

Μ 105

Ανάλυση και Μοντελοποίηση Δικτύων

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Στόχος

Παρουσίαση

Θεωρίας, εργαλείων και μεθοδολογίας
ανάλυσης και μοντελοποίησης
στοχαστικών συστημάτων
τηλεπικοινωνιακών δικτύων.

- **Basics of Stochastic Processes (including Markov)**
- **Basic Network Modeling, Performance Evaluation and Design**

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Υλικό Μαθήματος

Διαλέξεις

- .ppt διαφάνειες
- Βιβλιογραφία (συμπληρωματικά)

e-class link

<https://eclass.uoa.gr/courses/D209/>

- γραφτείτε αν δεν το έχετε κάνει ήδη!

Εξέταση/Αξιολόγηση

- Ερευνητικές εργασίες - παρουσίαση άρθρων (25%-30%)
- Γραπτές εξετάσεις (70%-75%) στο τέλος του εξαμήνου, Κατά κανόνα, γίνονται με κλειστά βιβλία.

PART A (3-4 classes)

Basics of Stochastic Processes

- Probability and random variables: Bernoulli trials; Poisson
- Stochastic Processes: independent increments; Wiener; stationarity & ergodicity;
- Poisson Process - Traffic modelling

PART B (8-9 classes)

Markov Processes - Papers

- Discrete / Continuous time Markov Processes - Markov Decision (?)
- Queuing models - performance evaluation
- Papers on modeling networking environments

Material e-class

- Introductory - review for probabilities and Stochastic Processes (notes. Reference book: ``Probability, Random Processes, and Estimation Theory for Engineers, H. Stark, J. Woods, Prentice Hall, 2nd edition, 1994.
- 3 chapters on Poisson / Markov processes (from "Introduction to Stochastic Processes", E. Cinlar)
- Possibly some material from Chapters 10, 11 from "Performance Analysis of Communications Networks and Systems", Piet Van Mieghem
- Research articles on Network modeling and performance evaluation
- Chapter on Markov Decision, probably from "Introduction to Operations Research", F. Hillier, G. Lieberman (?)

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RELATIVE FREQUENCY-BASED probability

$$P(A) = \lim_{n \rightarrow \infty} \frac{\# \text{ experiments in which A occurs}}{\text{total \# of experiment}}$$

Comments:

1. Requires experimentation
2. n is usually finite \Rightarrow approximation

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3. Convergence implies that as $n \rightarrow \infty$:

$$\bullet \left\{ \left| \frac{n_A}{n} - P(A) \right| > \delta \right\} \text{ occurs less \& less}$$

$$\bullet \text{ Not that: } \left| \frac{n_A}{n} - P(A) \right| < \delta \quad \forall n > n(\delta)$$

(Traditional definition of convergence)

AXIOMATIC DEFINITION OF PROBABILITY - PROBABILITY SPACE

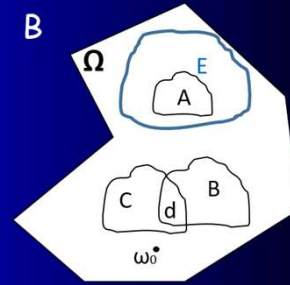
- Probability Space: $\{\Omega, \mathcal{F}, P\}$
- Ω : Sample space
- \mathcal{F} : σ -field generated by Ω
- P : A probability measure. It is a set function on \mathcal{F} satisfying:

- $P(A) \geq 0 \quad \forall A \in \mathcal{F}$
- $P(\Omega) = 1$
- $P(A \cup B) = P(A) + P(B) \quad \forall A, B \in \mathcal{F}, A \cap B = \emptyset$

\mathcal{F} : Contains all "good" subsets of Ω , for which the bullets hold; there are some subsets (e.g., limits of operations on subsets of Ω) of mostly mathematical (not engineering) interest for which the bullets do not hold; such subsets of Ω are not part of \mathcal{F}

Events in Ω - Probabilities of events

- Events are sets in Ω . For example: ω_0 , A, B, C, E
- An event occurs if ω (outcome of experiment) falls in the set defining the event
- If $\omega \in d$, then events C and B which contain ω have both occurred
- Since $A \subset E$, if event A has occurred, then event E has also occurred. Thus, $P(A \cap E) = P(A, E) = P(A)$
- Events A and B are mutually disjoint/exclusive (have no ω in common). Thus, $P(A \cup B) = P(A) + P(B)$



BERNOULLI TRIALS

Key Question: Prob. of k successes in n trials = ?

- Consider a simple experiment with $\Omega = \{s, f\}$, $P(s) = p$ & $P(f) = 1 - p = q$
- Consider 2 indep. experiments: $\Omega_2 = \Omega \times \Omega = \{ss, ff, sf, fs\}$, 2^2 possible outcomes
- Consider n indep. experiments / trials: $\Omega_n = \Omega \times \dots \times \Omega = \{\dots\}$, 2^n possible outcomes

$$\bar{\omega} = (\omega_1, \omega_2, \dots, \omega_n) \quad , \quad \omega_i \in \{s, f\}$$

Joint Probability:

$$P(\bar{\omega}) = P(\omega_1, \omega_2, \dots, \omega_n) = P(\omega_1)P(\omega_2)\dots P(\omega_n)$$

(indep. experiments)

Question: Prob. of k successes in n trials = ?

