UNIVERSITY OF ATHENS
Department of Economics
Course: Advanced Microeconomics Autumn-Winter 2008-2009
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Exercise 1.

A firm uses two inputs in the production of a single good. The input requirements per unit of output for a number of alternative techniques are given by the following table:

Process	1	2	3	4	5	6
Input 1	9	15	7	1	3	4
Input 2	4	2	6	10	9	7

The firm has exactly 140 units of input 1 and 410 units of input 2 at its disposal.

1. Discuss the concepts of technological and economic efficiency with reference to this example.

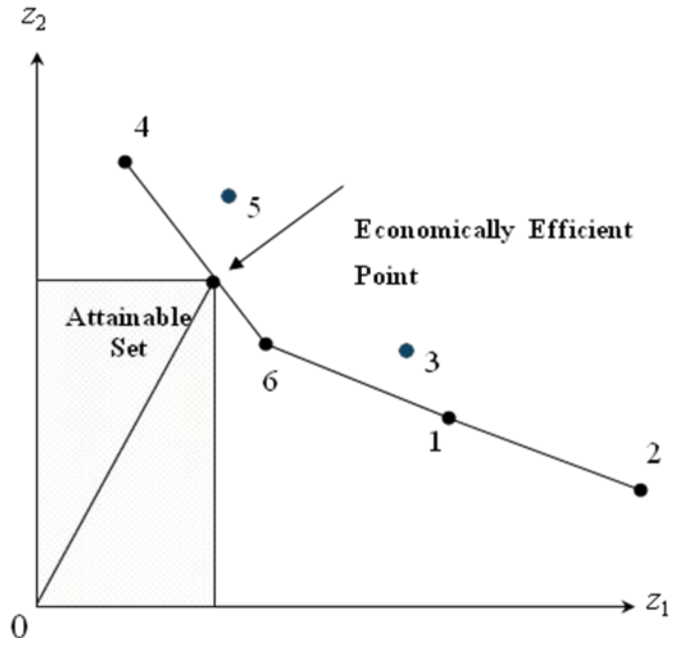
2. Describe the optimal production plan for the firm.

3. Would the firm prefer 10 extra units of input 1 or 20 extra units of input2?

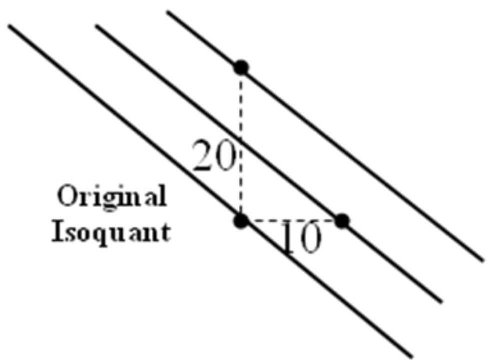
Exercise 1. Answer

1. As illustrated in figure 2.4 only processes 1,2,4 and 6 are technically efficient.

2. Given the resource constraint (see shaded area), the economically efficient input combination is a mixture of processes 4 and 6.



3. Note that in the neighbourhood of this efficient point MRTS=1. So, as illustrated in the enlarged diagram in Figure 2.5, 20 extra units of input 2 clearly enable more output to be produced than 10 extra units of input 1.



Exercise 2.

Suppose a .rm.s production function has the Cobb-Douglas form

$q = z_1^{\alpha_1} z_2^{\alpha_2}$

where z_1 and z_2 are inputs, q is output and α_1, α_2 are positive parameters.

1. Draw the isoquants. Do they touch the axes?

2. What is the elasticity of substitution in this case?

3. Using the Lagrangean method and the cost-minimising values of the inputs and the cost function.

4. Under what circumstances will the production function exhibit (a) decreasing (b) constant (c) increasing returns to scale? Explain this using first the production function and then the cost function.

