Cuisenaire - from Early Years to Adult
This publication is the result of collaboration between Mike Ollerton, Simon Gregg and Helen Williams. The idea arose from a workshop run by Mike and Helen at the ATM Conference, exploring the use of Cuisenaire\(^1\) with learners from Early Years to adult. As a result of the ideas generated during this session, we contacted Simon Gregg, a primary teacher in Toulouse, who Mike and Helen had ‘met’ through Twitter; Simon tweets regularly about his use of Cuisenaire with 8 year-olds and more recently in Kindergarten with 5 year-olds.

Mike and Helen have worked together at ATM conferences offering practical, hands-on workshops for some years. Our sessions do not stop at the doing; they are just as much about the craft and pedagogy of teaching and responding to the learning, as about the tasks and the ensuing activities themselves. This is because we firmly believe that mathematics is a human endeavour. Thus we began the conference session as we might begin with any learner new to Cuisenaire, by asking: “What can you do with a pile of Cuisenaire rods?”

While this way of working is common to Early Years classrooms, we believe that although the mathematics clearly increases in complexity, it is a way of working which is of equal importance throughout the key stages. We invited participants, therefore, as well as trying out some of the ideas in this publication, to play with Cuisenaire, to talk to each other, to ask questions, to scribble and to think. Together we generated more ideas ripe for further mathematical work. Subsequently some delegates kindly sent their notes for us to use in this publication. Most groups made posters and other ideas have been taken from these, as well as ideas of our own.

\(^1\) http://www.cuisenaire.co.uk/ is the official site of the Cuisenaire Company.
1.1 Cuisenaire and Gattegno: A very brief history

Cuisenaire rods were invented in 1945 by a Belgian teacher, Georges Cuisenaire. He was concerned that children didn’t take the same delight in mathematics as a mathematician might. As a music teacher, he saw too that mathematics learning lacked the sensory experience that learning music with an instrument gave. Children could for instance, learn about musical relationships through playing the keys on a keyboard that are a certain distance apart. His idea was to provide a similar way of showing numerical relationships through rods of different length and colour. Caleb Gattegno (founder member of the ATM) saw children using Cuisenaire rods and described their actions as an ‘illumination’ (p158). Gattegno subsequently set out on a long programme to explore and propagate their use worldwide.

Although Gattegno went on to write books that outline in detail a progression of learning with the rods, his idea was that educators themselves, with their students, would find ways of using the rods: “just as I did not want to interfere with the learning processes of children, so I did not wish to interfere with the teachers ‘freedom of work’.”, ibid (p167)

It is our experience that we ourselves, and teachers we’ve worked with, exploring with their students regularly discover new avenues to explore. So what follows is far from exhaustive. Hopefully instead, it will be a provocation to dip in, get the rods out and do some hands-on exploration.

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1.2 The structure of this book

Our underlying pedagogy is that play doesn’t begin and end with the Early Years. It is simply that we tend to call it investigating or exploring with older and adult learners. Thus, for anyone new to Cuisenaire, the first step is to play with a box, to make pictures and patterns and, for older learners, to make a note of any connections you are noticing. Everyone needs to play. And play periodically. So, if you are starting a new maths topic with a class, we have found it helpful to begin by playing with the Cuisenaire ourselves with the question in mind: “How might this help us explore the topic of, say, addition with these children?” and then inviting our learners to play with a question something like: ‘What can you find out about addition and subtraction using Cuisenaire?’

This book develops the kind of tasks we worked with at the ATM conference; starting from Early Years. To begin with we have suggested some starting points for beginning your work with Cuisenaire. Following this, we have collected ideas together which focus on specific curriculum themes, such as Counting, Sequences, Patterns and Algebraic Reasoning, Fractions, Ratio and Proportion, Addition and Subtraction (additive reasoning), Multiplication and Division (multiplicative reasoning) and finally present some problems that help learners to see the interconnectedness of mathematics. We’ve used photographs of children’s work with the rods; remember, these are often ‘work in progress’ rather than perfect finished pieces.

Enjoy your Cuisenaire!
2.1 Working with ‘older’ learners

In Early Years settings children’s engagement with mathematics is often described as a process of ‘play’. However, we believe this play approach to mathematics is no different whatever the age of the learner; perhaps the only difference is one might call this process “exploration” or “investigation”. Nevertheless, the process by which mathematics is learnt can be more powerful if learners are enabled to play with ideas first of all; whether it be Cuisenaire, linking cubes, Geoboards, Geostrips, Dienes base 10 apparatus, ATM MATs or dynamic software such as GeoGebra.

As an example, using a geoboard (and an elastic band) for the first time with a class; giving them five or so minutes to play with the equipment and observing what happens and pretending not to hear any questions they might ask. Then at the end of this period of play asking them to explain what they had been doing, picking up on their explanations and making questions for them to work on during the rest of the lesson and in subsequent lessons.

For us it hardly matters whatever age the learner are! What differs is likely to be the complexity of the mathematics about to be worked on.

2.2 Working with younger learners

With the youngest children there are two aspects to Cuisenaire work that work alongside each other. Familiarity through play and discussion, and ‘circle sessions’ led by an adult. The informal play is where we observe what children are doing and what interests them (standing the rods up, building, picture-making, organising etc.). We can build on these observations encouraging them to vocalise their ideas, extend and develop them (Have you used every rod? Do you think you could build a house using only two colours of rod? etc.), and try what others are exploring. The informal play does not stop when the circle tasks are introduced; rather it is essential this continues alongside more adult-led sessions aimed at familiarising children with the relative sizes of the rods and introducing new ideas.
2.2.1 Play

Place a full box of Cuisenaire on a tablecloth or picnic rug and encourage children to build and make patterns individually and in pairs; and to describe, reproduce and develop these another day. Taking photographs of what they have made and leaving these out with the Cuisenaire inspires more play and discussion.

In completely free play children will bounce ideas off one another, as well, of course, as coming up with ideas that are completely their own. We will watch closely, take photos and return to some of the ideas that are thrown up, asking the class to try to replicate or extend particular ideas.

A 5 year-old makes “a sandwich”. At first, it is not complete as the white rod on the right is missing. Upon asking: “What’s missing?” the reply came - “Cheese” and the child proceeded to add in the white rod:

We can see the mathematical possibilities here, as all the fillings have to be the same length as the bread and we can begin to work on equality (see Trains, section 2.4).

A little later, we return to this, borrow the idea and ask all the class to try to “make sandwiches”. Another day, we ask them to play a game: to make an incomplete sandwich and see if your partner can guess what’s missing.

In the same class, no staircases appear spontaneously for three, free play sessions.

