Markov Processes and Applications

- Discrete-Time Markov Chains
- Continuous-Time Markov Chains
- Applications
  - Queuing theory
  - Performance analysis

Discrete-Time Markov Chains

Books

- Introduction to Stochastic Processes (Erhan Cinlar), Chap. 5, 6
- Introduction to Probability Models (Sheldon Ross), Chap. 4
- Performance Analysis of Communications Networks and Systems (Piet Van Mieghem), Chap. 9, 11
- Introduction to Probability, D. Bertsekas & J. Tsitsiklis, Chap. 6

Contents

Example - A taxi driver conducts his business in three different towns: 1, 2 and 3.

- We assume that once the taxi driver is at a given town, he stays there until he picks up a passenger.
- We assume slotted time, that is, the taxi driver spends one hour in every town per passenger, fixed.
- On a given day, when he is in town 1, the probability that the next passenger he picks up is going to a place in town 1 is 0.3, to a place in town 2 is 0.2 and to a place in town 3 is 0.5. When he is in town 2, the probability that the next passenger he picks up is going to town 1 is 0.1, to town 2 is 0.8 and to town 3 is 0.1. When he is in town 3, the probability that the next passenger he picks up is going to town 1 is 0.4, to town 2 is 0.4 and to town 3 is 0.2.
- How would you model this problem ?

Questions that can be asked

- Assuming that the taxi driver starts in town 1, where will he most likely be after 4 passengers? (Chapman-Kolmogorov equations)
- For how long does he remain on a particular town (state) once he reaches it? (Sojourn times)
- What percentage of time does the taxi driver spend in a given town? (Limiting state probabilities)
- Assuming that the taxi driver gets 10 Euros on average when he picks a passenger in town 1, 20 Euros in town 2 and 30 Euros in town 3, how much does he earn on average per day? (Weighted average of limiting state probabilities)
- Assuming that the taxi driver starts in town 1, when is the first time he gets to town 2? (First-passage times)

Problem Formulation

Problem Formulation

- Let X_n denote the town at which the taxi driver is at time slot n. Hence, $X_n \in \{1, 2, 3\}$, $n = 0, 1, \ldots$.
- At the beginning (i.e. time $n = 0$), the taxi driver is in town i with probability:

$$P(X_0 = i) = \pi_i^{(0)}$$

- At time $n = 1$, the taxi driver is at town j with probability:

$$P(X_1 = j) = \sum_i P(X_1 = j | X_0 = i) P(X_0 = i) = \sum_i p_{ij} \pi_i^{(0)}$$

  where p_{ij} refers to the probability of moving from town i to town j.
- The town at which the taxi driver is at a given time (say time k) only depends on the town at which he was at the previous time $k - 1$ (Markov property).

INTRODUCTION :

nth order pdf of some stoc. proc. $\{X_t\}$ is given by

$$f(x_{t_1}, x_{t_2}, ..., x_{t_n}) = f(x_{t_n} \mid x_{t_{n-1}}, ..., x_{t_1}) f(x_{t_{n-1}} \mid x_{t_{n-2}}, x_{t_{n-3}}, ..., x_{t_1}) \\ ... f(x_{t_2} \mid x_{t_1}) f(x_{t_1})$$

very difficult to have it in general

- If $\{X_t\}$ is an indep. process:
  $$f(x_{t_1}, x_{t_2}, ..., x_{t_n}) = f(x_{t_n}) f(x_{t_{n-1}}) ... f(x_{t_1})$$
- If $\{X_t\}$ is a process with indep. increments:
  $$f(x_{t_1}, x_{t_2}, ..., x_{t_n}) = f(x_{t_1}) f(x_{t_2} - x_{t_1}) ... f(x_{t_n} - x_{t_{n-1}})$$

Note: First order pdf's are sufficient for above special cases

- If $\{X_t\}$ is a process whose evolution beyond t_0 is (probabilistically) completely determined by x_{t_0} and is indep. of x_t , $t < t_0$, given x_{t_0}, then:
  $$f(x_{t_1}, x_{t_2}, ..., x_{t_n}) = f(x_{t_n} \mid x_{t_{n-1}}) ... f(x_{t_2} \mid x_{t_1}) f(x_{t_1})$$
  This is a Markov process (nth order pdf simplified)

Definition of a Markov Process (MP)

A stoch. proc. $\{X_t; t \in I\}$ that takes values from a set E is called a Markov Process (MP) iff :

$$P(x_{t_n} \mid x_{t_{n-1}}, ..., x_{t_1}) = P(x_{t_n} \mid x_{t_{n-1}}) \qquad (E \text{ countable})$$

or

$$f(x_{t_n} \mid x_{t_{n-1}}, ..., x_{t_1}) = f(x_{t_n} \mid x_{t_{n-1}}) \qquad (E \text{ uncountable})$$

for all x_{t_n} and all $t_1 < t_2 < ... < t_n$ and all $n > 0$.

Notice: The "next" state x_{t_n} is indep. of the "past" $\{x_{t_1}, ..., x_{t_{n-2}}\}$ provided that the "present" is known.