(A) Consider a pattern Π (outcome in Ω_n) counting k successes

$$\Pi = \underbrace{ss\dots s}_{k-1} s f f s f f \dots f$$

$$P(\Pi) = P\{ss\dots s f f s f f \dots f\} = p^k q^{n-k}$$

This prob. is the same for ANY pattern with k successes & n-k failures

(B) Consider all possible legitimate patterns. There are:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Call them $\Pi_1, \Pi_2, \dots, \Pi_{\binom{n}{k}}$

(Binomial coefficients)

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Answer:

$$P\{\text{outcome has k successes in n trials}\} = P\left(\bigcup_{i=1}^{\binom{n}{k}} \Pi_i\right) =$$

$$(\text{mutually disjoint}) = \sum_{i=1}^{\binom{n}{k}} P(\Pi_i) = \binom{n}{k} p^k q^{n-k} = b(k, n, p)$$

(Binomial distribution)

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POISSON PROBABILITY LAW

(Modeling large number of small contributions)

Assume $n \rightarrow \infty$, $p \rightarrow 0$ such that $np = a$

($a = \text{rate of success}$)

$$P\{k \text{ successes}\} = \lim_{\substack{n \rightarrow \infty \\ p \rightarrow 0, np = a}} b(k, n) = \frac{a^k}{k!} e^{-a}$$

Comment:

- If n large and $p \ll 1$ approximate $b(k, n, p) \approx e^{-np} (np)^k / k!$
- Cumulative arrivals contributed by a large # of independent sources can be modeled as Poisson
- Above plus other properties of Poisson process, make it a good traffic (event generator) model

RANDOM VARIABLES (RV)

A mapping X from Ω into the real numbers satisfying:

$$(1) \quad X^{-1}\{(-\infty, x)\} = C \in \mathcal{F}$$

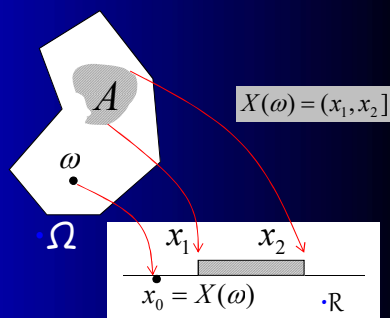
$$(2) \quad P(X = \pm\infty) = 0$$

Measures for a RV:

$$\text{PDF: } F_X(x) = P\{X \leq x\}$$

$$\text{pdf: } f_X(x) = \frac{dF_X(x)}{dx}, \quad F_X(x) = \int_{-\infty}^x f_X(y) dy$$

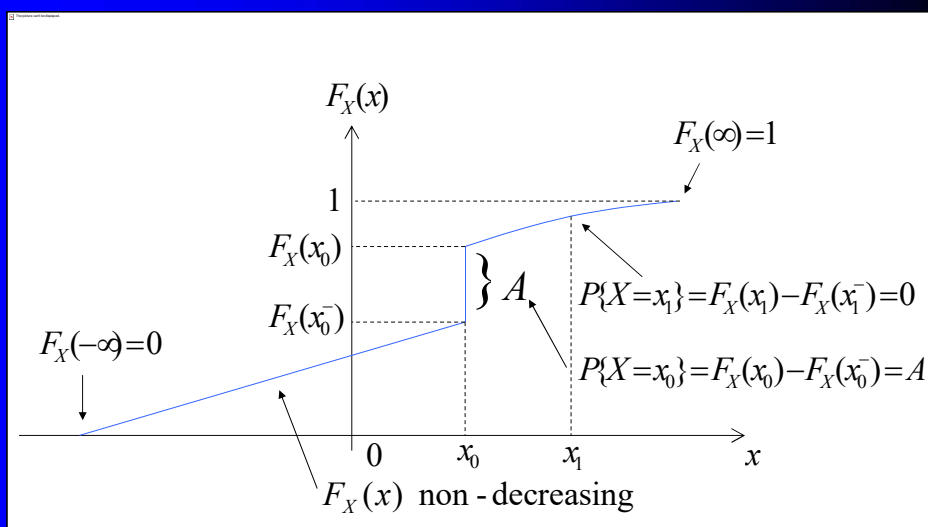
$$\text{PMF: } P\{X = x_i\} \quad \forall i \quad \text{for discrete - value RV}$$



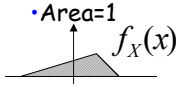
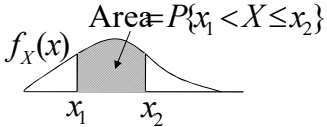
SOME PROPERTIES OF PDF

- $F_X(\infty) = 1, F_X(-\infty) = 0$
- $F_X(x)$ non-decreasing
- $F_X(x)$ is right continuous
- $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$
- $P\{X = x\} = F_X(x) - F_X(x^-)$
 ($P\{X = x\} = 0$ unless $F_X(x) \neq F_X(x^-)$, $F_X(x)$ has a jump)
- $P\{x_1 \leq X \leq x_2\} = F_X(x_2) - F_X(x_1^-)$

EXAMPLE-PDF



SOME PROPERTIES OF pdf

- $\int_{-\infty}^{\infty} f_X(x) dx = 1$ ($= F_X(+\infty)$) 
- $f_X(x) \geq 0$ (since $F_X(x)$ is non-decreasing)
- $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(t) dt$ 

MORE ON RVs

- Conditional & joint PDF/pdf/PMF; expectations; moments, moment generating functions & characteristic functions

EXAMPLE:

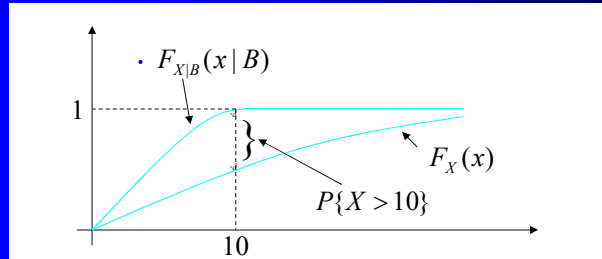
Given $F_X(x)$ and $B=\{X \leq 10\}$ calculate $F_{X/B}(x/B)$

