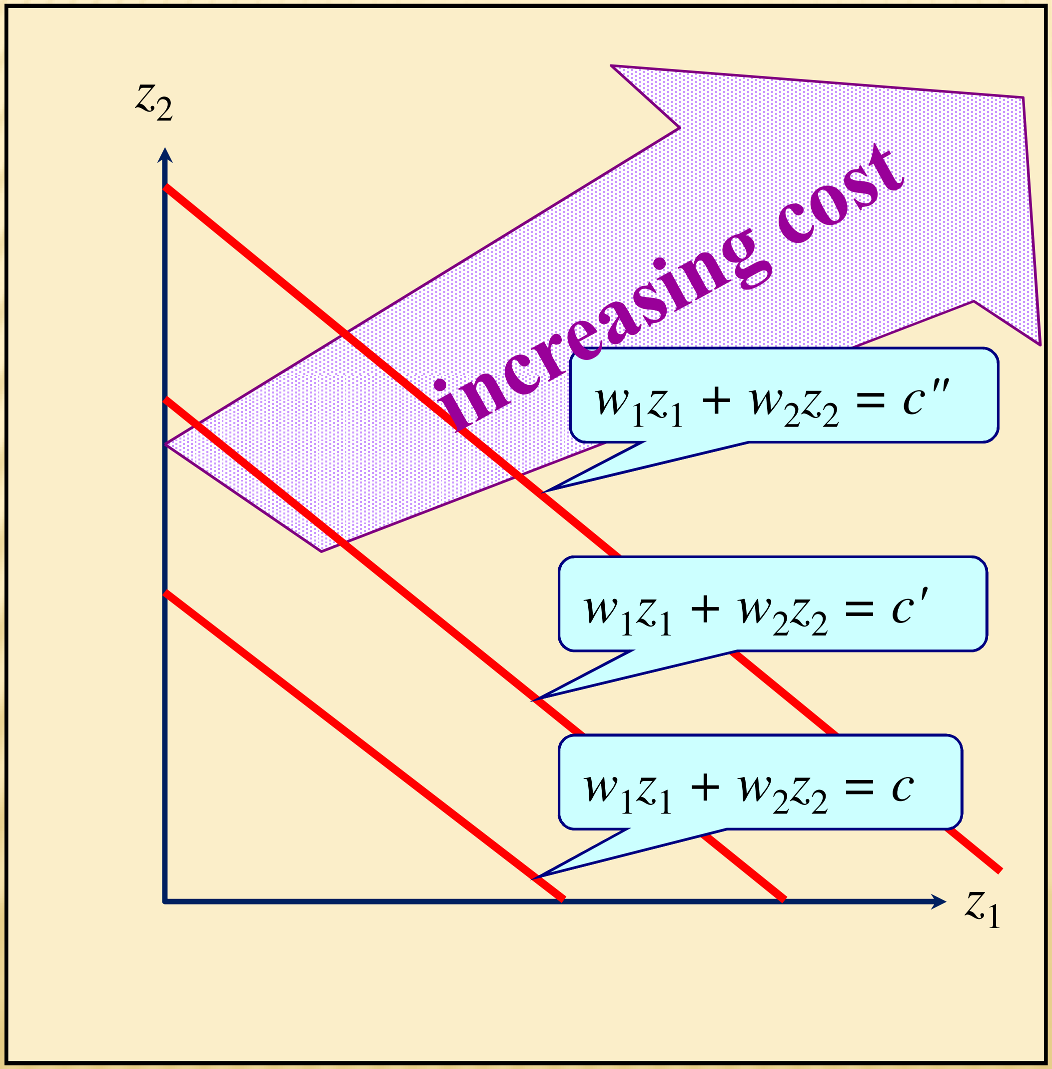
STAGE 1 OPTIMISATION

- Pick a target output level q
- Take as given the market prices of inputs $\mathbf{w}$
- Maximise profits...
- ...by minimising costs $\sum_{i=1}^{m} w_i z_i$

A USEFUL TOOL

- For a given set of input prices **w**...
- …the *isocost* is the set of points **z** in input space...
- ...that yield a given level of factor cost.
- These form a hyperplane (straight line)...
- ...because of the simple expression for factor-cost structure.