We are waiting for a staircase to emerge (Why? See below, Section 2.3). Then, one child, T, makes a rocket. His neighbour, A, made an alien:

Another day, the teacher shows the pictures of their creations: “Do you see that pattern that T and A are creating. Can you try and make that pattern?”

And another day, as the class are reading about going to the moon in a rocket, or climbing up a ladder to the moon, we might ask: “Can you make a way of getting to the moon?” We might ask them to describe what they have made, to explain their choices of rods and to look for similarities and differences.
The unconstrained play of children throws up all sorts of starting points. Adding constraints challenges pupils to try something different. Choosing a mathematical feature to focus on and extend allows the class to explore new areas, while acknowledging their creativity as crucial for both starting and for sustaining investigations.

One of Simon’s 8 year-old learners made a face:
Although, if we take the white rod to have a value of 1, this face was worth somewhat more than 100, the idea of making a face worth 100 was born, simply asking learners to create a face that "adds up" to a hundred. Once made, can they demonstrate clearly their faces do indeed total one hundred? Younger children might of course start building faces with smaller totals.

Questions to ask to develop learners’ play:

- What can you make?
- Why did you choose to put that rod there?
- What can you tell me about the rods?
- What have you found out?
- What do you know?
- What is the same about …?
- What is different about …?
- “Can I make one like …?”
- “Can I make one different to …?”
- “Can I make one bigger than …?”
- “Can I make one that …?”

Handy Hint: Remember to allow plenty of time to return the rods to the tray. Most children love doing this and it is important they are given time to do this and talk together as they do so.

Another day, add plain paper and coloured pens or pencils and encourage children to record what they make: "How can you record that so you can make it again tomorrow / show us later?"

Free writing and drawing is important to begin to make links between symbolism and the actual rods. Cover the Cuisenaire table with large sheets of drawing paper, coloured pens or pencils and encourage children to record what they make. Display these for later discussion. After they have finished and all the rods are back in the box, give out whiteboards or paper and pens and ask children to freely draw and write what they have been learning.

Another day, as an extra challenge, provide centimetre-squared paper and show them how the rods fit into the squares. Can they record what they make, accurately, using the squares?
Codes: Using their free writing, together you can begin to come up with a code to represent the rods with the children; e.g. white = w, orange = o etc.

The most commonly-used codes are:

<table>
<thead>
<tr>
<th>Color</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>w</td>
</tr>
<tr>
<td>Red</td>
<td>r</td>
</tr>
<tr>
<td>Green</td>
<td>g</td>
</tr>
<tr>
<td>Pink</td>
<td>p</td>
</tr>
<tr>
<td>Yellow</td>
<td>y</td>
</tr>
<tr>
<td>Dark Green</td>
<td>d</td>
</tr>
<tr>
<td>Black</td>
<td>b</td>
</tr>
<tr>
<td>Tan</td>
<td>t</td>
</tr>
<tr>
<td>Blue</td>
<td>b</td>
</tr>
<tr>
<td>Orange</td>
<td>o</td>
</tr>
</tbody>
</table>

We encourage them to use this code when they write about what they have been working on and thinking about. Caroline Ainsworth has done much work with primary children symbolising mathematics in this way. An article and some videos are available at: https://www.ncetm.org.uk/resources/28795

Caroline says: “Importantly, the labelling of an operation with a mathematical sign can be agreed between teacher and children in the context of the shared experience of constructing an arrangement of rods.

... A shift of focus is all that is needed to re-label the same arrangement with a different mathematical sign. By encouraging independent ‘free’ mathematical writing, children then explore the algebra of operations (often as a series of transformations), working on all operations at once, and express their discoveries for themselves.”

Thoughts of a 3 year-old (C):

With thanks to Alison Parish, who contributed this idea at the conference workshop.

C was playing with Dienes Base-10 materials:

A truck

A train

A train with ten trucks

C decided to split up the trucks:

Not all the same

Not all the same

Not all the same

The same

Not all the same

Alison added: “This was done with Dienes apparatus but would have been more colourful with Cuisenaire – bigger trucks – and more combinations... Links to Thomas (the tank engine) – Thomas is a small tank engine, Gordon is stronger, so would carry more trucks.”
Goldilocks and the 3 Bears: Read the story to the class. At the next sitting, show the book and ask each child to choose one rod to represent Mummy Bear, one rod for Daddy Bear and one rod for Baby Bear. They each hide their ‘three bears’ behind their backs. You read the story aloud and every time you say “Daddy Bear”, they have to feel and hold up their Daddy Bear rod, when you say “Mummy Bear” they have to feel and hold up their Mummy Bear rod, and so on. This sounds dull but in fact some interesting things happen. Firstly, obviously not all the children choose the same rods and when they hold them up and they are not the same, this stimulates surprise and comment, such as “THAT’S not Baby!” which you can invite them to expand on and justify.

At some point, you will find it useful to ask the whole group to lay out their 3 bears in front of them for comparisons to be made. Play this on more than one occasion and a discussion can be had about what they have chosen differently this time, what they have kept the same and why. Leave a voice-recording of the story on a table with a box of rods for them to play independently.

What is important about this activity? Children make their own decisions and justify these; they begin to feel the relative sizes of the rods as well as look at them and make comparisons of size. They become sure – ask them to explain how they know one rod from another.

Hands behind backs: Each child takes one each of, w, r and g. A caller calls out the colour of a rod in the staircase or tray and participants hold up the rod called out. Callers might call out a colour that is not hidden! Lots of games can be played in a circle over time, with each player taking a few consecutive rods to hide and find by feel alone. Leaving a tray of rods out for them to see makes it easier. These games follow on well from making staircases (see Section 2.3) as you can leave the staircases out for them to visually check what they are feeling.
Identifying rods by touch alone, without a visual clue, is harder. Play this with different sets of three consecutive rods, then extend the hidden set to five consecutive rods.

Once the roll-and-name game (see later section 3.1) has been played and number names attached to the w, r, g, p, y, ... o staircase; the caller can roll a die and call out a number for us to find by feel only (hard).

A film made in 1961 of Caleb Gattegno working with young children shows him asking the children to find rods by feel alone, identifying them by both colour and number value from a complete set of w - o. Don’t underestimate how challenging this is. You may want to watch Gattegno at work: https://www.youtube.com/watch?v=JrMty8v2DqI

Practice makes perfect!

Secret Rod: Hide a rod. Children work in pairs to decide on questions to ask you to discover which rod you have hidden. You can only answer yes or no. Model good questions like “Is your rod longer than the y rod?” and discuss why this is better than asking “is it y?”