Definition of a Markov Chain (MC)

(Discrete - time & discrete - value MP)

If I is countable and E is countable then a MP is called a MC and is described by the transition probabilities:

$$p(i,j) = P\{X_{n+1} = j \mid X_n = i\} \qquad i,j \in E$$

(indep. of n for a time - homogeneous MC). Assume $E = \{0,1,2,...\}$ (state - space of the MC)

Transition matrix :

$$P = \begin{bmatrix} P(0,0) & P(0,1) & \dots & P(0,n) & \dots \\ P(1,0) & P(1,1) & \dots & P(1,n) & \dots \\ \vdots & \vdots & & \vdots & \\ P(n,0) & P(n,1) & \dots & P(n,n) & \dots \\ \vdots & \vdots & & \vdots & \end{bmatrix}$$

P is non - negative, $\sum_j P(i,j) = 1 \quad , \quad \forall i$ (stochastic matrix)

For a given P (stoch. matrix) a MC may be constructed

Chain rule :

If $\bar{\pi}$ is a PMF on E s.t. $\pi(i) = P\{X_0 = i\}$, $i \in E$, then

$$P\{X_0 = i_0, X_1 = i_1, X_2 = i_2, ..., X_n = i_n\} = \pi(i_0)P(i_0,i_1)...P(i_{n-1},i_n)$$

$$\forall n \in N \quad , \quad i_0, i_1, ..., i_n \in E$$

k - step transitions :

$\forall\, k \in \mathrm{N}$,

$$P\{X_{n+k} = j \mid X_n = i\} = P^k(i,j)$$

$\forall i,j \in E$, $\forall\, k \in \mathrm{N}$; $P^k(i,j)$ is the (i,j) entry of the kth power of the transition matrix P.

Proof : For $k = 3$ (general n through iterations)

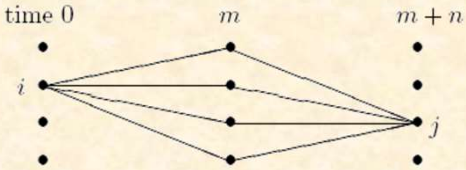
$$P\{X_{n+3} = j \mid X_n = i\} = \underbrace{\sum_{l_1 \in E} P(i,l_1) \underbrace{\sum_{l_2 \in E} P(l_1,l_2)P(l_2,j)}_{P^2(l_1,j)}}_{P^3(i,j)}$$

Chapman Kolmogorov Equations :

From previous,

$$P^{m+n}(i,j) = \sum_{k \in E} P^m(i,k) P^n(k,j) \qquad i,j \in E$$

In order for $\{X_n\}$ to be in j after $m+n$ steps and starting from i , it will have to be in some k after m steps and move then to j in the remaining n steps.



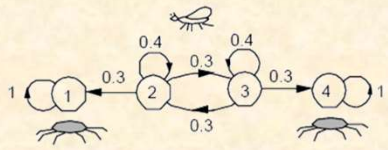
Bertsekas & Tsitsiklis

Example 6.2. A fly moves along a straight line in unit increments. At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4, independently of the past history of movements. A spider is lurking at positions 1 and m: if the fly lands there, it is captured by the spider, and the process terminates. We want to construct a Markov chain model, assuming that the fly starts in one of the positions $2, \ldots, m-1$.

Let us introduce states $1, 2, \ldots, m$, and identify them with the corresponding positions of the fly. The nonzero transition probabilities are

$$p_{11} = 1, \qquad p_{mm} = 1,$$

$$p_{ij} = \begin{cases} 0.3 & \text{if } j = i-1 \text{ or } j = i+1, \\ 0.4 & \text{if } j = i, \end{cases} \qquad \text{for } i = 2, \ldots, m-1.$$



	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

p_{ij}

Figure 6.2: The transition probability graph and the transition probability matrix in Example 6.2, for the case where $m = 4$.

Example: # of successes in Bernoulli process

$\{N_n; n \geq 0\}$, N_n = # of successes in n trials

$N_n = \sum_{i=0}^{n} Y_i$, $n \geq 0$, Y_i indep. Bernoulli, $P\{Y_i = 1\} = p$

Notice: $N_{n+1} = N_n + Y_{n+1} \Rightarrow$ evolution of $\{N_n\}$ beyond n does not depend on $\{N_i\}_{i=0}^{n-1}$ (given N_n) and thus $\{N_n\}$ is a M.C.

$$P\{N_{n+1} = j \mid N_0, N_1, ..., N_n\} = P\{Y_{n+1} = j - N_n \mid N_0, N_1, ..., N_n\}$$

$$= \begin{cases} p & \text{if } j = N_n + 1 \\ q = 1 - p & \text{if } j = N_n \\ 0 & \text{otherwise} \end{cases} \quad \text{and } P = \begin{bmatrix} q & p & 0 & ... & & \\ 0 & q & p & 0 & ... & \\ 0 & 0 & q & p & 0 & ... \\ \vdots & & & & & \end{bmatrix}$$

Notice: $\{N_n\}$ is a special M.C. whose increment is indep. both from present and past (process with indep. increments)

Example: Sum of i.i.d. RV's with PMF $\{p_k; k = 0, 1, 2, ...\}$

$$X_n = \begin{cases} 0 & n = 0 \\ Y_1 + Y_2 + ... + Y_n & n \geq 1 \end{cases}$$

$$X_{n+1} = X_n + Y_{n+1}$$

$$P\{X_{n+1} = j \mid X_0, ..., X_n\} = P\{Y_{n+1} = j - X_n \mid X_0, ..., X_n\} = p_{j-X_n}$$

Thus $\{X_n\}$ is a M.C. with $P(i,j) = P\{X_{n+1} = j \mid X_n = i\} = p_{j-i}$

$$P = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & p_0 & p_1 & \cdots \\ 0 & 0 & 0 & p_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example : Independent trials

$X_0, X_1, \ldots$ i.i.d. with $\pi(k) \quad , \quad k = 0,1,2,\ldots$

$P\{X_{n+1} = j \mid X_0, \ldots, X_n\} = P\{X_{n+1} = j\} = \pi(j)$

$\{X_n\}$ is a M.C.

$$P = \begin{bmatrix} \pi(0) & \pi(1) & \cdots \\ \pi(0) & \pi(1) & \cdots \\ \vdots & \vdots & \\ \pi(0) & \pi(1) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Notice that rows are identical and $P^m = P \quad \forall m \geq 1$

(If P has all rows identical then $X_0, X_1, \ldots$ are i.i.d.)