5. Find the conditional demand curve for input 1.

Exercise 2. Answer

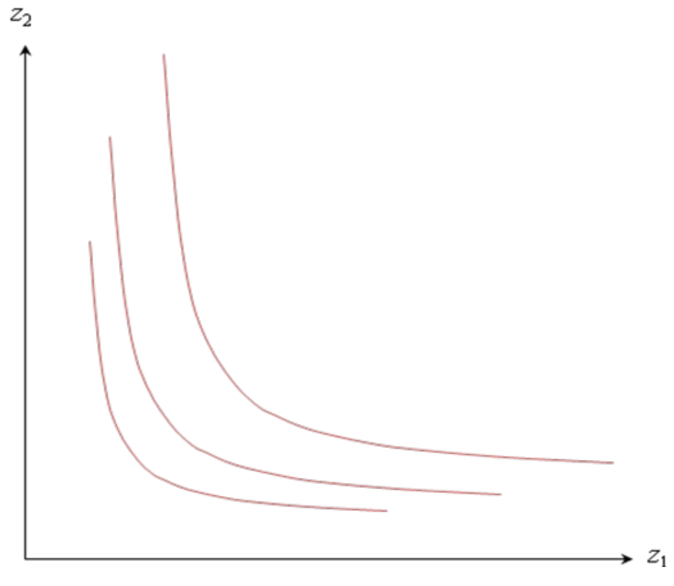
1. The isoquants are illustrated in Figure 2.7. They do not touch the axes.

2. The elasticity of substitution is defined as

$$\sigma_{ij} := -\frac{\partial \log\left(z_j/z_i\right)}{\partial \log\left(\phi_j(\mathbf{z})/\phi_i(\mathbf{z})\right)}$$

which, in the two input case, becomes

$$\sigma = -\frac{\partial \log\left(\frac{z_1}{z_2}\right)}{\partial \log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)}$$



In case 1 we have $\phi(\mathbf{z}) = z_1^{\alpha_1} z_2^{\alpha_2}$ and so, by differentiation, we find:

$$\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})} = \frac{\alpha_1}{\alpha_2} / \frac{z_1}{z_2}$$

Taking logarithms we have

$$\log\left(\frac{z_1}{z_2}\right) = \log\left(\frac{\alpha_1}{\alpha_2}\right) - \log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)$$

or

$$u = \log\left(\frac{\alpha_1}{\alpha_2}\right) - v$$

where $u := \log(z_1/z_2)$ and $v := \log(\phi_1/\phi_2)$.

Differentiating u with respect to v we have

$$\frac{\partial u}{\partial v} = -1. \tag{2.2}$$

So, using the definitions of u and v in equation (2.2) we have

$$\sigma = -\frac{\partial u}{\partial v} = 1.$$

3. This is a *Cobb-Douglas production function*. This will yield a unique interior solution; the Lagrangean is:

$$\mathcal{L}(\mathbf{z},\lambda) = w_1 z_1 + w_2 z_2 + \lambda \left[q - z_1^{\alpha_1} z_2^{\alpha_2}\right] \ , \tag{2.3}$$

and the first-order conditions are:

$$\frac{\partial \mathcal{L}(\mathbf{z},\lambda)}{\partial z_1} = w_1 - \lambda \alpha_1 z_1^{\alpha_1 - 1} z_2^{\alpha_2} = 0 \, , \tag{2.4}$$

$$\frac{\partial \mathcal{L}(\mathbf{z},\lambda)}{\partial z_2} = w_2 - \lambda \alpha_2 z_1^{\alpha_1} z_2^{\alpha_2 - 1} = 0 \, , \tag{2.5}$$

$$\frac{\partial \mathcal{L}(\mathbf{z},\lambda)}{\partial \lambda} = q - z_1^{\alpha_1} z_2^{\alpha_2} = 0 \, . \tag{2.6}$$

Using these conditions and rearranging we can get an expression for minimized cost in terms of and q:

$$w_1 z_1 + w_2 z_2 = \lambda \alpha_1 z_1^{\alpha_1} z_2^{\alpha_2} + \lambda \alpha_2 z_1^{\alpha_1} z_2^{\alpha_2} = [\alpha_1 + \alpha_2] \lambda q.$$