ISO-COST LINES

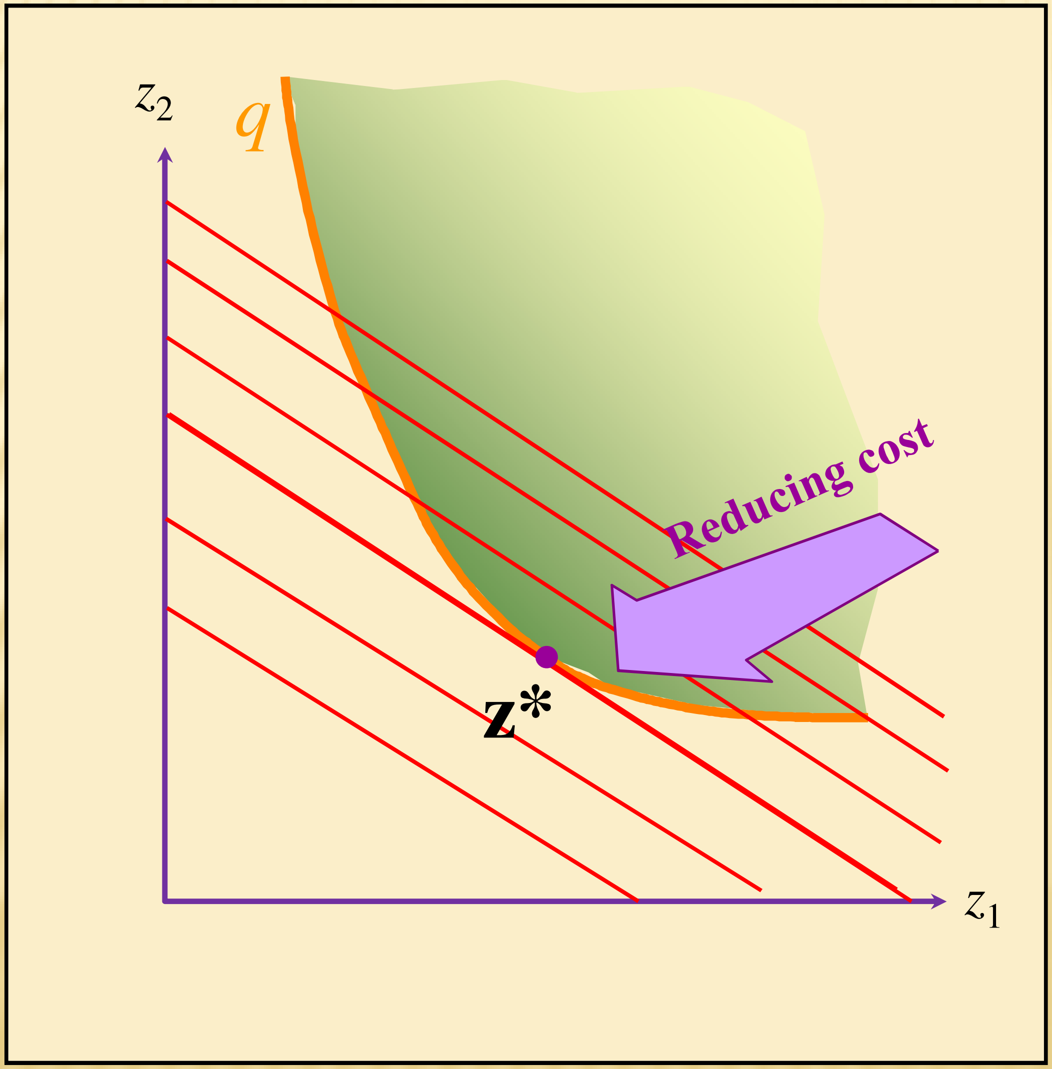


- ***Draw set of points where cost of input is c, a constant***
- ***Repeat for a higher value of the constant***
- ***Imposes direction on the diagram...***

slope w_1/w_2

Use this to derive optimum

COST-MINIMISATION



- *The firm minimises cost...*
- *Subject to output constraint*
- *Defines the stage 1 problem.*
- *Solution to the problem*