Example : $\{Y_n\}$ are i.i.d. $\quad Y_n \in \{0,1,2,3,4\}$ with $\{p_0, p_1, p_2, p_3, p_4\}$

$X_{n+1} = X_n + Y_{n+1}$ (modulo 5) $\quad , \quad \{X_n\}$ is a M.C.

$$P = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & p_4 \\ p_4 & p_0 & p_1 & p_2 & p_3 \\ p_3 & p_4 & p_0 & p_1 & p_2 \\ p_2 & p_3 & p_4 & p_0 & p_1 \\ p_1 & p_2 & p_3 & p_4 & p_0 \end{bmatrix}$$

$\sum$ rows $= 1$ (stoch. matrix)

$\sum$ columns $= 1$ (here)

(double - stochastic matrix)

Example : Remaining lifetime

An equipment is replaced by an identical as soon as it fails

$p_k = \Pr\{\text{a new equip. lasts for } k \text{ time units}\} \qquad k = 1, 2, \ldots$

X_n = remaining lifetime of equip. at time n

$$X_{n+1}(\omega) = \begin{cases} X_n(\omega) - 1 & \text{if } X_n(\omega) \ge 1 \\ Z_{n+1}(\omega) - 1 & \text{if } X_n(\omega) = 0 \end{cases}$$

$Z_{n+1}(\omega)$ is the lifetime of equip. installed at time n

It is independent of $X_0, X_1, \ldots, X_n$

X_n is a M.C.

- $i \ge 1$:

$$P(i,j) = P\{X_{n+1} = j \mid X_n = i\} = P\{X_n - 1 = j \mid X_n = i\}$$
$$= P\{X_n = j+1 \mid X_n = i\} = \begin{cases} 1 & \text{if } j = i-1 \\ 0 & \text{if } j \ne i-1 \end{cases}$$

- $i = 0$:

$$P(0,j) = P\{X_{n+1} = j \mid X_n = 0\} = P\{Z_{n+1} - 1 = j \mid X_n = 0\}$$
$$= P\{Z_{n+1} = j+1\} = p_{j+1}$$

$$P = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Theorem : (conditional indep. of future from past given present)

Let Y be a bounded function of $X_n, X_{n+1}, \ldots$. Then

$$E\{Y | X_0, X_1, \ldots, X_n\} = E\{Y | X_n\}$$

Current and future states

Proposition :

$$E\{f(X_n, X_{n+1}, \ldots) | X_n = i\} = E\{f(X_0, X_1, \ldots) | X_0 = i\}$$

Corollary : f a bounded function on $E \times E \times \ldots$

Let $g(i) = E\{f(X_0, X_1, \ldots) | X_0 = i\}$.

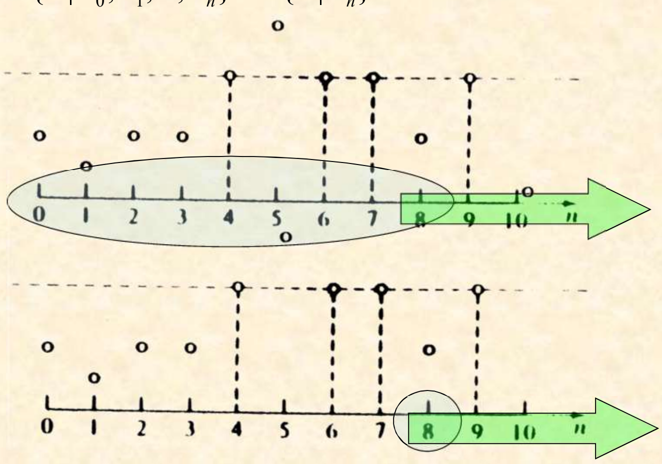
Then $\forall n \in N$ $\quad E\{f(X_n, X_{n+1}, \ldots) | X_0, X_1, \ldots, X_n\} = g(X_n)$

A function that depends on current and the future states to be visited, is determined fully by the current state only. Future states are probabilistically determined by the MP, while states visited in the past do not impact on future ones given the current.