We can then eliminate λ:

$$\left.\begin{array}{l} w_1 - \lambda \alpha_1 \frac{q}{z_1} = 0 \\ w_2 - \lambda \alpha_2 \frac{q}{z_2} = 0 \end{array}\right\}$$

which implies

$$\left.\begin{array}{l} z_1^* = \frac{\alpha_1}{w_1} \lambda q \\ z_2^* = \frac{\alpha_2}{w_2} \lambda q \end{array}\right\}. \tag{2.7}$$

Substituting the values of z_1^* and z_2^* back in the production function we have

$$\left[\frac{\alpha_1}{w_1}\lambda q\right]^{\alpha_1} \left[\frac{\alpha_2}{w_2}\lambda q\right]^{\alpha_2} = q$$

which implies

$$\lambda q = \left[q \left[\frac{w_1}{\alpha_1}\right]^{\alpha_1} \left[\frac{w_2}{\alpha_2}\right]^{\alpha_2}\right]^{\frac{1}{\alpha_1 + \alpha_2}} \tag{2.8}$$

So, using (2.7) and (2.8), the corresponding cost function is

$$\begin{aligned} C(\mathbf{w}, q) &= w_1 z_1^* + w_2 z_2^* \\ &= [\alpha_1 + \alpha_2] \left[q \left[\frac{w_1}{\alpha_1}\right]^{\alpha_1} \left[\frac{w_2}{\alpha_2}\right]^{\alpha_2}\right]^{\frac{1}{\alpha_1 + \alpha_2}} . \end{aligned}$$

4. Using the production functions we have, for any $t > 0$:

$$\phi(t\mathbf{z}) = [tz_1]^{\alpha_1} [tz_2]^{\alpha_2} = t^{\alpha_1 + \alpha_2} \phi(\mathbf{z}).$$

Therefore we have DRTS/CRTS/IRTS according as $\alpha_1 + \alpha_2 \lesseqqgtr 1$. If we look at average cost as a function of q we find that AC is increasing/constant/decreasing in q according as $\alpha_1 + \alpha_2 \lesseqqgtr 1$.

5. Using (2.7) and (2.8) conditional demand functions are

$$H^1(\mathbf{w}, q) = \left[q\left[\frac{\alpha_1 w_2}{\alpha_2 w_1}\right]^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}}$$

$$H^2(\mathbf{w}, q) = \left[q\left[\frac{\alpha_2 w_1}{\alpha_1 w_2}\right]^{\alpha_1}\right]^{\frac{1}{\alpha_1+\alpha_2}}$$

and are smooth with respect to input prices.

Exercise 3.

Suppose a firm's production function has the Leontief *form*

$$q = \min\left\{\frac{z_1}{\alpha_1}, \frac{z_2}{\alpha_2}\right\}$$

where the notation is the same as in Exercise 2.4.

1. *Draw the isoquants.*
2. *For a given level of output identify the cost-minimising input combination(s) on the diagram.*
3. *Hence write down the cost function in this case. Why would the Lagrangean method of Exercise* $\boxed{2}$ *be inappropriate here?*
4. *What is the conditional input demand curve for input 1?*
5. *Repeat parts 1-4 for each of the two production functions*

$$\begin{aligned} q &= \alpha_1 z_1 + \alpha_2 z_2 \\ q &= \alpha_1 z_1^2 + \alpha_2 z_2^2 \end{aligned}$$

   Explain carefully how the solution to the cost-minimisation problem differs in these two cases.

Exercise 3. Answer

1. The Isoquants are illustrated in Figure 2.8 – the so-called *Leontief* case,

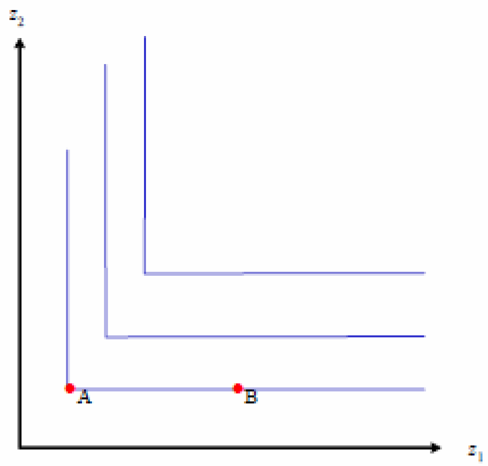


Figure 2.8: Isoquants: Leontief

2. If all prices are positive, we have a unique cost-minimising solution at A: to see this, draw any straight line with positive finite slope through A and take this as an isocost line; if we considered any other point B on the isoquant through A then an isocost line through B (same slope as the one through A) must lie above the one you have just drawn.