ANSWER:

$$F_{X/B}(x/B) = P\{X \leq x | B\} = \frac{P\{X \leq x, B\}}{P\{B\}} = \frac{P\{X \leq x, X \leq 10\}}{P\{X \leq 10\}}$$

$$(i) \text{ if } x \geq 10 : F_{X/B}(x/B) = \frac{P\{X \leq 10\}}{P\{X \leq 10\}} = 1$$

$$(ii) \text{ if } x < 10 : F_{X/B}(x/B) = \frac{P\{X \leq x\}}{P\{X \leq 10\}} = \frac{1}{P\{X \leq 10\}} F_X(x)$$



Note:

Given $X \leq 10$, cond. PDF must reach 1 at $x=10$
 $P\{X > 10\}$ is "distributed" to values of $X \leq 10$ by
increasing the original mass by $1/P\{X \leq 10\}$

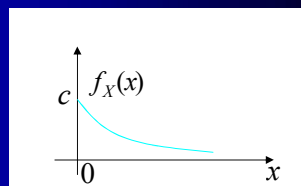
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EXAMPLE: memoryless property of exponential RV

$$F_X(x) = 1 - e^{-cx}, \quad x \geq 0$$

$$f_X(x) = ce^{-cx}, \quad x \geq 0$$



Consider that duration of calls X is exp

QUESTION: Given that a call is still on t units after initiation (i.e., $X \geq t$), calculate the probability it will still be on after s time units (i.e., find $A = P\{X > t+s | X > t\}$)

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ANSWER:

$$\begin{aligned} A &= \frac{P\{X > t+s, X > t\}}{P\{X > t\}} = \frac{P\{X > t+s\}}{P\{X > t\}} = \\ &= \frac{1 - (1 - e^{-c(s+t)})}{1 - (1 - e^{-ct})} = \frac{e^{-cs} e^{-ct}}{e^{-ct}} = e^{-cs} = P\{X > s\} \end{aligned}$$

That is :

$$P\{X > t+s \mid X > t\} = P\{X > s\}$$

(the history factor (i.e. already on for t units)
does not alter the result)

SEQUENCES OF RVs $\{X_1, X_2, \dots, X_n, \dots\}$

SOME KEY QUESTIONS:

- What is the behavior of X_n as $n \rightarrow \infty$ or n is very large?
- What is $\gg \gg$ of $\sum_{i=1}^n X_i \gg \gg \gg \gg ?$

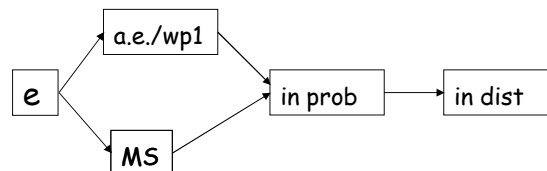
USEFUL IN:

- Determining a behavior in steady-state (i.e. $X_n, n \rightarrow \infty$); also its existence or not
- Dealing with averages of large number of samples

SENSES OF CONVERGENCE OF $\{X_n(\omega)\} \rightarrow X(\omega)$

- (a) Everywhere (e.): iff it holds for all ω in Ω
- (b) Almost Everywhere or with probability 1 (a.e./wp1): iff it does not hold ONLY for all ω in A with $P(A)=0$
- (c) In Probability: iff $\lim_{n \rightarrow \infty} P\{|X_n(\omega) - X(\omega)| \geq \varepsilon\} = 0, \varepsilon > 0$
- (d) Mean Square (M.S.): iff $\lim_{n \rightarrow \infty} E\{|X_n(\omega) - X(\omega)|^2\} = 0$
- (e) In distribution: iff $\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$; point x of continuity

COMPARISON OF SENSES OF CONVERGENCE



COMPARISON OF SENSES OF CONVERGENCE

EXAMPLE/APPLICATION: LAW OF LARGE NUMBERS: (Averaging)

- WLLN: $\{X_i\}$ indep. RVs, $E\{X_n\} < \infty, \sigma^2 < \infty; S_n = \sum_{i=1}^n X_i$; then

$$Y_n = \frac{S_n - E\{S_n\}}{n} \rightarrow Y = 0 \text{ in prob. (weak Law of Large Numbers - WLLN)}$$
 (if $\{X_i\}$ are i.i.d., $\frac{S_n}{n} \rightarrow E\{X_n\}$ in prob;
 could be used to estimate the mean of unknown RV)
- SLLN: $\{X_i\}$ indep. and (a) i.i.d. with $E\{X_i\} < \infty$ or (b) $\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty$; then

$$Y_n = \frac{S_n - E\{S_n\}}{n} \xrightarrow[n \rightarrow \infty]{\text{a.e.}} 0 \text{ (Strong Law of Large Numbers - SLLN)}$$

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EXAMPLE: Proof of WLLN for i.i.d. RVs $\{X_i\}$

Using Chebyshev's inequality for $Y_n = S_n/n$:

$$A = P\{|Y_n - E\{Y_n\}| \geq \delta\} \leq \frac{\text{Var}\{Y_n\}}{\delta^2}$$

$$E\{Y_n\} = E\{X_i\}$$

$$\text{Var}\left\{\frac{1}{n}S_n\right\} = \frac{1}{n^2}\text{Var}\{S_n\} \stackrel{\text{uncorrelated } \{X_i\}}{=} \frac{1}{n}\text{Var}\{X_i\}$$

$$\text{Thus: } A \leq \frac{\text{Var}\{X_i\}}{n\delta^2} \xrightarrow[n \rightarrow \infty]{} 0$$

$$\text{and } \frac{S_n}{n} \rightarrow E\{X_i\} \text{ in prob.}$$

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EXAMPLE/APPLICATION: CENTRAL LIMIT THEOREM (normalizing)