Theorem : (conditional indep. of future from past given present)

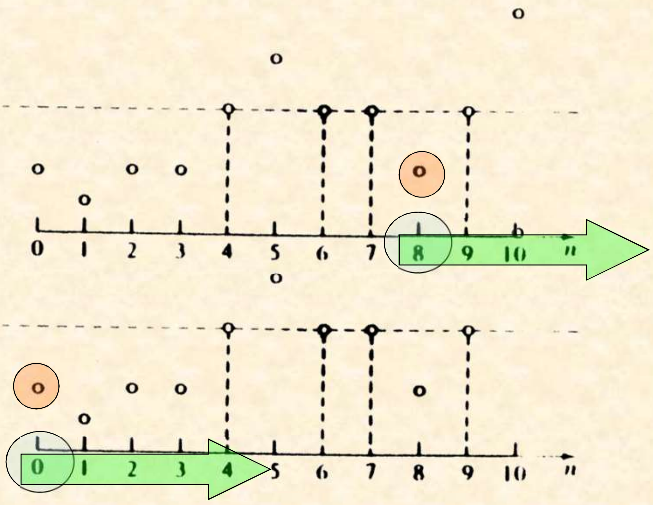
Let Y be a bounded function of $X_n, X_{n+1}, \ldots$. Then

$$E\{Y | X_0, X_1, \ldots, X_n\} = E\{Y | X_n\}$$



Proposition :

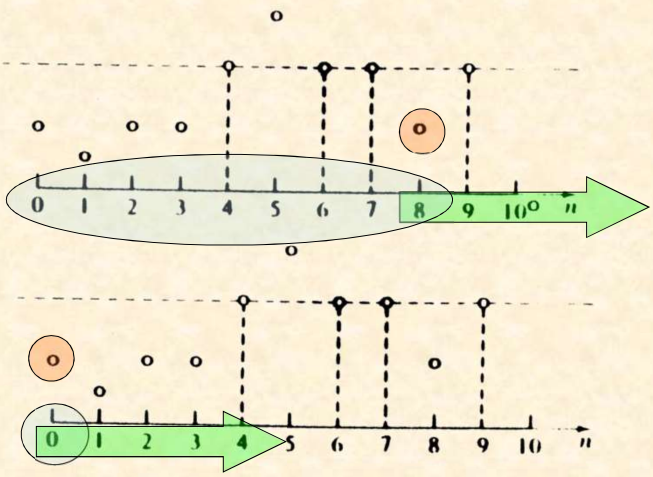
$$\mathrm{E}\{f(X_n, X_{n+1},...)|X_n = i\} = \mathrm{E}\{f(X_0, X_1,...)|X_0 = i\}$$



Corollary : f a bounded function on $E \times E \times ...$

Let $g(i) = \mathrm{E}\{f(X_0, X_1,...)|X_0 = i\}$.

Then $\forall n \in N$ $\mathrm{E}\{f(X_n, X_{n+1},...)|X_0, X_1,...,X_n\} = g(X_n)$



Stopping Times :

Previous results derived for fixed time $n \in N$

What if time is an RV instead?

- If for a RV T, the past $\{X_m; m \leq T\}$ and the future $\{X_m; m \geq T\}$ are conditionally indep. given present X_T, then the strong Markov property is said to hold at T.
- If T is a stopping time, then above hold true (T is a stopping time if the event $\{T \leq n\}$ can be determined by looking at $X_0, X_1, ..., X_n$)

For any stopping time T :

- $E\{f(X_T, X_{T+1}, ...) \mid X_n, n \leq T\} = E\{f(X_T, X_{T+1}, ...) \mid X_T\}$
- For $g(i) = E\{f(X_0, X_1, ...) \mid X_0 = i\}$

  $E\{f(X_T, X_{T+1}, ...) \mid X_n; n \leq T\} = g(X_T)$

  e.g., if $f(a_0, a_1, ...) = \begin{cases} 1 & \text{if} \quad a_m = j \\ 0 & \text{if} \quad a_m \neq j \end{cases} \qquad j \in E, m \in N$

  $E\{f(X_0, X_1, ...) \mid X_0 = i\} = P\{X_m = j \mid X_0 = i\} = P^m(i, j)$

  $E\{f(X_T, X_{T+1}, ...) \mid X_n, n \leq T\} = P\{X_{T+m} = j \mid X_n; n \leq T\}$

- Strong Markov property at T:

  $P\{X_{T+m} = j \mid X_n; n \leq T\} = P^m(X_T, j)$

Visits to a state

$X = \{X_n; n \in N\}$ MC, State space E, Transition matrix P.
Notation: $P_i\{A\} = P\{A \mid X_0 = i\}$ and $E_i[Y] = E[Y \mid X_0 = i]$
Let $j \in E$, $\omega \in \Omega$ and Define:

$N_j(\omega)$ = total number of times state j appears in $X_0(\omega), X_1(\omega), \ldots$.

♣ $N_j(\omega) < \infty$, X eventually leaves state j never to return.

♣ $N_j(\omega) = \infty$, X visits j again and again.

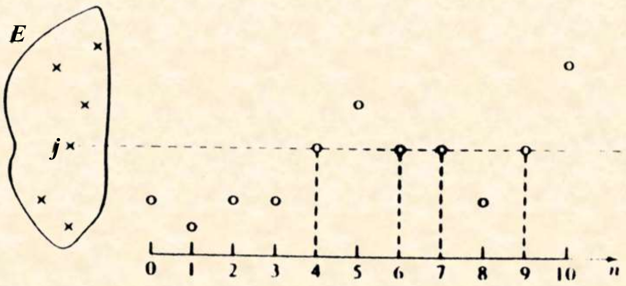
Let $T_1(\omega), T_2(\omega), \ldots$ the successive indices $n \geq 1$ for which $X_n(\omega) = j$.

$\forall n \in N$, $\{T_m(\omega) \leq n\}$ is equivalent to j appears in $\{X_1(\omega), \cdots, X_n(\omega)\}$ at least m times.

T_m is a stopping time.