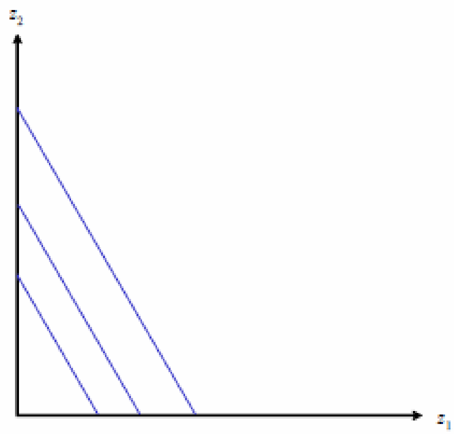


Figure 2.9: Isoquants: linear

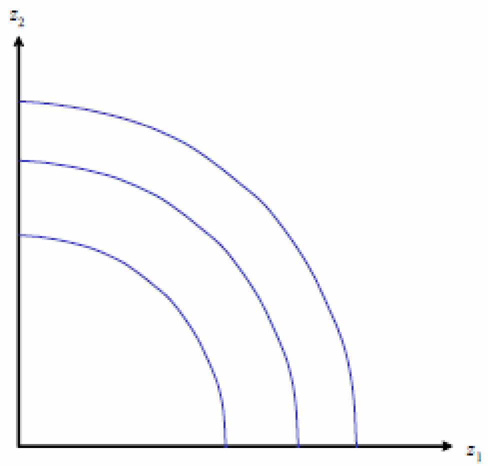


Figure 2.10: Isoquants: non-convex to origin

3. The coordinates of the corner A are $(\alpha_1 q, \alpha_2 q)$ and, given $\mathbf{w}$, this immediately yields the minimised cost.

$$C(\mathbf{w}, q) = w_1 \alpha_1 q + w_2 \alpha_2 q.$$

The methods in Exercise 2.4 since the Lagrangean is not differentiable at the corner.

4. Conditional demand is constant if all prices are positive

$$\begin{aligned} H^1(\mathbf{w}, q) &= \alpha_1 q \\ H^2(\mathbf{w}, q) &= \alpha_2 q. \end{aligned}$$

5. Given the linear case

$$q = \alpha_1 z_1 + \alpha_2 z_2$$

- Isoquants are as in Figure 2.9.
- It is obvious that the solution will be either at the corner $(q/\alpha_1, 0)$ if $w_1/w_2 < \alpha_1/\alpha_2$ or at the corner $(0, q/\alpha_2)$ if $w_1/w_2 > \alpha_1/\alpha_2$, or otherwise anywhere on the isoquant
- This immediately shows us that minimised cost must be.

$$C(\mathbf{w}, q) = q \min\left\{\frac{w_1}{\alpha_1}, \frac{w_2}{\alpha_2}\right\}$$

- So conditional demand can be multivalued:

$$H^1(\mathbf{w},q) = \begin{cases} \frac{q}{\alpha_1} & \text{if } \frac{w_1}{w_2} < \frac{\alpha_1}{\alpha_2} \\ z_1^* \in \left[0, \frac{q}{\alpha_1}\right] & \text{if } \frac{w_1}{w_2} = \frac{\alpha_1}{\alpha_2} \\ 0 & \text{if } \frac{w_1}{w_2} > \frac{\alpha_1}{\alpha_2} \end{cases}$$

$$H^2(\mathbf{w},q) = \begin{cases} 0 & \text{if } \frac{w_1}{w_2} < \frac{\alpha_1}{\alpha_2} \\ z_2^* \in \left[0, \frac{q}{\alpha_2}\right] & \text{if } \frac{w_1}{w_2} = \frac{\alpha_1}{\alpha_2} \\ \frac{q}{\alpha_2} & \text{if } \frac{w_1}{w_2} > \frac{\alpha_1}{\alpha_2} \end{cases}$$