$\{X_i\}$ indep. RVs; $\mu_i; \sigma_i^2$

$$Y_n = \frac{\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i}{\left[\sum_{i=1}^n \sigma_i^2 \right]^{\frac{1}{2}}}, \quad E\{Y_n\} = 0, \text{Var}\{Y_n\} = 1$$

under general conditions $F_{Y_n}(x) \rightarrow N(0,1)$ (Normal PDF)
(iff X_i i.i.d. and $\sigma < \infty$, it always holds)

COMMENT ON CLT

To hold, look for a lot of RVs with good variability

$$X_i \text{ indep.} \Rightarrow w_i = \frac{X_i - E\{X_i\}}{\left(\sum \sigma_i^2\right)^{\frac{1}{2}}} \text{ indep.}$$

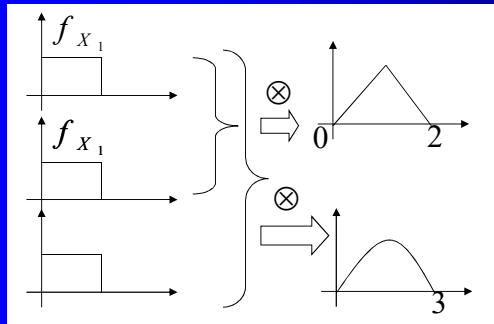
$$Y_n = \sum w_i \Rightarrow f_{Y_n} = f_{w_1} \otimes f_{w_2} \otimes \dots \otimes f_{w_n}$$

$$\Rightarrow \Phi_{Y_n}(\omega) = \Phi_{w_1}(\omega)\Phi_{w_2}(\omega)\dots\Phi_{w_n}(\omega)$$

$$\Phi_{w_i}(\omega) = E\{e^{j\omega x}\} = \text{characteristic function of } w_i$$

$$\text{(for normal } Y : \Phi_Y = e^{-\frac{\omega^2}{2}})$$

- if $f_{w_i}(\cdot)$ are delta fns (0 variance) $\Rightarrow \Phi_{w_i}(\omega) = 1$
 $\Rightarrow \Phi_{Y_n}(\omega) = 1 \neq e^{-\frac{\omega^2}{z}}$ (CLT cannot hold)
- if only m RVs have variance and rest have almost zero,
 $\Phi_{Y_n}(\omega) \approx \Phi_{w_1}(\omega) \dots \Phi_{w_m}(\omega) \cdot 1 \cdot 1 \cdot 1 \dots 1 \neq e^{-\frac{\omega^2}{z}}$
- Same conclusions in time domain as well:



Alternatively, the above variance requirement can be stated as:

$$\sigma_i^2 \ll \sum_{i=1}^n \sigma_i^2$$

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STOCHASTIC PROCESSES

Prob space $\{\Omega, \mathcal{F}, P\}$

A mapping $X(t, \omega), \omega \in \Omega, t \in I$

(index set e.g. time
(discrete or continuous))

is a stochastic process iff $X(t_0, \omega)$ is a RV for any fixed t_0

Interpretation 1 :

It is a collection of RVs $\{X_t(\omega); t \in I\}$
 defined on common prob space

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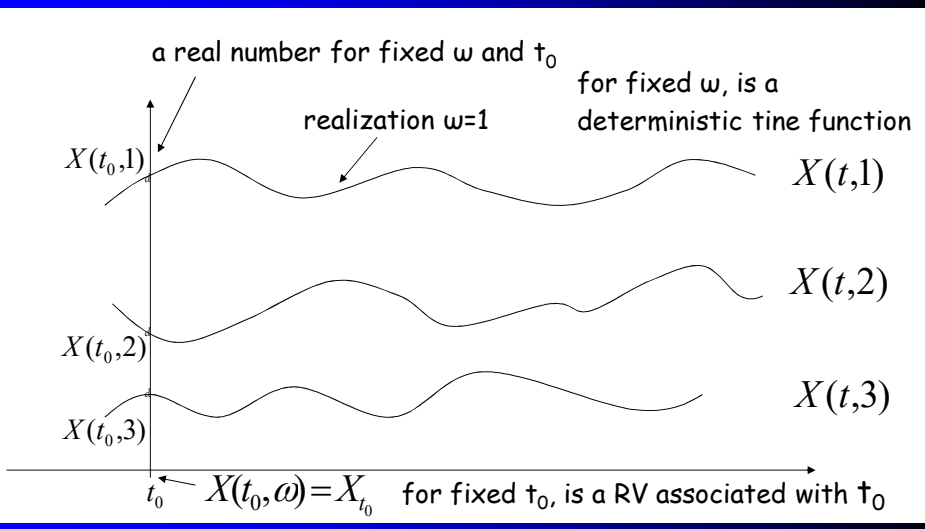
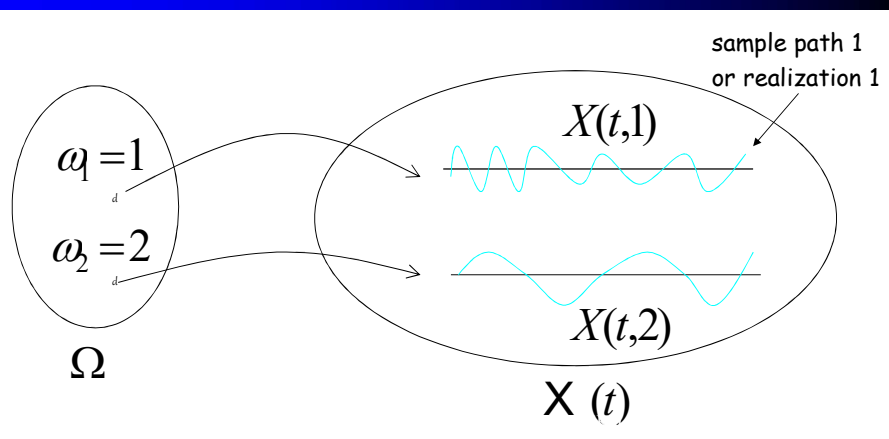
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Interpretation 2 :