Example

$T_1(\omega) = 4, \quad T_2(\omega) = 6, \quad T_3(\omega) = 7, \quad T_4(\omega) = 9, \ldots$



Proposition: $\forall i \in E, \quad k,m \geq 1$

$$P_i\{T_{m+1} - T_m = k \,|\, T_1,\ldots,T_m\} = \begin{cases} 0 & \{T_m = \infty\} \\ P_j\{T_1 = k\} & \{T_m < \infty\} \end{cases}$$

Computation of $P_j\{T_1 = k\}$. Let $F_k(i,j) = P_i\{T_1 = k\}$

$$k = 1 \Rightarrow F_k(i,j) = P_i\{T_1 = 1\} = P_i\{X_1 = j\} = P(i,j)$$

$$k \geq 2 \Rightarrow F_k(i,j) = P_i\{X_1 \neq j, \cdots, X_{k-1} \neq j, X_k = j\}$$

$$= \sum_{b \in E-\{j\}} P_i\{X_1 = b\} P_i\{X_2 \neq j, \cdots, X_{k-1} \neq j, X_k = j \,|\, X_1 = b\}$$

$$= \sum_{b \in E-\{j\}} P_i\{X_1 = b\} P_b\{X_1 \neq j, \cdots, X_{k-2} \neq j, X_{k-1} = j\}$$

Thus,

$$F_k(i,j) = \begin{cases} P(i,j) & k = 1 \\ \sum_{b \in E-\{j\}} P(i,b) F_{k-1}(b,j) & k \geq 2 \end{cases}$$

Probability to visit (be at) state j **for the first time** at step k, starting from state i

Example: Let $j = 3$ and the transition matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 3/5 & 1/15 \end{pmatrix}$

Find $f_k(i) = F_k(i,j), \quad i = 1,2,3$

- $k = 1$. In this case f_1 is the 3^{rd} column of matrix P.

  Hence, $f_1(1) = F_1(1,j) = 0$, $f_1(2) = F_1(2,j) = 1/3$, $f_1(3) = F_1(3,j) = 1/15$

Column vector

- $k \geq 2$. In this case $f_k = \begin{pmatrix} F_k(1,j) \\ F_k(2,j) \\ F_k(3,j) \end{pmatrix} = \begin{pmatrix} \sum_{b \in E-\{j\}} P(1,b) F_{k-1}(b,j) \\ \sum_{b \in E-\{j\}} P(2,b) F_{k-1}(b,j) \\ \sum_{b \in E-\{j\}} P(3,b) F_{k-1}(b,j) \end{pmatrix} = Q \cdot f_{k-1}$ where $Q = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/6 & 0 \\ 1/3 & 3/5 & 0 \end{pmatrix}$

After some algebra $f_1 = \begin{pmatrix} 0 \\ 1/3 \\ 1/15 \end{pmatrix}$ $f_2 = \begin{pmatrix} 0 \\ 1/18 \\ 1/5 \end{pmatrix}$ $f_3 = \begin{pmatrix} 0 \\ 1/108 \\ 1/30 \end{pmatrix}$ $f_4 = \begin{pmatrix} 0 \\ 1/648 \\ 1/180 \end{pmatrix}$ $\cdots$

and in general

$$F_k(1,3) = 0, \qquad F_k(2,3) = \frac{1}{3}\left(\frac{1}{6}\right)^{k-1}, \qquad F_k(3,3) = \begin{cases} \frac{1}{15} & k = 1 \\ \frac{3}{5}\left(\frac{1}{6}\right)^{k-2}\frac{1}{3} & k \geq 2 \end{cases}$$

$$F_k(1,3)=0, \qquad F_k(2,3)=\frac{1}{3}\left(\frac{1}{6}\right)^{k-1}, \qquad F_k(3,3)=\begin{cases}\frac{1}{15} & k=1\\ \frac{3}{5}\left(\frac{1}{6}\right)^{k-2}\frac{1}{3} & k\geq 2\end{cases}$$

Now we can state:

- Starting at state 1, X never visits 3 with probability: $P_1\{T_1=+\infty\}=1$
- Starting at state 2, X first visits 3 at k with probability: $\frac{1}{3}(\frac{1}{6})^{k-1}$
- Starting at state 2, X never visits 3 with probability: $P_2\{T_1=+\infty\}=1-P_2\{T_1<+\infty\}=1-\sum_{k=1}^{\infty}\frac{1}{3}(\frac{1}{6})^{k-1}=\frac{3}{5}$
- Starting at state 3, X never visits 3 again with probability: $P_3\{T_1=+\infty\}=1-P_3\{T_1<+\infty\}=\frac{52}{75}$

Now, for every i, j we define

$$F(i,j)=P_i\{T_1<+\infty\}=\sum_{k=1}^{\infty}F_k(i,j)$$

♣ $F(i,j)$ expresses the probability: starting at i the MC will ever visit state j.

$$F(i,j)=P(i,j)+\sum_{b\in E-\{j\}}P(i,b)F(b,j), \qquad i\in E$$

If by N_j we denote the total number of visits to state j, then

$$P_j\{N_j=m\}=F(j,j)^{m-1}\left(1-F(j,j)\right)$$

and for $i\neq j$,

$$P_i\{N_j=m\}=\begin{cases}1-F(i,j) & m=0\\ F(i,j)F(j,j)^{m-1}\left(1-F(j,j)\right) & m=1,2,\ldots\end{cases}$$