Secret Rule: Decide on a secret rule, e.g. every even rod, or every rod longer than y. Draw two columns on a large sheet of paper. Write YES at the top of one column and NO on the other. Children offer rods and you place them in the YES or NO column, depending on whether they fit your rule or not.

2.3 Staircases

2.3.1 Making staircases

Making staircases that go up regularly and back down using one of every rod is a fundamental and important stage in playing with Cuisenaire. A staircase made using one of every rod illustrates clearly each rod in relation to the next. At first with the younger learners, staircases might come in all shapes and sizes.
Encourage description and ask “What is the same … what is different about these two staircases?”

Staircases can be made freely as well as using a base-sheet.

One day you will assign number names to the stairs: “If w is 1, what do we call r? … g? …” etc. “Explain why?”

Play ‘What has changed?’ with a completed staircase. Children cover their eyes while one person does something to change the staircase: either removing a stair (and closing the gap) or swapping the position of two rods. Ask: “What has changed and how do you know?”

2.3.2 Extending staircases above ten

This creates a strong image of larger counting numbers from 11 to 20:

2.3.3 Playing with the idea of ‘stairs’

An 8 year-old child working with Helen built the following 3D staircase.
I might ask: “What is the value of each stage?”
With thanks to Keranjit Kaur for contributing these ideas:

“I began by making a staircase, starting with the smallest and putting the rods in ascending order.”

In pairs, each build a staircase to 10 like this. By turning one staircase around and placing the white on top of the blue we can make an interlocking pair of staircases to make all the number bonds to 10.

We can also make a pyramid with two sets of stairs.

If the top row has a total of 2, what are the other rows worth?

What would happen if I added another staircase in a different orientation?

In pairs, each build a staircase to 10 like this. By turning one staircase around and placing the white on top of the blue we can make an interlocking pair of staircases to make all the number bonds to 10.

This can easily be extended to number bonds of 11, simply by placing the white rod on top of the orange rod, the red rod on top of the blue etc.

How many ways this could be done? What shape has been created?

If the area of the 2 smallest shapes in the arrangement above equals 2 square centimetres, then what is the area of the other L-shapes? What if the L-shapes were part of a rectangle? In Section 3.5 we look at sequences made in different ways. Some of these are the triangular number sequence, the square number sequence and the cube number sequence.
2.4 Trains

Building trains (rods placed end-to-end) is an important aspect of using Cuisenaire rods with any age. This is because learners can see how number sequences based upon the same size rods can be constructed. There are also opportunities for some ‘same’ numbers being made by different coloured trains. We explore this idea a little later.

To help maintain strong links between image and equation it is interesting to ask older learners to represent a written equation with rods. Rod-work need not stop when written equations begin.

We may want to ask them to start with numbers they can make just using white rods. Clearly they can make every number; this could lead to numbers being infinite.

Invite the youngest learners to “make trains” and allow them to select different combinations of rods to string multi-coloured trains together.

Here are some trains which were restricted to using rods in two colours for each different train.

![Image of trains](image)

Use these for comparison and discussion (Longest? Shorter? Same/different?). This can be followed by the introduction of a restriction, such as “What different length trains can you make with only these rods?”

Another development would be to provide lengths of string or wool and invite children to make trains to exactly match its length. Which rods ‘work’ for this length?

The most important ideas to develop in these early explorations are the ability to order more than two items, to recognise and explain differences and similarities, and the development of a shared language for describing what you see: “Say what you see”.

2.4.1 Same-rod trains (early multiplication)

Give a lot of the same colour of rod to a pair of children. Give another pair lots of another colour. And another... and another. What lengths of trains can different pairs make?

For example, those with lots of green rods they can make chains such as:

- Length 9
- Length 12
- Length 6

What lengths can children make if you have a lot of reds, or a lot of yellows?

2.4.2 Different coloured trains of the same length

Both these trains have length 6:

- Red
- Green

Extending this, what is the next time red and green trains make the same length? This would, of course, be at 12. A useful task can be for learners to predict what the next few lengths would be.
Can the same length be made from red and yellow trains?

Try greens and blacks.
Try three colours.
What other questions could we ask about this?

### 2.4.3 Trains of only two rods
(combinations of single-digit additions and sequencing)

Give pairs of pupils one each of the following rods: \{w, r, g, p, y, d\}

The idea is to make ‘trains’ of rods by placing any two, but only two, rods in a line and working out the length of the train.
What is the shortest train? What is the longest train?
Can all the different trains between the shortest and the longest be made?

### 2.4.4 Trains of only three rods

This task is to choose three of the rods from \{w, r, g, p, y, t\}
What are the shortest and the longest trains?
Can all the different length trains between the minimum (6) and maximum (21) be found using three different rods only each time?

Some ‘in-between’ totals can be found in different ways.

For example, to make a train of length 9 we can have:
\[ w + r + y \quad w + g + p \quad r + g + p \]
(as shown in the picture below).

You might want to play with the following two ideas yourself before trying them with your learners!

### 2.4.5 Advanced train play – a coding problem

This idea has been adapted from Jackie Fairchild’s amazing opening keynote at the ATM conference.

When working in base 3 the first three column headings are, in place value order, Nines, threes and ones (\(9\ 3\ 1\)). As with base 10, where the largest number we can have in any column is 9, then for base 3, the largest number we can have in any column is 2.

So, in base three, 222 is worth: two nines + two threes + two ones which is equal to 26 in base ten. 26 of course just happens to be the number of letters in the alphabet.

For this task we can use up to two of each of the three colours B (9), g (3) and w (1), and, using only addition, therefore, we can make all the numbers up to 26. For example, the letter H, which is the 8th letter of the alphabet, would be represented by:
This idea can be further adapted to using just one each of the four colours $t$, $p$, $r$ and $w$ to make all the counting numbers up to 15; thus the emergence of binary arithmetic. Here is a ‘tidied up’ set of solutions from 1 to 8:

2.4.6 More advanced train play

Making the counting numbers by addition trains:

For example, by allowing ‘repeats’ or rearrangements such as 2,1 and 1,2, the total number of possible arrangements doubles with each counting number. Thus the powers of 2 emerge from this pattern.

What do you notice? Why do you think this is? See a blog post by Simon on this:

http://seekecho.blogspot.co.uk/2015/11/ways-of-adding-up.html

However, by not allowing repeats we find the following ‘partitions’:

This is starting to look like the Fibonacci sequence; but is it?

These could be questions for GCSE or A Level students to explore.

An article from the late Dick Tahta from MT113, December 1985, and reproduced below, illustrates these compositions beautifully:
Commonly, particularly with older learners, number sequences and patterns are explored purely numerically. But by simply looking at strings of numbers you are missing a trick. Investigating and representing patterns such as Fibonacci using Cuisenaire brings to us a whole new level of awareness and depth of understanding.