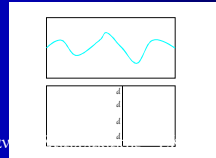
It is a mapping $\Omega \rightarrow X(t)$

$X(t)$ contain time functions instead of real numbers (case of RV)

For fixed t_0 , $X(t_0)$ is a RV



Oscilloscope "view"
 Freeze image -> Sample path
 Block screen except a slit



DESCRIPTION OF A STOCH PROCESS

Joint PDF: $F_{\bar{X}}(X_1, X_2, \dots, X_n, \dots)$

Mean: $\mu_X(n) = E\{X_n\}$

Autocorrelation fn: $R_X(k, l) = E\{X_k X_l^*\}$

Autocovariance fn: $K_X(k, l) = E\{(X_k - \mu_X(k))(X_l - \mu_X(l))^*\}$
 $= R_X(k, l) - \mu_X(k)\mu_X(l)$

Can be facilitated by:

- Taking advantage of special structure of some processes (indep. increments, Markov, etc.)
- Stationarity
- Ergodicity

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PROCESS WITH INDEPENDENT INCREMENTS

$\forall N > 1$, all $n_1 < n_2 < \dots < n_N$

$Y_{n_1}, Y_{n_2} - Y_{n_1}, \dots, Y_{n_N} - Y_{n_{N-1}}$ are indep.

then for $\{Y_n\}_n$ with indep. increments:

$F_{\bar{Y}}(y_1, y_2, \dots, y_N) = F_{Y_1}(y_1)F_{Y_2 - Y_1}(y_2 - y_1) \dots F_{Y_1}(y_1)F_{Y_N - Y_{N-1}}(y_N - y_{N-1})$

Example: BINOMIAL COUNTING PROCESS

$S_n = \sum X_i$, X_i i.i.d. wp p

$E\{S_n\} = np$ (S_n not WSS)

Let $I_{k_n} = S_n - S_k = X_{k+1} + \dots + X_n$, $n > k$

I_{k_n} indep. of S_k

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Stationarity

- Strict Sense Stationarity (SSS):

$$f_X(x_0, \dots, x_{n-1}) = f_X(x_k, \dots, x_{k+(n-1)}) \quad \forall k, \forall n \geq 1$$

(Neither prob behavior of each x_k
nor interdependencies change with k)

- Wide Sense Stationarity (WSS):

$$\mu_X(n) = \text{constant} \quad \forall n \geq 1$$

$$K_X(k, l) = K_X(k + \rho, l + \rho) \quad \forall \rho \geq 1$$

(both mean and covariance are time-shift invariant)

Note: if $\{x_n\}$ is not WSS it is also not SSS

THE WIENER PROCESS

$$\Omega = \{s, f\} \quad , \quad p_f = 1 - p_s$$

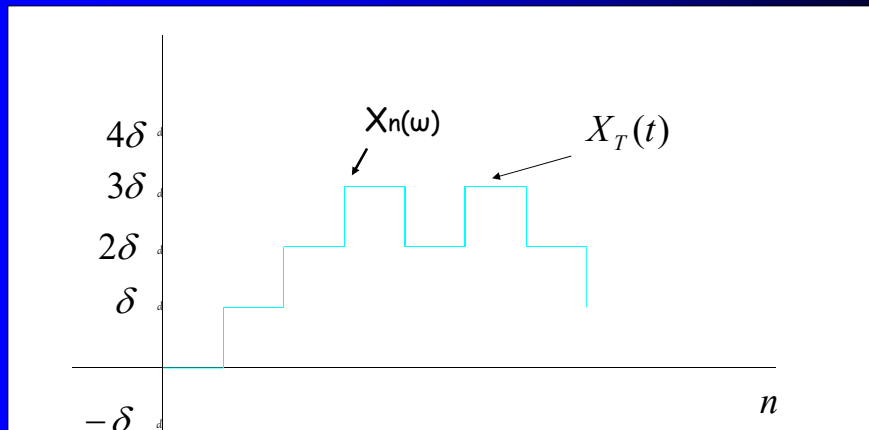
$$\text{Let } w: \Omega \rightarrow \{+\delta, -\delta\}$$

$$\text{where } w = \begin{cases} +\delta & \text{if } w = s \\ -\delta & \text{if } w = f \end{cases}$$

$$\text{Let } X_n = w_1 + w_2 + \dots + w_n$$

$\{X_n(\omega)\}$ is a random sequence which increases
by $+\delta$ or decreases by δ in each step. (Random walk)

THE WIENER PROCESS



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For fixed n , $X_n(\omega)$ is a RV with prob. :

$$\begin{aligned}
 P\{X_n = r\delta\} &= \Pr\{\# \text{ successes} - \# \text{ failures} = r\} = \\
 &= P\{k - (n - k) = r\} = P\{k = \frac{n+r}{2}\} \\
 &= \binom{n}{(n+r)/2} p_s^{(n+r)/2} p_f^{n-(n+r)/2} \quad \underset{p_f = p_s = 1/2}{=} \\
 &= \binom{n}{(n+r)/2} 2^{-n} \quad \left(\frac{n+r}{2} \text{ integer}\right)
 \end{aligned}$$