>From the previous we obtain the Corollary:

$$P_j\{N_j<+\infty\}=\begin{cases}1 & F(j,j)<1\\ 0 & F(j,j)=1\end{cases}$$

$$\sum_{m=1}^{\infty} P_j\{N_j = m\}$$
$$= \sum_{m=1}^{\infty} F(j,j)^{m-1}\left(1 - F(j,j)\right)$$
$$= \frac{1}{1 - F(j,j)}\left(1 - F(j,j)\right) = 1$$

$$j) = P_i\{T_1 < +\infty\} = \sum^{\infty} F_k(i,j)$$

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}, \quad |x| < 1$$

starting at ... ate j.

$$i,j) + \sum_{b \in E - \{j\}} P(i,b)F(b,j), \qquad i \in E$$

If by N_j we denote t... number of visits to state j, then

$$P_j ... m\} = F(j,j)^{m-1}\left(1 - F(j,j)\right)$$

and for $i \neq j$,
$$P_i\{N ... n\} = \begin{cases} 1 - F(i,j) & m = 0 \\ F(i,j)F(j,j)^{m-1}\left(1 - F(j,j)\right) & m = 1,2,\ldots \end{cases}$$

>From the previous we obtai the Corollary:

$$P_j\{N_j < +\infty\} = \begin{cases} 1 & F(j,j) < 1 \\ 0 & F(j,j) = 1 \end{cases}$$

$$\sum_{m=1}^{\infty} m\, P_j\{N_j = m\}$$
$$= \sum_{m=1}^{\infty} m\, F(j,j)^{m-1}(1 - F(j,j))$$
$$= \frac{1}{(1 - F(j,j))^2}(1 - F(j,j)) = \frac{1}{(1 - F(j,j))}$$

$$\sum_{m=1}^{\infty} m\, x^{m-1} = \frac{1}{(1-x)^2}, \quad |x| < 1$$

- If $F(j,j) = 1 \Rightarrow N_j = +\infty$ w.p.1. Therefore, if $X_0 = j \Rightarrow E_j[N_j] = +\infty$
- If $F(j,j) < 1$ then N_j follows geometric distribution with probability of success $p = 1 - F(j,j)$. Hence, $E_j[N_j] = \frac{1}{p} = \frac{1}{1 - F(j,j)}$

Let $R(i,j) = E_i[N_j]$ (R is called the **potential** matrix of X)

Then,

$$R(j,j) = \frac{1}{1-F(j,j)}$$

$$R(i,j) = F(i,j)\, R(j,j)\,, \quad (i \neq j)$$

$$R(i,j) = F(i,j)\, R(j,j) + (1 - F(i,j))\, 0$$

Computation of $R(i,j)$ first and then $F(i,j)$

Define:

$$1_j(k) = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \Rightarrow 1_j(X_n(\omega)) = \begin{cases} 1, & X_n(\omega) = j \\ 0, & X_n(\omega) \neq j \end{cases}$$

Then,

$$N_j(\omega) = \sum_{n=0}^{\infty} 1_j(X_n(\omega))$$

$$R(i,j) = E_i\left[\sum_{n=0}^{\infty} 1_j(X_n)\right] = \sum_{n=0}^{\infty} E_i\left[1_j(X_n)\right] = \sum_{n=0}^{\infty} P_i\{X_n = j\} = \sum_{n=0}^{\infty} P^n(i,j)$$

In matrix notation:

$$R = I + P + P^2 + \cdots \Rightarrow RP = PR = P + P^2 + \cdots = R - I$$

from which we obtain

$$R(I-P) = (I-P)R = I$$

Classification of states

X: MC, with state space E, transition matrix P

T : The time of first visit to state j

N_j : The total number of visits to state j

Definition

♣ State j is called **recurrent** if $P_j\{T < \infty\} = 1$

♣ State j is called **transient** if $P_j\{T = \infty\} > 0$

♣ A recurrent state j is called **null** if $E_j[T] = \infty$

♣ A recurrent state j is called **non-null** if $E_j[T] < \infty$

♣ A recurrent state j is called **periodic** with period δ, if $\delta \geq 2$ is the greatest integer for which

$$P_j\{T = n\delta \text{ for some } n \geq 1\} = 1$$

R(j,j) = 1 / {(1-F(j,j) }

- If j is recurrent then starting at j the probability of returning to j is 1.

$$F(j,j)=1 \Rightarrow R(j,j)=E_j[N_j]=+\infty \Longleftrightarrow P_j\{N_j=+\infty\}=1$$

- If j is transient then there exists a positive probability $1-F(j,j)$ of never returning to j.

$$F(j,j)<1 \Rightarrow R(j,j)=E_j[N_j]<\infty \Longleftrightarrow P_j\{N_j<\infty\}=1$$

In this case $R(i,j)=F(i,j)R(j,j)<R(j,j)<\infty$ and since $R(i,j)=\sum_n P^n(i,j)$ we conclude that

$$\lim_{n\to\infty} P^n(i,j)\to 0$$

Theorem:

♣ If j <u>transient</u> or recurrent <u>null</u> (for which $E_j[T]=\infty$) then

$$\forall i\in E, \qquad \lim_{n\to\infty} P^n(i,j)\to 0$$

♣ If j recurrent <u>non-null</u> then

$$\pi(j)=\lim_{n\to\infty} P^n(j,j)>0 \quad \text{and} \quad \forall i\in E, \qquad \lim_{n\to\infty} P^n(i,j)=F(i,j)\pi(j)$$