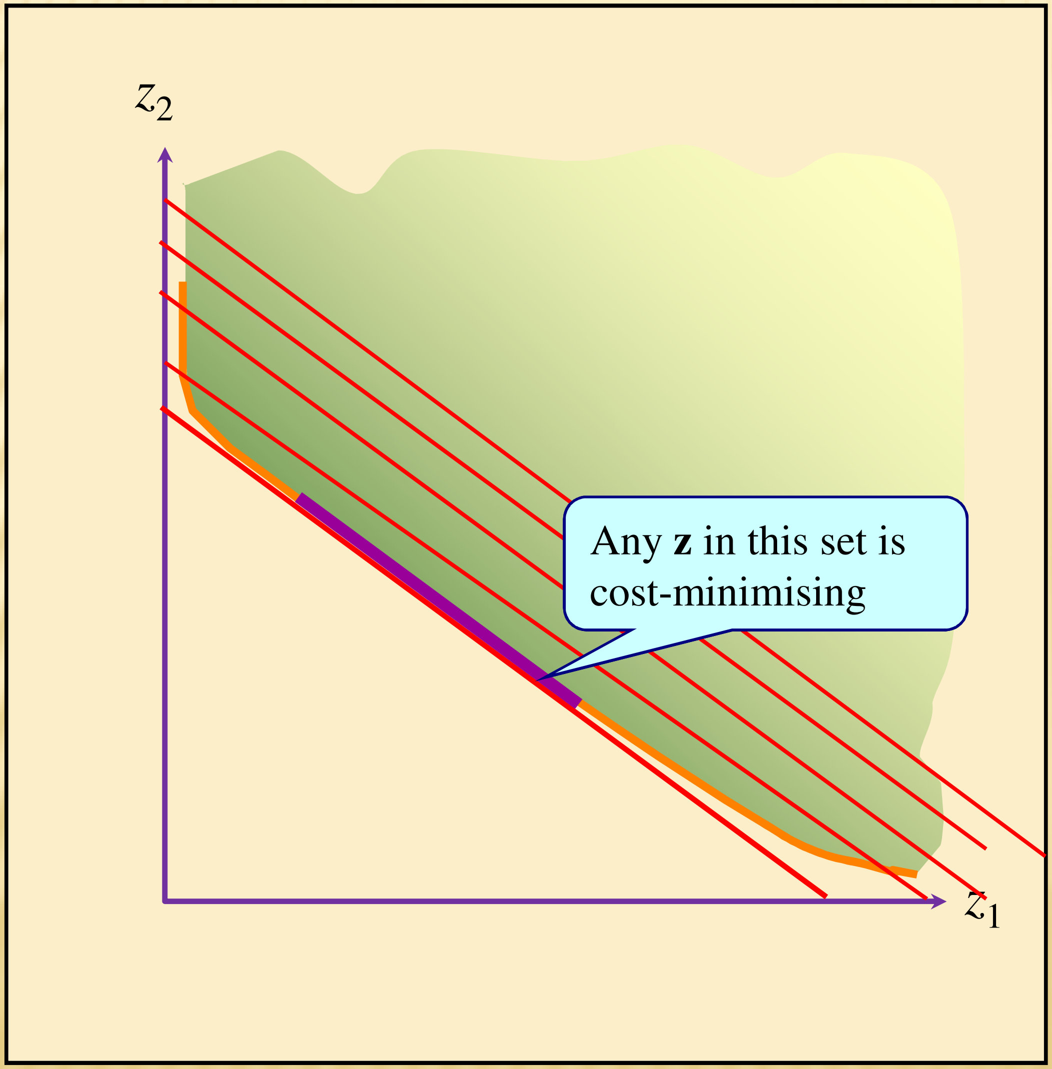
minimise

$$\sum_{i=1}^{m} w_i z_i$$

subject to $\phi(\mathbf{z}) \geq q$

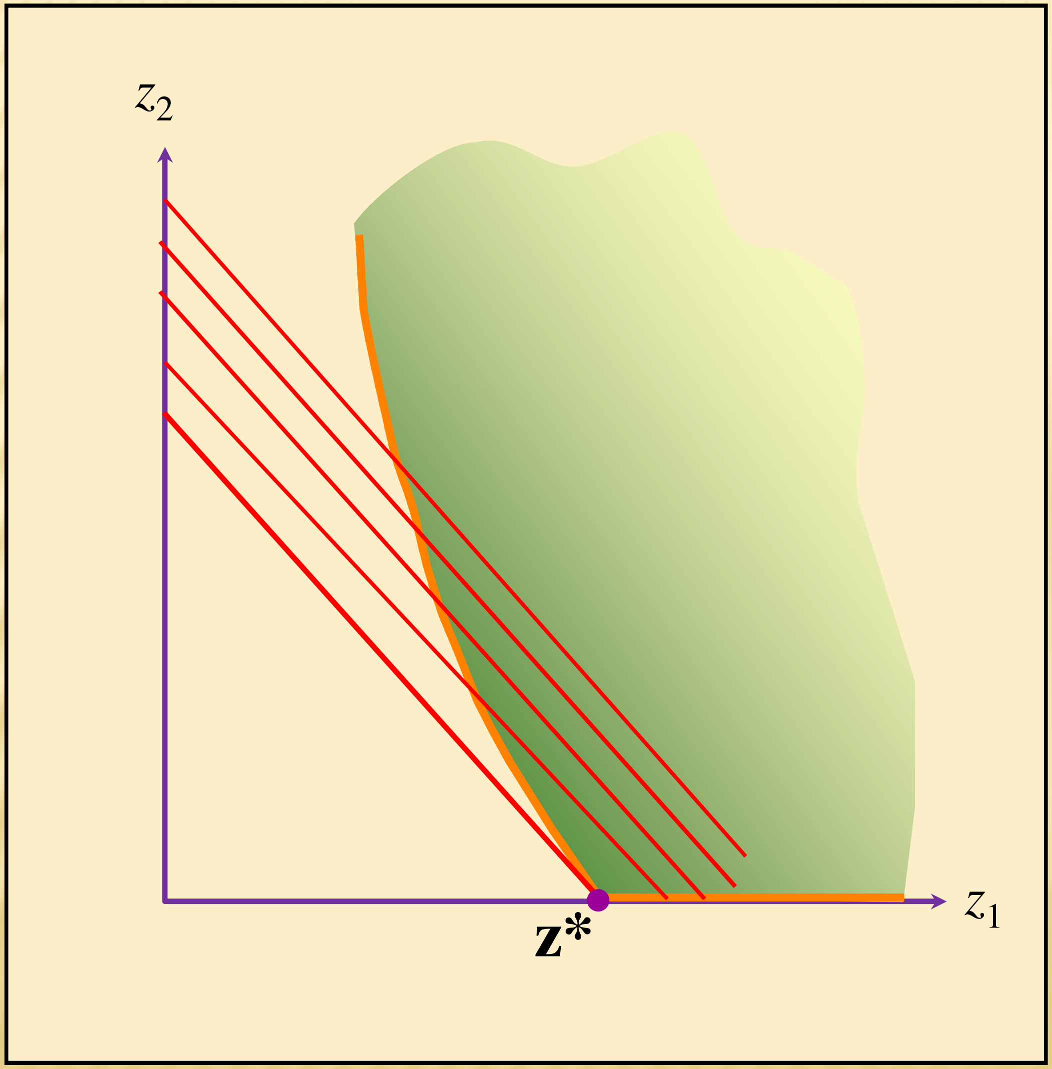
- *But the solution depends on the shape of the input-requirement set Z.*
- *What would happen in other cases?*