In this section, the ideas are arranged broadly from less to more ‘difficult’ in terms of the mathematics involved.

3.1 Assigning numerical values

At first some adults express discomfort that the rods are not marked in some way to indicate their value. In fact, you begin to realise this is their very power. To find what the \( y \) rod is worth we have to relate one rod to another, for example, by encouraging children to place other rods next to the \( y \) rod until they are sure what each is worth. Later we might work ‘as if’ the white (\( w \)) rod was worth 1. Even later we may want to call other rods “1”, e.g. the orange (\( o \)) = 1 or the pink (\( p \)) = 1. I can ask: “What does that make the other rods worth and how do you know?” This idea is developed further in Section 4.

Roll and name: Each player takes it in turns to roll a 1-6 die. They read the number shown on the die roll and take that many \( w \) rods, laying them as a train (line). They say the colour of rod that they think matches the length of their \( w \) train before taking it from the tray to check. If they are correct, they return the \( w \) rods to the tray and keep the coloured rod. Maybe the longest train after five turns each wins?

Developments: Roll two dice and collect in the same way. The object is to make a staircase to ten. Or roll three dice and make a staircase to 18.

Boats (With thanks to Janine Blinko): A ‘boat’ is made of any three rods on top of each other, with the shortest on the top and the longest on the bottom.

What different boats can we make? Can we use all the colours? What is the same about any two boats? Now tell me what is different?

Boat Race: Consider \( w \) to be equal to 1. Take it in turns to roll a die and take the matching rod. Start to build a boat from the bottom up. You can only keep the rod if it fits on top of the rod(s) you already have (to make a boat consisting of three different lengths of rod) or you can start a new boat. The winner is the first to make five perfect boats. Talk about and compare the boats.

3.2 Counting in tens

Putting lots of orange rods end-to-end in a train is a strong image to support counting in 10s up to and over 100. Then count backwards by removing the rods. Implicit in this removal of rods is the concept of subtraction. To count in 10s from different starting points, start your train with one shorter rod and add on orange (\( o \)) rods; e.g. 2, 12, 22, 32, 42 …
3.3 Cuisenaire with tracks or metre sticks

Try laying numbers of one colour of rod into a Cuisenaire track:
https://www.youtube.com/watch?v=jl-FhxOXaRQ

Alternatively, lay them alongside a metre stick, to illustrate multiplying a number of reds to make, say, 20; then relate to division by asking questions such as, “So, how many 2s are there in 20?” Encourage children to write the related equations:

\[10 \times 2 = 20\]
\[20 \div 2 = 10\]

3.4 Building and interpreting a growing pattern

How, with a staircase, could you show the way the Romans wrote numbers?
Here’s one way an 8-year-old child chose to show it:

As part of an exploration of different base systems, children could build a staircase that shows how numbers would be counted in base 6, 7, 8 or 9. For instance, here is another 8 year-old’s pattern in base 6:

Create a growing pattern of Cuisenaire rods:
This is an invitation that can be interpreted widely by learners. Keep each step of the sequence. Label them Step Number 1, Step Number 2, …
“What would the first step in the sequence be? How does the pattern grow? What number does it grow by each time?”

Later, we could ask older learners to plot their sequence on a graph – the Desmos Online Graphing Calculator is a free, user-friendly graphing tool that could be used to do this – https://www.desmos.com/calculator

They could write, in symbols, the total rods for step ‘n’: “Can you work out what the 43rd step in the series would be?”

3.5 Triangular, square and cube numbers

Play with staircases can be developed to access triangular numbers, square numbers and cube numbers. Triangle numbers:

Of course the number 1 is also a square and a cube number.

The triangle drawn around the outside of each arrangement is only intended to illustrate the triangular-ness of 1, 3, 6 and 10. As such these triangles have been removed in the diagrams below.

By revisiting the process of turning a staircase around and placing it at right-angles to the original staircase, we gain the following rectangles:

This brings out the basic structure of how any triangle number is half of the area of a rectangle formed by joining two of that triangle number, so $3 = 3 \times 2 \div 2$ and $6 = 4 \times 3 \div 2$. 
Here are similar but different staircase diagrams to aid children’s understanding of square numbers.

So staircase 1 = 1
Staircase 2 = 1 + 2 + 1 (which is 4)
Staircase 3 = 1 + 2 + 3 + 2 + 1 (which is 9)
There are, of course, other ways of building square numbers with the rods:

And

Normally cube numbers appear in the secondary curriculum, but, interestingly, by using Cuisenaire rods, the idea of cube numbers can become accessible for younger learners to explore.

Build successive cubes like this:

How might all these cubes be combined into a square?
We could start by surrounding the white with the four reds:

“How might this continue?”
“If I make a cube from pink rods, can I make a square from all the w, r, g and p rods?”
“What do you notice about the square?”
3.6 Making equations

Creating things physically and visually can quickly lead on to representing them with symbols, even for very young children. This allows them to easily record what they have found, and after a while, to express mathematical ideas without needing the rods. Pairs of pupils need to be provided with:
5 white rods, 2 reds, 1 green, 1 pink and 1 yellow; pens and paper.

This game is to find all the combinations of Cuisenaire rods that make the same length as the yellow rod. For example:

Or, emphasising the same length, we could place them on top of each other:

Young children might begin just by listing, or drawing, the rods that make trains the same length as the yellow rod, as in this recording of findings made by five-year-olds:

With a little more experience, children will be able to represent the ‘equation’ for this arrangement as, for instance, \( g + r = y \).
It could also be \( r + g = y \) or indeed \( y = g + r \) or \( y = r + g \).

How many different equations are there when the answer is yellow?
How many different equations are possible if the answer is pink?
This question could lead to subtraction:
\( y - w = p \).

Making two trains the same length, students could write equations that represent them. How many ways can they find?
They could then write equations without using the rods.
For older learners, this idea could be turned into a ‘Boggle’ type game, where pupils, say in groups of threes, could be asked to write as many different equations as possible when a specific colour is the given ‘answer’ and in a given length of time (say 5 minutes). Each group then reads out their collections of equations and marks are given for unique solutions. Higher marks could be given for longer equations.

3.6.1 Mystery Equations

In pairs or small groups, children choose one secret rod. They write clues to identify the secret rod. Children swap clue sheets with another pair/group and solve try to solve each other’s mystery equations ...

3.6.2 Squares and equations

A lot of interesting equations and patterns emerge from looking at squares made of the rods. Make equivalent squares. Record them.