♣ If j <u>periodic</u> with period δ, then a retutrn to j is possible only at steps numbered δ, 2δ, 3δ, ...

$$P^n(j,j)=P_j\{X_n=j\}>0 \text{ only if } n\in\{0,\delta,2\delta,\ldots\}$$

Recurrent non-null	Recurrent null	Transient
$P_j\{T<\infty\}=1$		$P_j\{T=\infty\}>0$
$E_j[T]<\infty$	$E_j[T]=\infty$	
$F(j,j)=1 \Rightarrow R(j,j)=E_j[N_j]=+\infty \Longleftrightarrow P_j\{N_j=+\infty\}=1$		$F(j,j)<1 \Rightarrow R(j,j)=E_j[N_j]<\infty \Longleftrightarrow P_j\{N_j<\infty\}=1$
$\pi(j)=\lim_{n\to\infty}P^n(j,j)>0$ and $\forall i\in E$, $\lim_{n\to\infty}P^n(i,j)=F(i,j)\pi(j)$	$\forall i\in E,\quad \lim_{n\to\infty}P^n(i,j)\to 0$	

♣ A recurrent state j is called **periodic** with period δ, if $\delta \geq 2$ is the greatest integer for which

$P_j\{T=n\delta \text{ for some } n\geq 1\}=1$

♣ If j periodic with period δ, then a return to j is possible only at steps numbered δ, 2δ, 3δ, ...

$$P^n(j,j)=P_j\{X_n=j\}>0 \text{ only if } n\in\{0,\delta,2\delta,\ldots\}$$

We say that state j can be reached from state i $i\to j$, if $\exists n\geq 0: P^n(i,j)>0$

$i\to j$, iff $F(i,j)>0$

Definition:

- A set of states is **closed** if no state outside it can be reached from any state in it.
- A state forming a closed set by itself is called an **absorbing** state
- A closed set is called **irreducible** if no proper subset of it is closed.
- A MC is called irreducible if its only closed set is the set of all states

Comments:

♣ If j is absorbing then $P(j,j)=1$.
♣ If MC is irreducible then all states can be reached from each other.
♣ If $C=\{c_1,c_2,\cdots\}\in E$ is a closed set and $Q(i,j)=P(c_i,c_j)$, $c_i,c_j\in C$, then Q is a Markov matrix.
♣ If $i\to j$ and $j\to k$ then $i\to k$.

To find the closed set C that contains i we work as follows:

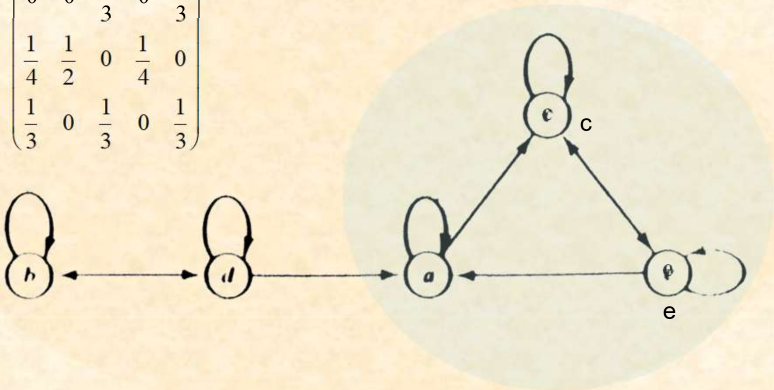
- Starting with i we include in C all states j that can be reached from i: $P(i,j)>0$.
- We next include in C all states k that can be reached from j: $P(j,k)>0$.
- We repeat the previous step

Example: MC with state space $E=\{a,b,c,d,e\}$ and transition matrix

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \quad P=\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

Comments:

- Closed sets: $\{a,c,e\}$ and $\{a,b,c,d,e\}$
- There are two closed sets. Thus, the MC is not irreducible.



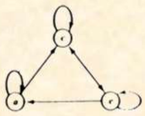
Example: MC with state space $E=\{a,b,c,d,e\}$ and transition matrix

$$P=\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

Comments:

- Closed sets: $\{a,c,e\}$ and $\{a,b,c,d,e\}$
- There are two closed sets. Thus, the MC is not irreducible.
- If we delete the 2^{nd} and 4^{th} rows we obtain the Markov matrix:

$$Q=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$



If we relabel the states $1=a$, $2=c$, $3=e$, $4=b$ and $5=d$ we get

$$\overline{P}=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

<u>Lemma</u> If j recurrent and $j \to k \Rightarrow k \to j$. Thus, $F(k,j)=1$.

<u>Proof:</u> If $j \to k$ then k is reached without returning to j with probability **a**. Once k is reached, the probability that j is never visited again is $1-F(k,j)$. Hence,

$$1-F(j,j) \geq \text{a}(1-F(k,j)) \geq 0$$

But j is recurrent, so that $F(j,j)=1 \Rightarrow F(k,j)=1$

♠ As a result: If $j \to k$ but $k \not\to j$, then j **must** be **transient**.

Theorem: From recurrent states only recurrent states can be reached.
Theorem: In a Marcov chain the recurrent states can be divided in a unique manner, into irreducible closed sets C_1, C_2, ..., and after an appropriate arrangement:

$$P = \begin{pmatrix} P_1 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ 0 & 0 & P_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ Q_1 & Q_2 & Q_3 & \cdots & Q \end{pmatrix}$$

Theorem: Let X an <u>irreducible</u> MC. Then, one of the following holds:

- All states are transient.
- All states are recurrent null
- All states are recurrent non-null
- Either all aperiodic or if one is periodic with period δ, all are periodic with the same period.