CONVEX, BUT NOT STRICTLY CONVEX Z



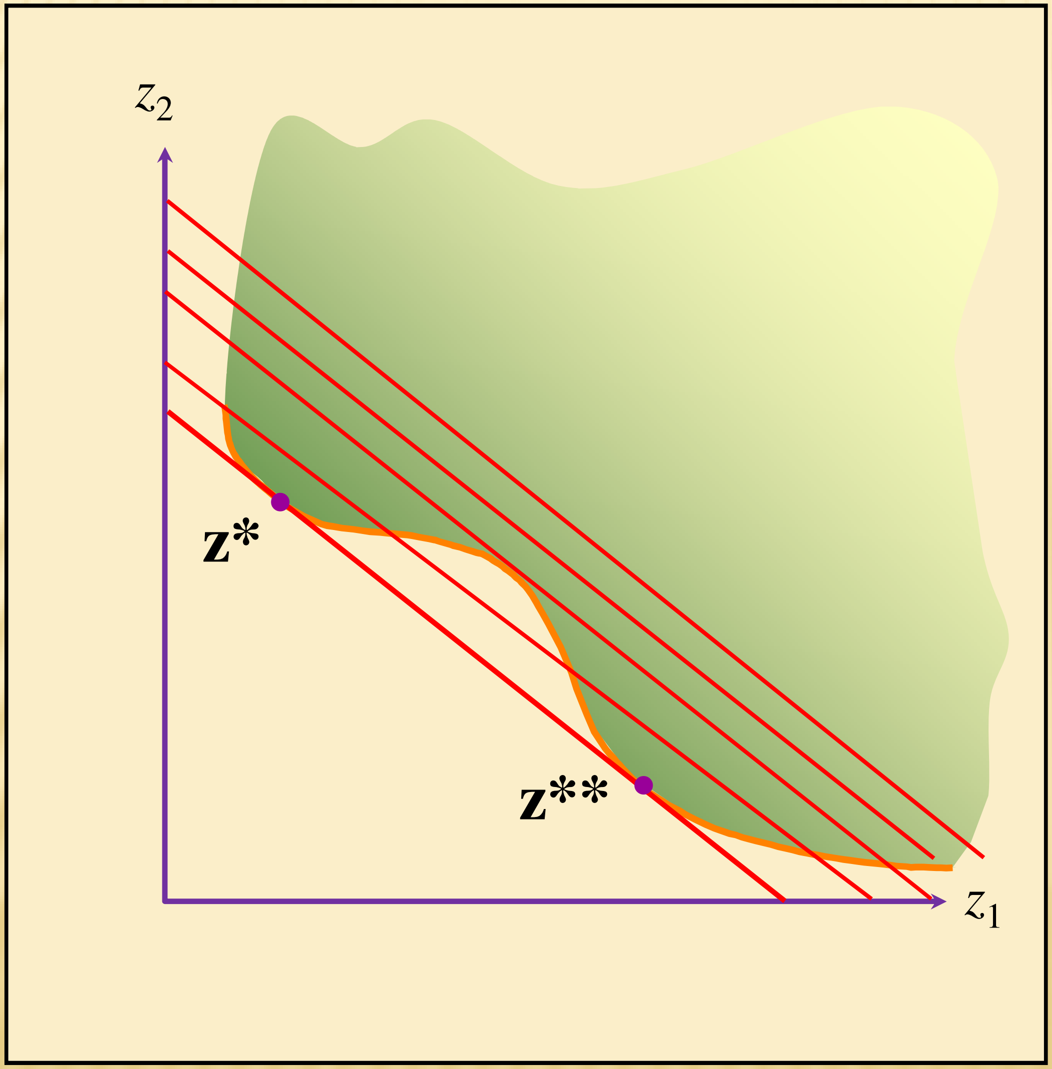
- *An* _*interval*_ *of solutions*

CONVEX Z, TOUCHING AXIS



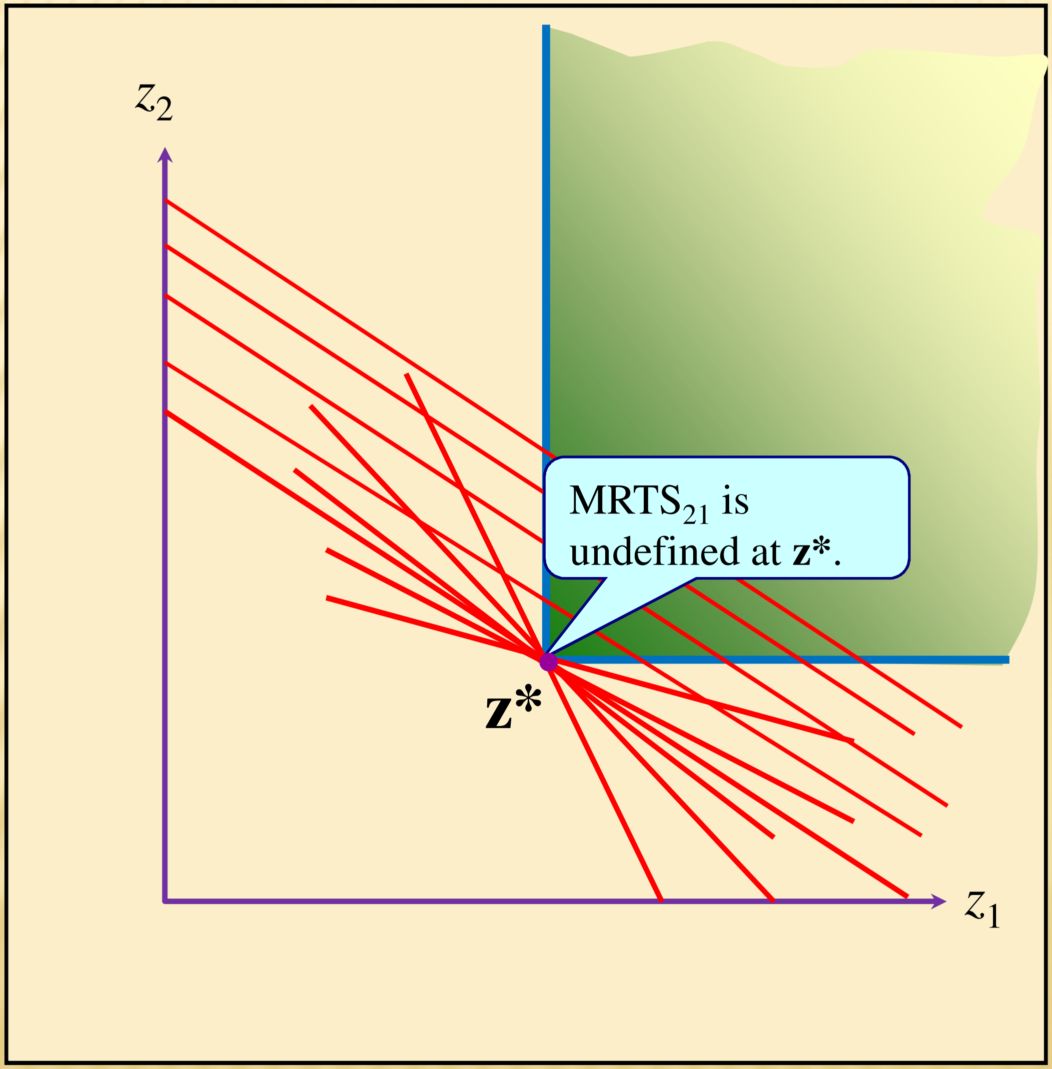
- *Here $MRTS_{21} > w_1 / w_2$ at the solution.*
- *Input 2 is "too expensive" and so isn't used: z_2*=0.*

NON-CONVEX Z



- *There could be multiple solutions.*
- *But note that there's no solution point between z* and z**.*

NON-SMOOTH Z



- *z^* is unique cost-minimising point for q.*
- *True for all positive finite values of w_1, w_2*

COST-MINIMISATION: STRICTLY CONVEX Z

- Minimise

$$\sum_{i=1}^{m} w_i z_i + \lambda[q \leq \phi(\mathbf{z})]$$

Lagrange multiplier

- Because of strict convexity we have an interior solution.
- A set of $m+1$ First-Order Conditions

$$\left.\begin{aligned} \lambda^* \phi_1(\mathbf{z}^*) &= w_1 \\ \lambda^* \phi_2(\mathbf{z}^*) &= w_2 \\ \ldots \quad \ldots \quad &\ldots \\ \lambda^* \phi_m(\mathbf{z}^*) &= w_m \end{aligned}\right\}$$

one for each input

$$q = \phi(\mathbf{z}^*)$$

output constraint

- *Use the objective function*
- *...and output constraint*
- *...to build the Lagrangean*
- *Differentiate w.r.t. $z_1, \ldots, z_m$ and set equal to 0.*
- *... and w.r.t λ*
- *Denote cost minimising values with a * .*

IF ISOQUANTS CAN TOUCH THE AXES...

- Minimise

$$\sum_{i=1}^{m} w_i z_i \;+\; \lambda[q - \phi(\mathbf{z})]$$

- Now there is the possibility of corner solutions.
- A set of m+1 First-Order Conditions

$$\left.\begin{array}{l} \lambda^*\phi_1(\mathbf{z}^*) \le w_1 \\ \lambda^*\phi_2(\mathbf{z}^*) \le w_2 \\ \dots \quad \dots \quad \dots \\ \lambda^*\phi_m(\mathbf{z}^*) \le w_m \end{array}\right\}$$

$$q = \phi(\mathbf{z}^*)$$

Can get “<” if optimal value of this input is 0

Interpretation

FROM THE FOC

- If both inputs i and j are used and MRTS is defined then...

$$\frac{\phi_i(\mathbf{z}^*)}{\phi_j(\mathbf{z}^*)} = \frac{w_i}{w_j}$$

- MRTS = input price ratio
  - "implicit" price = market price

- If input i could be zero then...

$$\frac{\phi_i(\mathbf{z}^*)}{\phi_j(\mathbf{z}^*)} \leq \frac{w_i}{w_j}$$

- $\mathrm{MRTS}_{ji} \leq$ input price ratio
  - "implicit" price ≤ market price

Solution

PROPORTIES OF THE MINIMUM-COST SOLUTION

- (a) The cost-minimising output under perfect competition is technically efficient.
- (b) For any two inputs, i, j purchased in positive amounts MRTSij must equal the input price ratio w_j/w_i.
- (c) If i is an input that is purchased, and j is an input that is not purchased then $MRTS_{ij}$ will be less than or equal to the input price ratio w_j/w_i.