There are square trays for Cuisenaire rods (currently available from OUP: https://global.oup.com/education/product/9780198487128/). These are not really essential – you can make squares anyway – but they make all sorts of exploration easier. Explore possibilities with a set of squares. Record what you’ve found, what you’ve noticed. Reflect on what you have found.

Is there a general pattern in your squares? Does it continue? Can you make a general claim about this? Can you explain why it would always be true?
These three concepts are ways of comparing parts and wholes. Similar tasks can be done with fractions and ratios, although they are denoted differently when written.

### 4.1 Finding fractions

What different fractions can you find in a box of Cuisenaire?
Can you find halves? Thirds? ...
Development: Make fractions with pairs of the following colours.

- 
- 
- 
- 

For example this arrangement is one half. It could be

- 

This arrangement is three fifths:

- 

Why?

How many different fractions can be made?

Have you found them all? What about fractions greater than 1?
An extension task could be to order the fractions from smallest to largest. The largest fraction the children eventually found, in the image below, was $\frac{6}{1}$, and $\frac{5}{1}$ was the next largest.

Another approach to the same thing could be, to call a different rod to the white rod “one”. What fraction would the other rods be?
In the picture below the pink rod has been designated as “one”:

This is one of the powerful things about Cuisenaire rods, that ‘the white rod isn’t always, and shouldn’t always be, one’.

Are there fractions which can be written in more than one way?
4.2 Ratios

Rods can be compared in terms of ratio as follows:

\[
\begin{align*}
\text{white} & \text{ to } \text{ red} \text{ is } w : r \text{ which is } 1 : 2 \\
\text{red} & \text{ to } \text{ green} \text{ is } r : g \text{ which is } 2 : 1 \\
\end{align*}
\]

What is the ratio of \(w : g : p\) in these two pictures?

\[
\begin{align*}
\text{white} & \text{ to } \text{ green} \text{ to } \text{ pink} \text{ is } w : g : p \\
\end{align*}
\]

If we have \(r, g, p, y, d\), what ratios could you make? Which are equivalent?

This could be extended to many questions such as: What is the ratio of \(g\) to \((y + p)\)?

4.3 Proportionality

In the above ratio of 1:2, the proportion of white is 1 out of 3 and the proportion of red is 2 out of 3, we could try giving these proportional results to learners and ask them to turn other ratios similarly into proportional amounts.

For example the ratio of

\[
\begin{align*}
\text{green} & \text{ to } \text{ pink} \\
\end{align*}
\]

is 3:4 but the proportion of green is 3 out of 7 and the proportion of pink is 4 out of 7.

We may wish to develop this idea by using three rods, e.g. what would the proportional amounts of \(w, r\) and \(p\) be?

\[
\begin{align*}
\text{white} & \text{ to } \text{ red} \text{ to } \text{ green} \text{ to } \text{ pink} \text{ is } w : r : g : p \\
\end{align*}
\]

4.4 More ratio using Cuisenaire rods as bar models

If blue represents 18, what are the values of the pink rod and the yellow rod?

\[
\begin{align*}
\text{blue} & \text{ to } \text{ pink} \text{ to } \text{ yellow} \\
\end{align*}
\]

If blue now represents 36, what are the values for the pink and the yellow rods?

If blue represents 378 …

If I start with blue rod and subtract the yellow, what will remain?

How many different ways can you make the relationships between \(p, y\) and \(b\)?

What different whole numbers could the following rods represent?

Make up some numerical relationships between \(p, y, w\) and \(o\)? For instance:

\[
\begin{align*}
\text{purple} & \text{ to } \text{ yellow} \text{ to } \text{ orange} \text{ to } \text{ brown} \\
\end{align*}
\]
Play with ‘trains’ (section 2.4) is an important precursor to what follows here.

**5.1 Number bonds to 10**

Each child is given one orange rod and has to find all the possible ways of making any two rods the same length as the orange rod.

By providing white-boards or paper and pens we can encourage children to record what they are making.

They might be encouraged to place all their answers in a systematic order. Here we are looking for, and organising, all the pairs of numbers that sum to 10.

A next problem is to find all the different ways of making the same length as orange using any three colours (repeat colours are allowed). How many ways are there of using three colours? What about four colours?

**5.2 Differences**

Pairs of children take one rod each and lay them alongside each other in order to look at the gap between them and therefore find the ‘difference’. For example, lay the dark green rod and the pink rod alongside each other; the difference between them will be the red rod.

It’s often helpful to model the language we use to describe this:

“The difference between the dark green rod and the pink rod is the red rod.”

This could lead to a spot of algebraic coding, e.g. $d - p = r$. More on this below.

As children make the shift to numerical values for the rods the following game can be played:

Same-Difference game: Which pairs of rods can you find where red is the difference between them?

A next step could be to record these as subtraction equations:

$4 - 2 = 2$ or $7 - 5 = 2$ etc.

Children might be encouraged to place all their answers in a systematic order.

They might see a pattern in the pairs with the same difference. Can they explain how the pattern works? This might involve them in making a “claim” about equivalent subtractions. Can they explain why they think that claim is true? The image below shows, at the top, the teacher’s scribing of children’s description of what they had found with the rods with the difference of three. The teacher asked, “What do you notice?” Some children noticed the pattern could carry on “forever”; what Marie noticed she wrote in the speech bubble.
5.3 A problem using difference and addition

This problem is a development from the idea on p17. Here we can use both addition and differencing with combinations of one each of a Blue, a green and a white.

Using one Blue, one green and one white, it is possible to make a total of 13.

i.e. \( B + g + w \)

To make a total of, say, 6 we can use this arrangement, where the gap is 6
To make 1 we can just use the white

The challenge is to make all the counting numbers from 1 to 13 in this way.
Is there a pattern that emerges? If so, could it be extended, without rods if necessary?
It is possible choose a fourth number so you can make all the counting numbers up to 40; can you make them, as above, using adding and differencing?

5.4 Biggest number that cannot be made

This problem is about exploring which is the largest counting number that cannot be made when using different numbers of just two colours of rods and addition only.

For example, if I choose green and black rods:
I cannot make the number 1 or the number 2.
I can make the number 3 with one green rod.
I cannot make the numbers 4 or 5, but I can make 6 and 7

In the table below are these and some more results.