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$$E\{X_n\} = \sum_{k=1}^n E\{W_k\} = 0$$

$$E\{X_n^2\} = \sum_{k=1}^n E\{W_k^2\} = \sum_{k=1}^n \left\{ \frac{1}{2} \delta^2 + \frac{1}{2} \delta^2 \right\} = n \delta^2$$

Let $X_T(t)$ be the piecewise constant version of X_n . That is,

$$X_T(t) = \sum_{k=1}^{\infty} W_k \cdot u(t - kT) \quad , \quad W_k = \begin{cases} +\delta & wp \quad 1/2 \\ -\delta & wp \quad 1/2 \end{cases}$$

Clearly

Let $X_T(t)$ be the piecewise constant version of X_n . That is,

$$X_T(t) \Big|_{t=kT} = X_k = \sum_{i=1}^k w_i$$

The PMF and moments of $X_T(t)$ can be obtained from those of X_k

Definition of Wiener Process

- The Wiener process (or Wiener-Levy or Brownian motion) is the process whose distribution is the limiting distribution of $X_T(t)$ as $T \rightarrow 0$; the jump size δ goes to zero as well and $X(t)$ is a continuous state continuous time stochastic process.

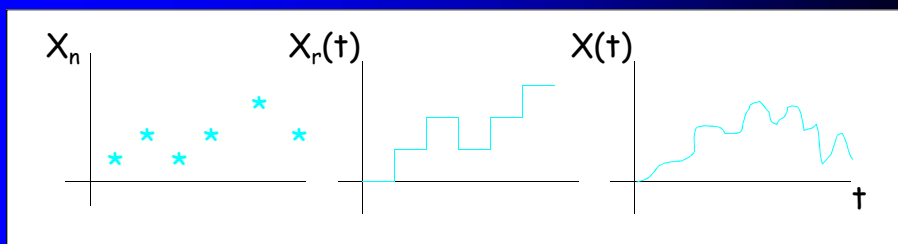
$$E\{X_T(nT)\} = E\{X_n\} = 0$$

$$E\{X_T^2(nT)\} = E\{X_n^2\} = n\delta^2 \Rightarrow \text{Var}\{X_T(nT)\} = n\delta^2$$

$$\text{and } \text{Var}\{X_T(t)\} = \frac{t}{T}\delta^2 \quad \text{for } t = nT$$

Suppose that $\delta^2 = aT$ so that the variance does not vanish as $T \rightarrow 0$. Then

$$X(t) = \lim_{\substack{T \rightarrow 0 \\ S^2 = aT}} X_T(t) \quad \text{and} \quad \text{Var}\{X(t)\} = at$$



Wiener process models the chaotic motion of gas molecules.

Note:

$\{X_n\}_n$ is the sum of n i.i.d. RVs w_i



(Central Limit Theorem) limiting process (as n increases) would be Gaussian, and the same would be expected for $X(t)$, the Wiener process (as $t \rightarrow 0$).

$$E\{X(t)\} = \mu_X(t) = 0 \quad , \quad \text{Var}\{X(t)\} = \sigma_X^2(t) = at$$

$$f_{X_t}(x_t) = \frac{1}{\sqrt{2\pi at}} e^{-\frac{x_t^2}{2at}} \quad t > 0$$

(notice that $X(t)$ is not SSS since $f_{X_t}(x_t)$ changes with time)

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- The Wiener process is an example of Gaussian process since all n^{th} -order pdf's are Gaussian. This can be seen intuitively as follows:

Let $\Delta = X(t) - X(s)$, $t > s$. By considering the infinite number of i.i.d. RVs between $X(s)$ and $X(t)$ and applying the CLT we have:

$$f_{\Delta}(y; t-s) = \frac{1}{\sqrt{2\pi a(t-s)}} e^{-\frac{y^2}{2a(t-s)}}$$

since $E\{X(t) - X(s)\} = E\{\Delta\} = 0$

and $E\{(X(t) - X(s))^2\} = a(t-s) \quad t > s$

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- Using the pdf for the increment Δ and the independent increment property of the Wiener process we can derive the n^{th} -order pdfs and show they are Gaussian.

$$(f_{X_{t_1} X_{t_2}}(x_{t_1}, x_{t_2}) = f_{X_{t_1}}(x_{t_1}) \cdot f_{X_{t_2} - X_{t_1}}(x_{t_2} - x_{t_1}))$$

- Covariance or autocorrelation function:

$$\begin{aligned} K_X(t, s) &= E\{X(t)X^*(s)\} \stackrel{t>s}{=} E\{[X(t) - X(s) + X(s)]X^*(s)\} \\ &= E\{[X(t) - X(s)]X^*(s)\} + E\{X^2(s)\} = as \end{aligned}$$

Thus

$$K_X(t, s) = a \cdot \min(t, s)$$

MEAN SQUARE CALCULUS & ERGODICITY

- CALCULUS: Define limits, integrals & derivatives for SPs
- ERGODICITY: Condition under which time averages=ensemble (over Ω) averages

Mean Square Continuity

A SP $X(t)$ is continuous at $t \Leftrightarrow E\{|X(t+\varepsilon) - X(t)|^2\} \xrightarrow{\varepsilon \rightarrow 0} 0$

N & S condition : $R_X(t, t)$ is continuous at t

$(E\{|X(t+\varepsilon) - X(t)|^2\} =$

$$R_X(t+\varepsilon, t+\varepsilon) + R_X(t, t) - R_X(t, t+\varepsilon) - R_X(t+\varepsilon, t) \xrightarrow{\varepsilon \rightarrow 0} 0)$$

Note : N & S condition for WSS SP : $R_X(\tau)$ continuous at $\tau = 0$

Mean Square Derivative at t

Condition :

$$E \left\{ \left| \frac{X(t + \varepsilon_1) - X(t)}{\varepsilon_1} - \frac{X(t + \varepsilon_2) - X(t)}{\varepsilon_2} \right|^2 \right\} \xrightarrow[\varepsilon_2 \rightarrow 0]{\varepsilon_1 \rightarrow 0} 0$$

N & S condition : $\frac{d^2 R_X(t_1, t_2)}{dt_1 dt_2}$ exists at $t = t_1 = t_2$

Note : N & S condition for WSS SP : $\frac{d^2 R_X(\tau)}{d\tau^2}$ exists at $\tau = 0$.