THE SOLUTION...

- Solving the FOC, you get a cost-minimising value for each input...

$$\mathbf{z}_i^* = H^i(\mathbf{w}, q)$$

- ...for the Lagrange multiplier

$$\lambda^* = \lambda^*(\mathbf{w}, q)$$

- ...and for the minimised value of cost itself.
- The ***cost function*** is defined as

$$C(\mathbf{w}, q) := \min_{\{\phi(\mathbf{z}) \geq q\}} \Sigma\, w_i\, z_i$$

vector of input prices

Specified output level

INTERPRETING THE LAGRANGE MULTIPLIER

- The solution function:

$$C(\mathbf{w}, q) = \sum_i w_i \, z_i^*$$
$$= \sum_i w_i \, z_i^* - \lambda^* [\phi(\mathbf{z}^*) - q]$$

At the optimum, either the constraint binds or the Lagrange multiplier is zero

- Differentiate with respect to q:

Express demands in terms of $(\mathbf{w},q)$

$$C_q(\mathbf{w}, q) = \sum_i w_i H^i_q(\mathbf{w}, q)$$
$$- \lambda^* [\sum_i \phi_i(\mathbf{z}^*)$$

Vanishes because of FOC $\lambda^* \phi_i(\mathbf{x}^*) = w_i$

- Rearrange:

$$C_q(\mathbf{w}, q) = \Sigma_i [w_i - \lambda^* \phi_i(\mathbf{z}^*)] \, H^i_q(\mathbf{w}, q) + \lambda^*$$

Lagrange multiplier in the stage 1 problem is just marginal cost

$$C_q(\mathbf{w}, q) = \lambda^*$$

This result – extremely important in economics – is just an applications of a general "envelope" theorem.

THE COST FUNCTION IS A USEFUL CONCEPT

- Because it is a solution function...
- ...it automatically has very nice properties.
- These are true for *all* production functions.
- And they carry over to applications other than the firm.
- We'll investigate these graphically.

PROPERTIES OF THE MINIMUM-COST SOLUTION

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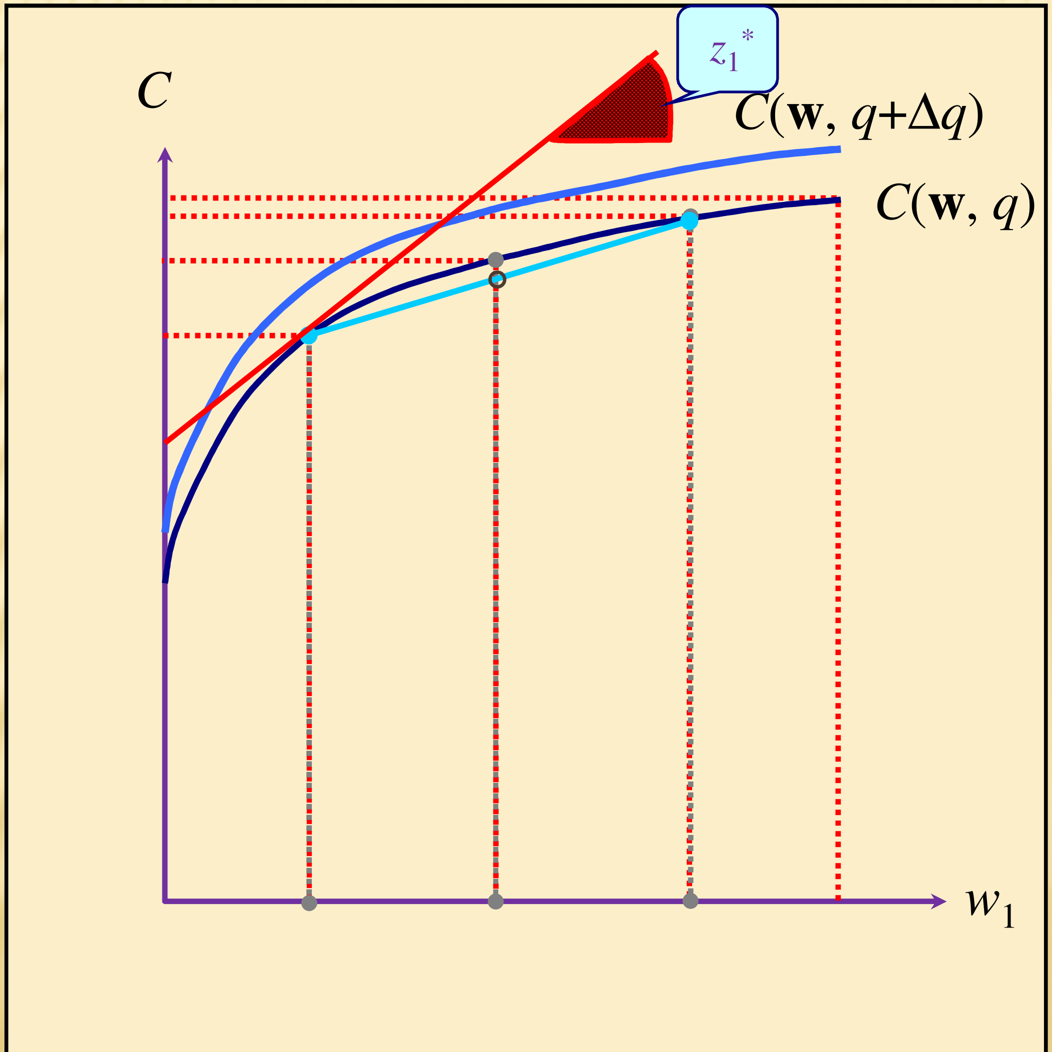
PROPERTIES OF C