<table>
<thead>
<tr>
<th>Numbers</th>
<th>Numbers I cannot make</th>
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It is beginning to look that after 11 I can carry on and make all the counting numbers. Is this true? If it is then 11 would be the biggest number we cannot make using green and black rods.
Explore the biggest number that cannot be made for other pairs of rods.
6.1 Multiplication ‘floors’

This idea is to make floors, using rods longer than the white (w) rod.
For example, we can lay three 5-rods together to make a ‘floor’ then look to make a same size floor from other rods that fit exactly on top. In this case, five 3-rods.
From this the equations $5 \times 3 = 15$ and $3 \times 5 = 15$ are created.
So, 15 has the following pair of ‘floors’:

6.2 Products and partner pairs (PPPs)

This idea builds upon the ‘floors’ idea above but includes the use of the white rod.
PPPs might be found using Cuisenaire rods.
Number 8 for example has the following PPPs:
$8 \times 1, 1 \times 8, 2 \times 4$ and $4 \times 2$

Moving on, finding pairs of whole numbers whose product is 24 will create more possible pairs and the following questions might be asked:
How many different pairs can you find? Are you sure you have found them all?
Can you write down your reasoning to explain why you are sure you have found them all?

The task is to explore how many pairs of floors different numbers can have.
6.3 Turning PPPs into rectangles

Make some rectangles with Cuisenaire rods which have an area of 24. Some examples could be:

- $8 \times 3$
- $6 \times 4$
- $4 \times 6$

What other rectangles can be made? As before, we can allow the use of white rods.

6.3.1 Turning PPPs into coordinate points and plotting them on a graph

Imagine laying the $4 \times 6$ PPP on a coordinate grid:

We mark on the coordinates of the top right corner of the PPP.

All the PPPs above can be turned into these coordinate pairs: $(8, 3)$, $(4, 6)$ and $(6, 4)$.

There are several other coordinate pairs to be found for 24, and by plotting these on square grid paper an interesting-shaped graph emerges about which we can ask questions as and when learners are ready to go deeper:

What observations do you have about the graph?

Allowing partner pairs which do not need to be whole numbers (and, therefore, moving away from using Cuisenaire), if one of the pairs of numbers is 10, what could its partner be?

If we know what the partner for 10 is, what would be the partner for 5?

What would be the partner for 2.5?

What would be the partner for 1.25?

How can we work out the product partners for 1.5 and 0.75?

What about the product partners for 0.6 and 1.2?

In KS4 and KS5 classrooms the name and the equation of the graph so produced can feature, as can the notion of asymptotes. Also, that there is a reflection of the graph in $y = -x$, the rest of the graph appearing in the third (negative, negative) quadrant. This is an example of how a simple idea, beginning with Cuisenaire rods can be developed through to an important A-level concept.
6.3.2 Heading towards the idea of a square root

The concept of a square root may be considered a relatively complex idea for KS2 children. However, the photograph below might stimulate some thoughts about working on this concept in a visual way:

Again, here we have a concept having its roots (!) in Cuisenaire rods yet leading to a concept frequently taught in KS3.

Furthermore, by returning to the partner pair coordinates in Section 6.3.1 to include the newer set of coordinates such as (10, 2.4), (5, 4.8), (2.5, 9.6) etc., plotting these will reveal a hyperbolic graph.

Can you find the partner pair where both values on the graph are the same?

Plot this partner pair on your graph.

What do you notice about the position of this point on the graph?

6.3.3 Examining the perimeters of rectangles with a constant area

Continuing further with rectangles with a constant area of 24 and examining the perimeters of these rectangles, we can find coordinate pairs formed by one of the dimensions with its perimeter.

In doing so the integer solutions are:

(1, 50), (2, 28), (3, 22), (4, 20), (6, 20), (8, 22), (12, 28), (24, 50)

A KS5 class could be presented with this problem in order to find the resulting function; which itself has an interesting asymptote.

6.3.4 Creating product trios

This is an extension task from partner pairs to partner trios.

Staring with 24, a partner trio could be 2, 2 and 6. There are of course others: using whole numbers only, what are they?

By turning each set of three numbers into the dimensions of a cuboid, the following questions could be asked:

What is the volume of each cuboid?

What is the surface area of each cuboid?

If we allow partner trios which do not have to be whole numbers, try to find which trio will produce the cuboid with the minimum surface area.
6.3.5 Cuisenaire and ‘hollow’ rectangles with a common perimeter

Make some hollow rectangles on cm square grid paper with a perimeter of 20

Some examples are:

This hollow rectangle has dimensions 8 by 2

This hollow rectangle has dimensions 6 by 4

How many hollow rectangles can be made with a perimeter of 20 using rods?
To develop this, turn the dimensions into coordinate pairs and plot these on a graph.
What kind of graph is made?
Again, moving away from whole numbers and from using rods, make some more rectangles which have a perimeter of 20.
What are the areas of these new hollow rectangles?
For example, a rectangle with dimensions $7\frac{1}{2}$ by $2\frac{1}{2}$ will have an area of $18\frac{3}{4}$ square units. This is the area the rectangle which, in turn, can be calculated using square grid paper and counting squares and part squares or by multiplying the dimensions together:

So, using the rectangle drawing approach, the quarter-size area shown by ■ and the half size area ■ can be seen as pictorial representations of the grid method. Given that the picture has the same structure as the multiplication method, it is a mathematical travesty how the grid method has, politically, become discouraged in some places!

As before, students can plot other, different coordinate pairs, this time using one of the dimensions and the area of the rectangle. So, with the two hollow rectangle pictures above we gain the coordinate pairs $(8, 16)$, $(2, 16)$, $(4, 24)$ and $(6, 24)$.

As further coordinate pairs are plotted, what does the graph look like?
What dimensions will maximise the area of the rectangle with a perimeter of 20?
Again here, gathering data and plotting coordinates is not beyond some KS2 learners. However, a final task, to determine the equation of the graph, would be more appropriate in KS4 classrooms.
6.4 *Remainders, divisibility, factors and primes*

Choose one of the longer rods and try to make same-colour trains using each smaller rod. If they don’t fit, fill in the gap with a remainder, e.g.

If we regard the orange as number 10, which numbers ‘fit’ exactly, without remainders?

The diagram above starts with the ten rod and shows how there are no remainders with trains made from the \(w\), the \(r\) or the \(y\) rods but there are remainders for trains made with the \(g\) and \(p\) rods. Clearly, all rods which are longer than yellow, but shorter than orange will, therefore, also leave remainders.

What do similar train arrangements for other rods, i.e. the Blue, the tan, black and the dark green, look like?

Some colours/numbers have very few trains without remainders, while others have several solutions. What do you notice?

Some colours/numbers only have two solutions without remainders.

For example, the yellow rod only has two solutions:

Again, when children have found all the colours/numbers up to orange, each of which have only two train solutions, i.e. \(r\), \(g\), \(y\) and \(b\), they can be challenged to find further two-train solutions of length greater than ten. For example, eleven:
Section 7 – Some problems that link many areas of mathematics

7. Three problems which bring together several areas of mathematical content knowledge and mathematical thinking

This final section offers ideas which begin with Cuisenaire rods and provide opportunities to extend into more complex areas of mathematics.