- Case *3* is a test to see if you are awake: the isoquants are not convex to the origin: an experiment with a straight-edge to simulate an isocost line will show that it is almost like case 2 – the solution will be either at the corner $(\sqrt{q/\alpha_1}, 0)$ if $w_1/w_2 < \sqrt{\alpha_1/\alpha_2}$ or at the corner $(0, \sqrt{q/\alpha_2})$ if $w_1/w_2 > \sqrt{\alpha_1/\alpha_2}$ (but nowhere else). So the cost function is :

$$C(\mathbf{w},q) = \min\left\{w_1\sqrt{\frac{q}{\alpha_1}}, w_2\sqrt{q/\alpha_2}\right\}.$$

The conditional demand function is similar to, but slightly different from, the previous case:

$$H^1(\mathbf{w},q) = \begin{cases} \frac{q}{\alpha_1} & \text{if } \frac{w_1}{w_2} < \sqrt{\frac{\alpha_1}{\alpha_2}} \\ z_1^* \in \left\{0, \frac{q}{\alpha_1}\right\} & \text{if } \frac{w_1}{w_2} = \sqrt{\frac{\alpha_1}{\alpha_2}} \\ 0 & \text{if } \frac{w_1}{w_2} > \sqrt{\frac{\alpha_1}{\alpha_2}} \end{cases}$$

$$H^2(\mathbf{w},q) = \begin{cases} 0 & \text{if } \frac{w_1}{w_2} < \sqrt{\frac{\alpha_1}{\alpha_2}} \\ z_2^* \in \left\{0, \frac{q}{\alpha_2}\right\} & \text{if } \frac{w_1}{w_2} = \sqrt{\frac{\alpha_1}{\alpha_2}} \\ \frac{q}{\alpha_2} & \text{if } \frac{w_1}{w_2} > \sqrt{\frac{\alpha_1}{\alpha_2}} \end{cases}$$

Note the discontinuity exactly at $w_1/w_2 = \sqrt{\alpha_1/\alpha_2}$

Exercise 4.

Assume the production function

$$\phi(\mathbf{z}) = \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}}$$

where z_i is the quantity of input i and $\alpha_i \geq 0$, $-\infty < \beta \leq 1$ are parameters. This is an example of the CES (Constant Elasticity of Substitution) production function.

1. *Show that the elasticity of substitution is $\frac{1}{1-\beta}$.*
2. *Explain what happens to the form of the production function and the elasticity of substitution in each of the following three cases: $\beta \to -\infty$, $\beta \to 0$, $\beta \to 1$.*

Exercise 4. Answer

1. Differentiating the production function

$$\phi(\mathbf{z}) := \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}}$$

it is clear that the marginal product of input i is

$$\phi_i(\mathbf{z}) := \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}-1} \alpha_i z_i^{\beta-1} \tag{2.9}$$

Therefore the MRTS is

$$\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})} = \frac{\alpha_1}{\alpha_2}\left[\frac{z_1}{z_2}\right]^{\beta-1} \tag{2.10}$$

which implies

$$\log\left(\frac{z_1}{z_2}\right) = \frac{1}{1-\beta}\log\frac{\alpha_1}{\alpha_2} - \frac{1}{1-\beta}\log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right).$$

Therefore

$$\sigma = -\frac{\partial\log\left(\frac{z_1}{z_2}\right)}{\partial\log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)} = \frac{1}{1-\beta}$$

2. Clearly $\beta \to -\infty$ yields $\sigma = 0$ ($\phi(z) = \min\{\alpha_1 z_1, \alpha_2 z_2\}$), $\beta \to 0$ yields $\sigma = 1$ ($\phi(z) = z_1^{\alpha_1} z_2^{\alpha_2}$), $\beta \to 1$ yields $\sigma = \infty$ ($\phi(z) = \alpha_1 z_1 + \alpha_2 z_2$).