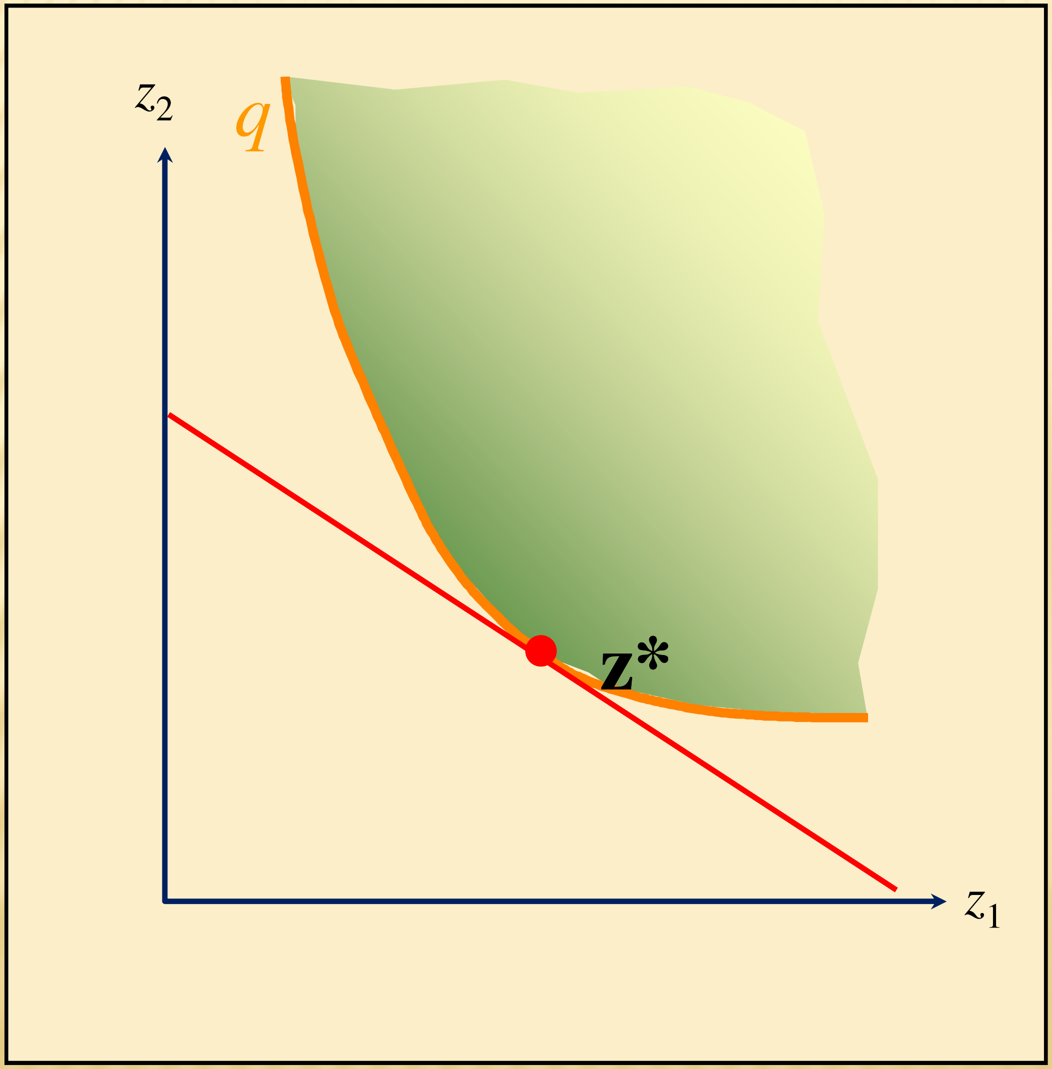


- *Draw cost as function of w_1*
- *Cost is non-decreasing in input prices .*
- *Cost is increasing in output.*
- *Cost is concave in input prices.*
- *Shephard's Lemma*

$$C(t\mathbf{w}+[1-t]\mathbf{w}',q) \geq tC(\mathbf{w},q) + [1-t]C(\mathbf{w}',q)$$

$$\frac{\partial C(\mathbf{w},q)}{\partial w_j} = z_j^*$$

WHAT HAPPENS TO COST IF **W** CHANGES TO *t***W**



- ***Find cost-minimising inputs for w, given q***
- ***Find cost-minimising inputs for tw, given q***

- *So we have:*

$$C(t\mathbf{w},q) = \Sigma_i\, t\, w_i z_i^* = t\, \Sigma_i w_i z_i^* = tC(\mathbf{w},q)$$

- *The cost function is homogeneous of degree 1 in prices.*

COST FUNCTION: 5 THINGS TO REMEMBER

- Non-decreasing in every input price.
  - Increasing in at least one input price.
- Increasing in output.
- Concave in prices.
- Homogeneous of degree 1 in prices.
- Shephard's Lemma.

EXAMPLE

Production function: $q \leq z_1^{0.1} \; z_2^{0.4}$

Equivalent form: $\log q \leq 0.1 \log z_1 + 0.4 \log z_2$

Lagrangean: $w_1 z_1 + w_2 z_2 + \lambda\,[\log q - 0.1 \log z_1 - 0.4 \log z_2]$

FOCs for an interior solution:

$w_1 - 0.1\,\lambda / z_1 = 0$

$w_2 - 0.4\,\lambda / z_2 = 0$

$\log q = 0.1 \log z_1 + 0.4 \log z_2$

From the FOCs:

$\log q = 0.1 \log (0.1\,\lambda / w_1) + 0.4 \log (0.4\,\lambda / w_2)$

$\lambda = 0.1^{-0.2}\, 0.4^{-0.8}\, w_1^{0.2}\, w_2^{0.8}\, q^2$

Therefore, from this and the FOCs:

$w_1 z_1 + w_2 z_2 = 0.5\lambda = 1.649\, w_1^{0.2}\, w_2^{0.8}\, q^2$

OVERVIEW…

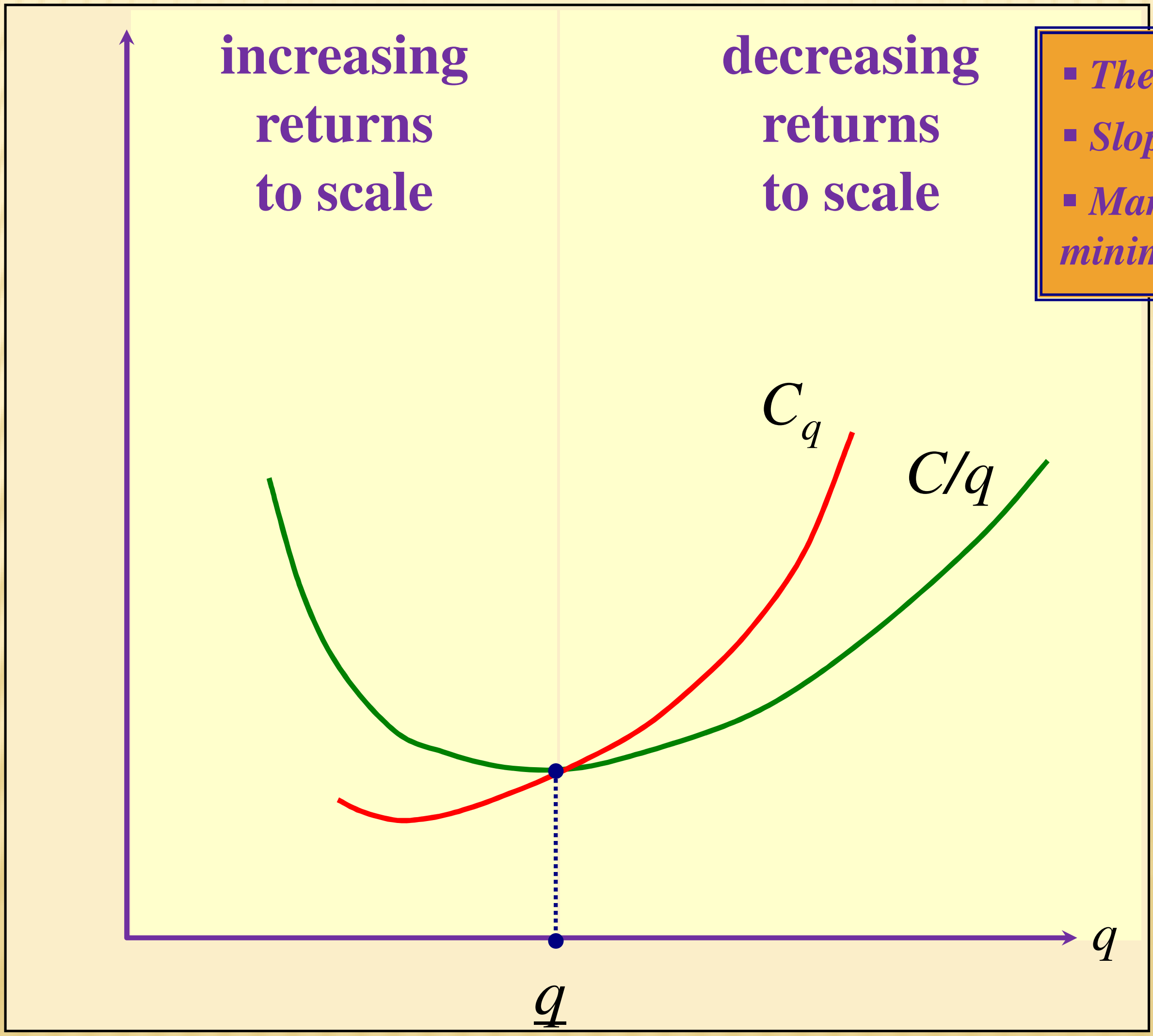
…using the results of stage 1

- Firm: Optimisation
  - The setting
  - Stage 1: Cost Minimisation
  - **Stage 2: Profit maximisation**

STAGE 2 OPTIMISATION

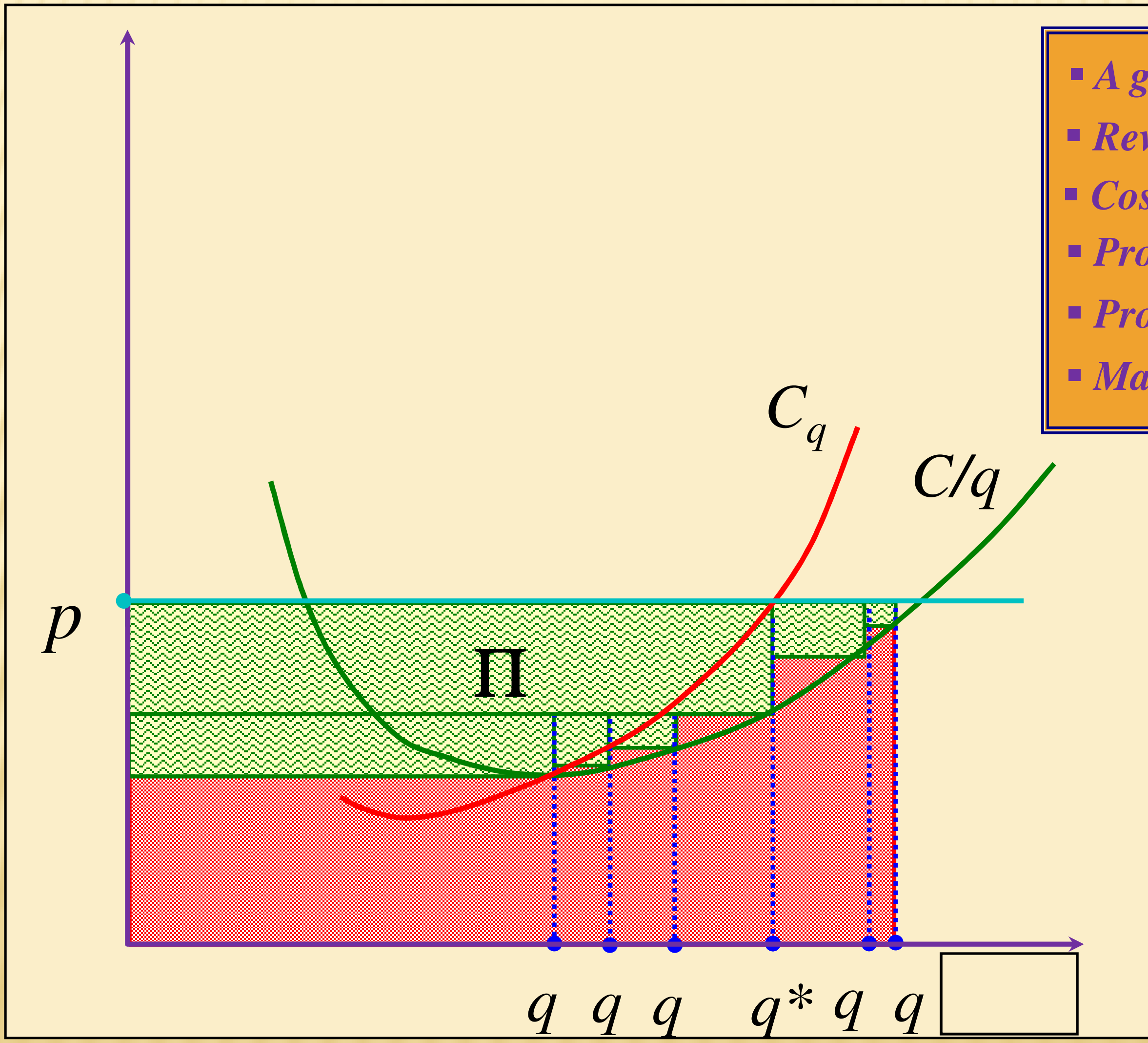
- Take the cost-minimisation problem as solved.
- Take output price p as given.
  - Use minimised costs $C(\mathbf{w},q)$.
  - Set up a 1-variable maximisation problem.
- Choose q to maximise profits.
- First analyse the components of the solution graphically.
  - Tie-in with properties of the firm introduced in the previous presentation.
- Then we come back to the formal solution.