7.1 Hollow triangles

This task is described in Points of Departure 4 (task 39) as ‘Sticky Triangles’:
https://www.atm.org.uk/Shop/Primary-Education/Primary-Education-Books/Books--PDF-Downloads/Points-of-Departure-4---PDF/dnl006

It has been adapted, here, for Cuisenaire rods.

The PoD4 description begins: ‘Using twelve cocktail sticks what triangles can you make?’

Using Cuisenaire rods instead of cocktail sticks, how many hollow triangles can we make with rods which total 12?

One example is to use one red and two yellow rods (r, y, y).

Another example, shown below, uses green, pink and yellow (g, p, y).

And another (p, p, p).

Discussing the names/types of triangles formed and trying to show there are, actually, only three solutions for a perimeter of 12 (P = 12) will promote students’ reasoning skills.

Interestingly, when P = 11 there are more triangles (well one more!) than for P = 12.

Exploring the number of different possible triangles presents an interesting challenge.

For a perimeter of 13, the following configurations are possible: (d,d,w or 6,6,1), (d,y,r or 6,5,2), (d,p,g or 6,4,3), (y,y,g or 5,5,3) and (p,p,y or 4,4,5). This produces three isosceles and two scalene triangles (see further idea re Pythagorean triples below).

Searching for different numbers and types of triangles as the perimeter varies can present learners with much to think about.

One possible route could be to classify perimeters according to whether they have remainders of 0, 1 or 2 when divided by 3; thus modular arithmetic (Mod 3). Without wishing to give too much away, perimeter values with remainder 0, in mod 3, will always produce one equilateral triangle...

For post-trigonometricians, working out the angles and the areas of triangles can provide plenty of opportunities for students to use and apply trigonometry in a problem solving context.
7.2 Pythagorean triples

Thanks to Mary O’Connor who contributed this idea in the workshop.

Mary produced the picture below on the left:

The picture on the right is a reconfiguration of the picture on the left. It is ‘the’ iconic 3, 4, 5 right-angled triangle and the first Pythagorean triple.

For the 5, 12, 13 Pythagorean triple, the square on the hypotenuse can be formed from 5 yellows + 12 orange and 12 red rods (so $5 \times 5 + 12 \times 12 = 169$)

By examining the areas of squares drawn on the triangles, created from hollow triangles, KS3/KS4 students can work on the inequalities when the areas built on two sides of a scalene triangle are greater than or less than the area built on the long side of the (scalene) triangle. Of course Pythagoras’ theorem is based upon the equality of the areas of the sum of the squares built on the sides of right-angled triangles, whether scalene or isosceles.

Exploring Pythagorean triples in greater depth can lead to much conjecturing and generalising; thus, starting with the 3, 4, 5, followed by the 5, 12, 13, what comes next, and next and so on … and what about the 8, 15, 17 triple?

7.3 Modular Arithmetic leading to Group theory

This final idea could easily sit within the domain of sequences. It has been used in mathematics classrooms from Year 3 to undergraduate. We have chosen to place it here, at the end, as a reminder of the power of Cuisenaire rods as a tool which can span the age and mathematical complexity ranges.

Again the central task has a beautiful simplicity, yet engaging with ‘group theory’ is not usually thought of as a concept for young children to engage with. It is described in the ATM publication: Fletcher, T., J., (1964) Some lessons in mathematics: A handbook on the teaching of ‘modern mathematics’ by members of the Association of Teachers of Mathematics, Cambridge University Press.

The task proceeds as follows:

Everybody sits in a ring of chairs (all desks pushed around the sides of the room)

Numbers from 1 upwards written on a piece of A5 card and box of Cuisenaire rods

Everybody is given a number from 1 upwards. Number 1 is given a white rod. Number 2 is given a red rod. Number 3 is given a green rod. Number 4 is given a pink rod. Number 5 is given a yellow rod. Number 6 is given a white rod. Number 7 is given a red rod … and so on.

Once everybody has a number, ask all
the whites to stand up and call out their numbers, which should be 1, 6, 11, 16, 21 …

What does anyone notice about these numbers?

What colour would 91 be? What colour would 396 be? etc

In turn all the reds, greens, pinks and yellows are asked to stand up and call out their numbers. The teacher might choose to record the first few results on the board:

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There are clear opportunities here for pattern spotting and generality and some questions could be:

- What colour would number 96 be?
- How do you know?
- What number would a red rod be just less than 1000?
- How do you know?

If we want students to experience a context involving negative numbers a further question could be:

What happens if we write new rows underneath the 1, 2, 3, 4, 5 row?

One ‘nice’ way of describing the pink numbers is they are all one less than their immediate yellow neighbour.

Given the yellows are multiples of 5, then the pinks are multiples of 5 less 1.

Likewise the greens are multiples of 5 less 2.

By drawing up a two-way addition table all possible pairings can be shown:

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Similarly, with a multiplication table:

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Using the same idea with KS2 or KS3 students, they might colour in the grids instead of filling them with symbols and this nicely brings out the properties of each table.

Suppose we extend the number of colours to mod 6: \{w, r, g, p, y, d\}?

What is the same and what is different to the addition and multiplication grids in mod 5?

As things stand, there is nothing here that would be beyond many in KS2 or KS3 students, however, as mentioned earlier, this could be a beginning of the study of group theory for undergraduates.
And finally

We hope you too can appreciate the role manipulatives in general, and Cuisenaire rods in particular, play in the teaching and learning process for all learners, and you too can feel what we feel - the power of Cuisenaire rods and the joy of teaching, which can arise from seeing all learners develop as problem-solvers and mathematicians.

About the authors

Helen has a special interest in Early Years mathematics and all mathematics teaching that starts with the learner.

Simon is a primary teacher, especially interested in how to set up lessons where children can build understanding for themselves.

Mike, having spent 23 years in secondary classrooms, has become interested in the development of conceptual understandings which emerge in primary classrooms.

About this Book

Cuisenaire - from Early Years to Adult
Published March 2017
ISBN 978 1 898611 97 4

Association of Teachers of Mathematics
Vernon House 2A Vernon Street Derby DE1 1FR

An additional set of whiteboard slides have been written to accompany this book, as part of a bundle or available to buy separately.

Acknowledgement

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(Educational Solutions (UK) Ltd).
Photographer: Owen Lucas Photography.

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Cuisenaire - from Early Years to Adult

Mike Ollerton, Simon Gregg and Helen Williams

Whether you work with young children or older learners the use of practical materials is fundamental to developing conceptual understanding.