Moments of the Derivative SP

$$E\{X'(t)\} = \frac{d\mu_X(t)}{dt}, \quad R_X(t_1, t_2) = \frac{d^2 R_X(t_1, t_2)}{dt_1 dt_2}$$

Example - Wiener SP

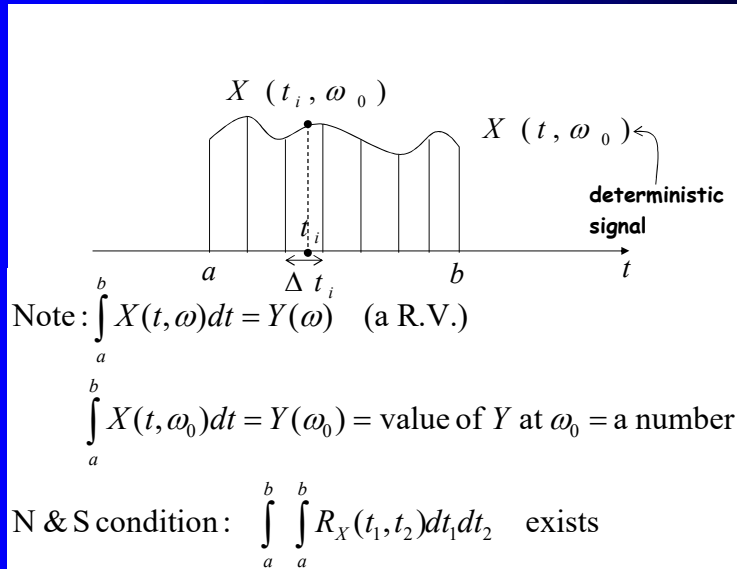
$$R_X(t_1, t_2) = a \min(t_1, t_2) = \begin{cases} at_1 & \text{if } t_1 < t_2 \\ at_2 & \text{if } t_2 < t_1 \end{cases}$$

$$\frac{dR_X(t_1, t_2)}{dt_2} = \begin{cases} 0 & \text{if } t_1 < t_2 \\ a & \text{if } t_2 < t_1 \end{cases} = a \cdot u(t_1 - t_2)$$

$$\frac{d^2 R_X(t_1, t_2)}{dt_1 dt_2} = \frac{d a \cdot u(t_1 - t_2)}{dt_1} = a \delta(t_1 - t_2) \quad \text{i.e. it is white (uncorrelated) and since Gaussian } \Rightarrow \text{ indep.}$$

(as expected for a SP with indep. increments)

Mean Square Integral



For $Y = \int_a^b X(t) dt$

$$\mu_Y = \int_a^b \mu_X(t) dt \quad , \quad E\{Y^2\} = \int_a^b \int_a^b R_X(t_1, t_2) dt_1 dt_2$$

Example for $X(t)$ WSS SP $\left(Y = \int_{-T}^T X(t) dt \right)$:

$$\sigma_Y^2 = \int_{-T}^T \int_{-T}^T K_X(t_1 - t_2) dt_1 dt_2 = \dots = \int_{-2T}^{2T} K_X(\tau) [2T - |\tau|] d\tau$$

Example for $n(t)$ zero mean, white SP $\left(X(t) = \int_0^t n(\tau) d\tau \right)$:

$$K_n(t_1, t_2) = A\delta(t_1 - t_2)$$

$$\sigma_X^2 = \int_0^t \int_0^t A\delta(t_1 - t_2) dt_1 dt_2 = A \int_0^t dt_1 = At$$

(if $n(t)$ is also Gaussian, then $X(t)$ is Wiener)

ERGODICITY

$X(t)$ is a SP, $X(t, \omega_0)$ = a sample path/realization

Time Averages ($A\{\cdot\}$) of (functions of) $X(t, \omega_0)$ are possible

Ensemble Averages ($E\{\cdot\}$) are typically not possible but needed

Question :

$$A\{\cdot\} \stackrel{?}{=} E\{\cdot\}$$

Answer :

Yes if $X(t)$ ergodic in some sense.

Definitions :

$$A\{\cdot\} = \lim_{T \rightarrow \infty} A_T\{\cdot\} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cdot dt$$

$E\{\cdot\}$ = an average of \cdot over all ω

with the corresponding probability as weights

Examples:

For $\cdot = X(t, \omega_0)$

$$A\{X(t, \omega_0)\} \stackrel{?}{=} E\{X(t, \omega_0)\} = E_{\omega_0}\{X(t, \omega_0)\} = \mu_X$$

For $\cdot = X(t, \omega_0)X(t + \tau, \omega_0)$

$$A\{X(t, \omega_0)X(t + \tau, \omega_0)\} \stackrel{?}{=} E\{X(t, \omega_0)X(t + \tau, \omega_0)\} = R_X(\tau)$$

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IMPORTANT OBSERVATION A (need for some stationarity)

For $X(t)$ to be POSSIBLE to be ergodic it must be stationary
(time shift invariant) to a certain extent