AVERAGE AND MARGINAL COST



- *The average cost curve.*
- *Slope of AC depends on RTS.*
- *Marginal cost cuts AC at its minimum*

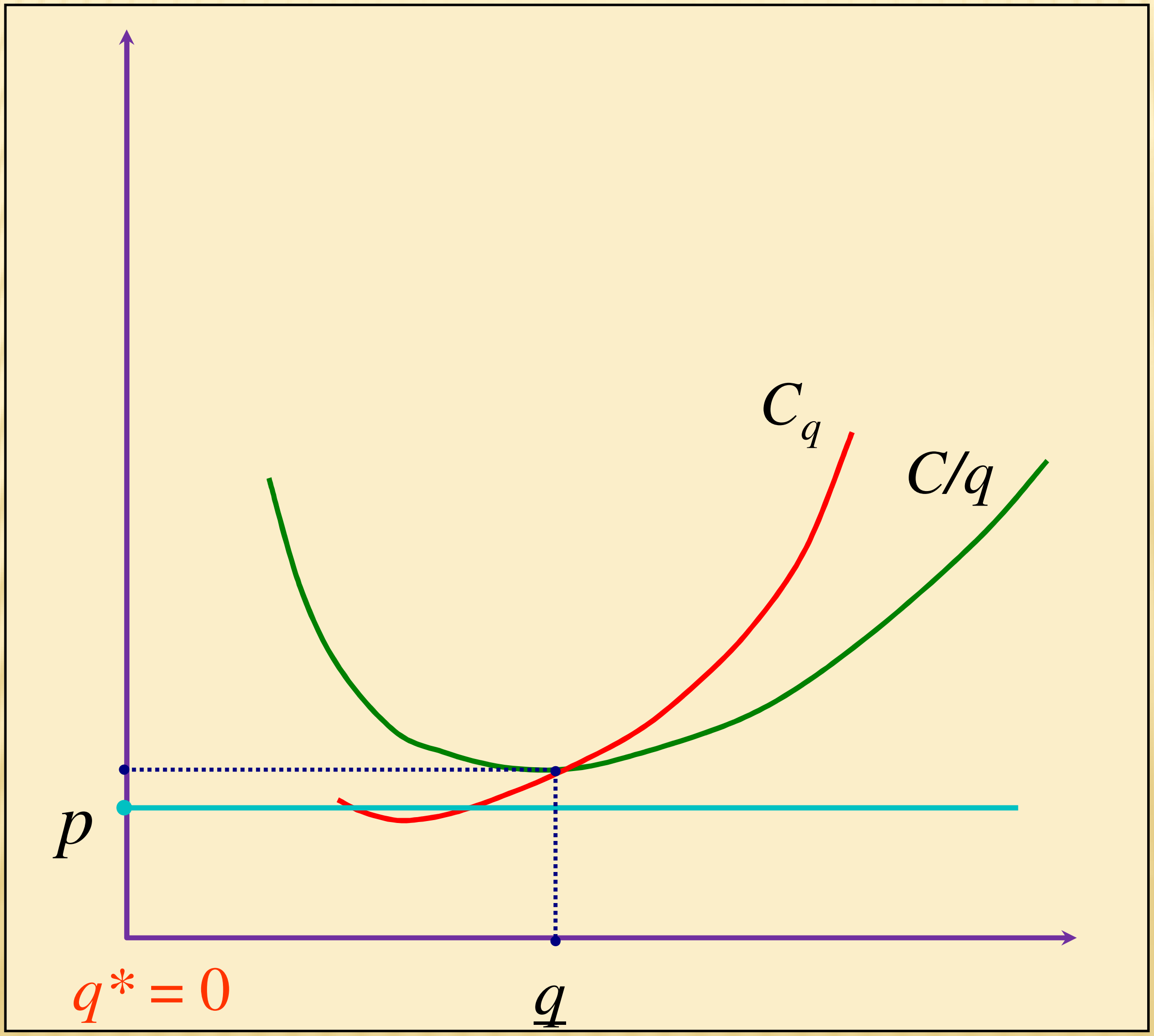
REVENUE AND PROFITS



- *A given market price p.*
- *Revenue if output is q.*
- *Cost if output is q.*
- *Profits if output is q.*
- *Profits vary with q.*
- *Maximum profits*

- *price = marginal cost*

WHAT HAPPENS IF PRICE IS LOW...



- *price < marginal cost*

PROFIT MAXIMISATION

- Objective is to choose q to max:

$$pq - C(\mathbf{w}, q)$$

"Revenue minus minimised cost"

- From the First-Order Conditions if $q^* > 0$:

$$p = C_q(\mathbf{w}, q^*)$$

"Price equals marginal cost"

$$p \geq \frac{C(\mathbf{w}, q^*)}{q^*}$$

"Price covers average cost"

- In general:

$$p \leq C_q(\mathbf{w}, q^*)$$

$$pq^* \geq C(\mathbf{w}, q^*)$$

covers both the cases:
$q^* > 0$ and $q^* = 0$

EXAMPLE (CONTINUED)

Production function: $q \le z_1^{0.1}\ z_2^{0.4}$

Resulting cost function: $C(\mathbf{w}, q) = 1.649\, w_1^{0.2}\, w_2^{0.8}\, q^2$

Profits:

$pq - C(\mathbf{w}, q) = pq - A\, q^2$

where $A := 1.649\, w_1^{0.2}\, w_2^{0.8}$

FOC:

$p - 2\, Aq = 0$

Result:

$q = p\, /\, 2A.$

$= 0.3031\, w_1^{-0.2}\, w_2^{-0.8}\, p$

SUMMARY

- *Key point:* Profit maximisation can be viewed in two stages:
  - Stage 1: choose inputs to minimise cost
  - Stage 2: choose output to maximise profit
- *What next?* Use these to predict firm's reactions