Cuisenaire rods predate the bar model by several decades and are the concrete model that underpins bar modelling. The book illustrates how you can use Cuisenaire rods with your learners, whatever age they may be. The diagrams give the detail of how the mathematical concepts are represented while the photos give authenticity, showing them being used by learners.

An additional set of whiteboard slides have been written to accompany this book, as part of a bundle or available to buy separately, to help you transform your learners’ understanding of the structure of number and calculation and focus on what is being learnt.

Use the ideas to deepen the understanding of your learners so they can master the structure of the number system.

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ISBN 978-1-898611-97-4

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www.atm.org.uk
What different ways can two staircases be joined together?
What do you notice about these trains?
When is the next time a red train and a yellow train make the same length?

When will a green train be the same length as a black train?
Choose three colours to make some equal length trains of your own
Use one each of the four colours each time: tan, pink, red and white
If \( w = 1 \), how can we make all the counting numbers to 15?
These are ‘boats’
Make some boats

What is the same about your boats?
What is different?
Here is a growing sequence
Make a growing sequence of your own

Describe it
What can you see?
What are the next numbers in this sequence?
What will the next few staircases look like?
If $w = 1$, what is the value of each stair?
Build successive cubes like this
How might this pattern continue?
Try to make some equations of your own
What do you notice?
What do you see?
Turn each into numbers
What else do you notice?
What will the next one be?
In this picture, $p = 1$

If $y = 1$, what would blue (B) be equal to?
What is the ratio of $w : g : p$ in each row?
What is the ratio of $w : g + p$?
What can you say about the proportions of each colour to each whole?
What are the proportions of pink, yellow and white in this picture?
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<td>5</td>
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<td>6</td>
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<td>9</td>
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<td>15</td>
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</table>
Which is the largest number that cannot be made when using combinations of green and/or black rods, and addition only?
Number 8 (tan) has the following partner product pairs:

- $8 \times 1$
- $1 \times 8$
- $2 \times 4$
- $4 \times 2$

Find the partner product pairs for other colours of rod.
These are rectangles made with Cuisenaire rods that have an area of 24

- $4 \times 6$
- $8 \times 3$
- $6 \times 4$

Can you make any more rectangles with an area of 24?
We can turn rectangles into coordinate pairs and plot these on a grid.
Here are some more coordinate pairs for 24:

\[(4, 6), (6, 4), (8, 3)\]

How many more pairs of coordinates can be drawn?
What might happen for rectangles with an area of ...?
Here are two hollow rectangles, both with the same internal perimeter.

- This hollow rectangle has dimensions 8 by 2.
- This hollow rectangle has dimensions 6 by 4.

How many more hollow rectangles can be made with the same internal perimeter?
Choose another perimeter and explore
If we call orange 10, explore which numbers fit exactly without remainders?
These are the different hollow triangles that can be made with an internal perimeter of 12.
Choose a perimeter less than 24

Partition this number into three in different ways

Which of these triples make triangles?
Cuisenaire rods

- White
- Red
- Green
- Pink
- Yellow
- Dark Green
- Black
- Tan
- Blue
- Orange
TRAINS

COMPOSITIONS
Choose a rod
Make equivalent trains

How many?

\[
\begin{array}{cccccc}
\text{n} & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{C(n)} & 1 & 2 & 4 & 8 & 16 & 32 \\
\text{G(n)} & 0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

General formula? Proof?

RESTRICTED COMPOSITIONS
Choose a rod
Make equivalent trains using just two rods

Try \( C_3(n) \) \ldots and so on

LIMITED COMPOSITIONS
Choose a rod
Make equivalent trains using only white rods and red rods

Try \( C_{w,r}(n) \) \ldots and so on

RESTRICTED COMPOSITIONS (sums)
...equivalent trains using up to two, or three, \ldots rods

\[
\begin{array}{cccc}
\text{n} & 1 & 2 & 3 \\
\text{G(n)} & 1 & 1 & 1 \\
\text{+C(n)} & 2 & 3 & 4 \\
\text{+G(n)} & 1 & 2 & 4 \\
\end{array}
\]

PARITIONS
Choose a rod
Make equivalent trains using any rods but not counting different orders

\[
\begin{array}{cccccc}
\text{n} & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{p(n)} & 1 & 2 & 3 & 5 & 7 & 11 \\
\end{array}
\]

Are these primes?

RESTRICTED PARTITIONS
Choose a rod
Make equivalent trains using only two rods and not counting different orders

\[
\begin{array}{cccccc}
\text{n} & 1 & 2 & 3 & 4 & 5 \\
\text{p_2(n)} & 0 & 1 & 1 & 2 & 2 \\
\end{array}
\]

Try \( p_2(n) \) \ldots and so on

LIMITED PARTITIONS
Choose a rod
Make equivalent trains using only white rods and red rods and not counting different orders

\[
\begin{array}{cccccc}
\text{n} & 1 & 2 & 3 & 4 & 5 \\
\text{p_{w,r}(n)} & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

Try \( p_{w,r}(n) \) \ldots and so on

Is there a connection? Yes! Here's Euler's argument

5 pegs can be arranged on a pegboard in

\[
\begin{array}{cccccc}
\text{1 row} & 2 rows & 3 rows & 4 rows & 5 rows \\
\end{array}
\]

Read acrossways \rightarrow \text{partitions into} \not more than 3 parts (restricted partitions (sum))

\[
\begin{array}{cccccc}
\text{p}(n,3) & = & p(n) + p_2(n) + p_3(n) \\
\end{array}
\]

In general, \( p(n,m) = p(n) + p_2(n) + \ldots + p_m(n) \)

Extensions of the partion problem can be made by various restrictions on the parts. For example, how many ways are there of giving change for £1 using 5p, 10p and 50p coins only?

Much of classical number theory was devoted to similar additive problems: What numbers can be expressed as a sum of two squares? Can a square number be expressed as a sum of triangular numbers? Can every even number be expressed as a sum of two primes? How many ways? And so on.

SPACE DIVISIONS
...regions created by n cuts

\[
\begin{array}{cccc}
\text{n} & 0 & 1 & 2 \\
\text{1 dim} & 1 & 2 & 3 \\
\text{2 dim} & 1 & 2 & 3 \\
\text{3 dim} & 1 & 2 & 3 \\
\end{array}
\]

SPAGHETTI
(i-3)

PANCAKE
(2-d)

BUTTER
(3-d)

Another additive problem, that can be intriguingly studied with coloured rods: make trains using only two chosen colours - what lengths can you make? What lengths can you not make?

(This is a problem of concern to architects when finding what spaces can be filled with modules of two standard lengths - or in deciding what two modules would be useful ones to choose.)