Argument : Since $A\{\cdot\}$ is indep. of time,

if $E\{\cdot\} = A\{\cdot\}$ then

$E\{\cdot\}$ must also be indep. of time (constant E)

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IMPORTANT OBSERVATION B (need for some regularity)

For $X(t)$ to be POSSIBLE to be ergodic it must be regular
(i.e. time averages be indep. of ω)

Argument : Since $E\{\cdot\}$ is indep. of ω ,

if $A\{\cdot\} = E\{\cdot\}$ then

$A\{\cdot\}$ must also be indep. of ω (constant A)

Comment on need for regularity :

▷ For any ω_0 , $X(t, \omega_0)$ should behave over time like any other

$X(t, \omega_i)$ so that ω is NOT identifiable by a time average

▷ For any ω_0 , $X(t, \omega_0)$ should exhibit over time all kinds

of behavior and with the proper "frequency & duration" as for

any ω , so that the time average becomes equivalent to ensemble average

DEFINITION : $X(t)$ is ergodic if :

(a) It is stationary to some extent ($E\{\cdot\} = E$ (constant))

(b) It is regular ($A\{\cdot\} = A$ (constant))

(c) Constants E and A are equal

((c) is met if $E\{\cdot\}$ & $A\{\cdot\}$ operators are interchangeable - Fubini's theorem)

EXAMPLE : $\Omega = \{-2, -1, 1, 2\}$, $X(t, \omega) = \omega$ for $-\infty < t < \infty$

$$, P(\omega) = \frac{1}{4} \text{ for } \omega \in \Omega$$

Ensemble Average: $E\{X(t, \omega)\} = E\{\omega\} = 0$ (time shift - invariant)

Time Average: $A\{X(t, \omega)\} = \omega$ (not regular)

$X(t, \omega)$ not ergodic.

M.S. ERGODICITY IN THE MEAN

Def : $A_T \{X(t)\} = \frac{1}{2T} \int_{-T}^T X(t) dt \xrightarrow{T \rightarrow \infty} \mu_X \quad (\text{m.s.}) \quad (*)$

N & S condition :

$$\lim_{T \rightarrow \infty} \left\{ \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) K_X(\tau) d\tau \right\} = 0 \quad (**)$$

Proof : $(*) \Leftrightarrow E\{|A_T \{X(t)\} - \mu_X|^2\} \xrightarrow{T \rightarrow \infty} 0 \Leftrightarrow \sigma_{A_T}^2 \xrightarrow{T \rightarrow \infty} 0$

$$\sigma_{A_T}^2 = \frac{1}{(2T)^2} \int_{-T}^T \int_{-T}^T K_X(t_1 - t_2) dt_1 dt_2 = (**)$$

Examples:

(a) if $\int_{-\infty}^{\infty} |K_X(\tau)| d\tau < \infty$ then $(**) < \frac{1}{2T} \int_{-\infty}^{\infty} |K_X(\tau)| d\tau \xrightarrow{T \rightarrow \infty} 0$

and $X(t)$ is ergodic in the mean

(b) if $K_X(0) < \infty$ & $K_X(\tau) \xrightarrow{\tau \rightarrow \infty} 0$ then $X(t)$ is ergodic in the mean

Proof : $(**) < \frac{1}{2T} \left\{ \int_{-\alpha}^{\alpha} |K_X(\tau)| d\tau + \int_{\alpha < |\tau| < 2T} |K_X(\tau)| d\tau \right\} \quad (a: |K_X(\tau)| < \varepsilon, |\tau| > a)$

$$< \frac{1}{2T} \{2a \cdot K_X(0) + 4T \cdot \varepsilon\} = \frac{2aK_X(0)}{2T} + 2\varepsilon$$

= arbitrarily small as $T \rightarrow \infty$

M.S. ERGODICITY IN THE AUTOCORRELATION

Def : $A_T \{X(t)X^*(t+\lambda)\} \xrightarrow{T \rightarrow \infty} R_X(\lambda) \quad (m.s.)$

[Equivalent: $A_T \{\Phi_\lambda(t)\} \xrightarrow{T \rightarrow \infty} E\{\Phi_\lambda(t)\} \quad (m.s.)$

for SP $\Phi_\lambda(t) = X(t)X^*(t+\lambda)$]

N & S condition :

$$(**) \Rightarrow \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) K_{\Phi_\lambda}(\tau) d\tau \xrightarrow{T \rightarrow \infty} 0$$

M.S. ERGODICITY IN DISTRIBUTION

Index function: $I_X(x, t) = \begin{cases} 1 & \text{if } X(t) \leq x \\ 0 & \text{otherwise} \end{cases}$

Def : $A_T \{I_X(x, t)\} = \frac{1}{2T} \int_{-T}^T I_X(x, t) dt \xrightarrow{T \rightarrow \infty} F_X(x)$

N & S condition :

$$(**) \Rightarrow \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) K_{I_X}(x, \tau) d\tau \xrightarrow{T \rightarrow \infty} 0$$

where $K_{I_X}(x, \tau) = E\{I_X(x, t)I_X^*(x, t+\tau)\} - [E\{I_X(x, t)\}]^2$
 $= F_{X_t, X_{t+\tau}}(x, x) - [F_{X_t}(x)]^2 \quad (***)$

Note : (**) implies that $K_{I_X}(x, \tau)$ should vanish to 0 as $T \rightarrow \infty$,

i.e. $F_{X_t, X_{t+\tau}}(x, x) \xrightarrow{T \rightarrow \infty} F_{X_t}(x) \cdot F_{X_t}(x)$

or $X_t, X_{t+\tau}$ should be asymptotically independent

intuitively expected for $A\{I_X\}$ to be equal to $E\{I_X\}$

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