<u>Proof:</u> Since X is irreducible then $j \to k$ and $k \to j$, which means that $\exists r,s$: $P^r(j,k)>0$ and $P^s(k,j)>0$. Pick the smallest r,s and let $\beta = P^r(j,k)P^s(k,j)$.

- If k recurrent $\Rightarrow j$ recurrent.
- If k transient $\Rightarrow j$ transient. (If it was recurrent then k would be recurrent)
- If k recurrent null then $P^m(k,k) \to 0$ as $m \to \infty$. But $P^{n+r+s}(k,k) \geq \beta P^n(j,j) \Rightarrow P^n(j,j) \to 0$

Corollary: If C irreducible closed set of **finitely** many states, then ∄ recurrent null states.

Proof: If one is recurrent null then all states are recurrent null.

Thus, $\lim_{n\to\infty} P^n(i,j)=0, \quad \forall i,j \in C$. But,

$$\forall i \in C, n \geq 0, \sum_{j\in C} P^n(i,j)=1 \Rightarrow \lim_{n\to\infty}\sum_{j\in C} P^n(i,j)=1$$

Because, we have finite number of states

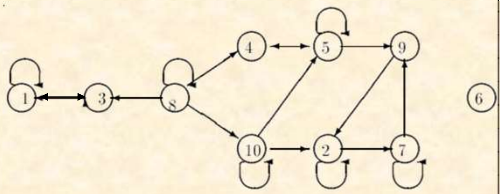
$$\lim_{n\to\infty}\sum_{j\in C} P^n(i,j)=\sum_{j\in C}\lim_{n\to\infty} P^n(i,j)=0$$

Corollary: If C is an irreducible closed set with finitely many states then there are **no** transient states

MC with Finite number of states - algorithm

- Identify irreducible closed sets.
- All states belonging to an irreducible closed set are recurrent positive
- The rest of the states are transient
- Periodicity is checked to each irreducible set

Example:
The irreducible closed sets are $\{1,3\}$, $\{2,7,9\}$ and $\{6\}$. The states $\{4,5,8,10\}$ are transient. If we relabel the states we obtain



$$P=\begin{pmatrix} 1 & & & & & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & & & & & \\ & 1 & 0 & & & & & & & \\ & & & \frac{1}{3} & \frac{2}{3} & 0 & & & & \\ & & & 0 & \frac{1}{4} & \frac{3}{4} & & & & \\ & & & 1 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

Example: Let N_n the number of successes in the first n Bernoulli trials. As we have seen

$$P(i,j)=P\{N_{n+1}=j \mid N_n=i\}=\begin{cases} p & j=i+1 \\ q & j=i \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$P=\begin{pmatrix} q & p & 0 & \cdots \\ 0 & q & p & \cdots \\ 0 & 0 & q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\forall j$ we have $j \to j+1$ but $j+1 \not\to j$. This means that j is **not** recurrent => j is **transient**.

Example: Remaining lifetime

Remember: $$X_{n+1}(\omega) = \begin{cases} X_n(\omega) - 1 & X_n(\omega) \geq 1 \\ Z_{n+1}(\omega) - 1 & X_n(\omega) = 0 \end{cases}$$

from which we obtain:

$i \geq 1$ $$P(i,j) = P\{X_{n+1} = j \mid X_n = j\} = P\{X_n - 1 = j \mid X_n = j\} = \begin{cases} 1 & j = i-1 \\ 0 & j \neq i-1 \end{cases}$$

$i = 0$ $$P(0,j) = P\{X_{n+1} = j \mid X_n = 0\} = P\{Z_{n+1} - 1 = j \mid X_n = 0\}$$
$$= P\{Z_{n+1} = j+1\} = p_{j+1}$$

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

>From state 0 we reach state j in one step. From j we can reach $j-1$, $j-2$, ..., 1, 0. Thus, all states can be reached from each other, which means that the MC is **<u>irreducible.</u>** Since, $P(0,0) > 0$ the MC is aperiodic. Return to state 0 occurs if the lifetime is finite:

$$\sum_j p_j = 1 \Rightarrow F(0,0) = \sum_j p_j = 1$$

Since state 0 is recurrent, all states are recurrent.
If the expected lifetime:

$$\sum_j jp_j = +\infty$$

then state 0 is null and all states are recurrent null.
If the expected lifetime:

$$\sum_j jp_j < \infty$$

then state 0 is non-null and all states are recurrent non-null.

MC with Infinite number of states - algorithm

Theorem: Let X an **irreducible MC**, and consider the system of linear equations:

$$\nu(j) = \sum_{i \in E} \nu(i) P(i,j), \qquad j \in E$$

Then all states are **recurrent non-null** iff there exists a solution ν with

$$\sum_{j \in E} \nu(j) = 1$$

Theorem: Let X an **irreducible MC** with transition matrix P, and let Q be the matrix obtained from P by deleting the k-row and k-column for some $k \in E$. Then all states are **recurrent** if and only if the only solution of

$$h(i) = \sum_{j \in E_0} Q(i,j) h(j), \qquad 0 \le h(i) \le 1, \qquad i \in E_0$$

is $h(i) = 0$ for all $i \in E_0$. $E_0 = E - \{k\}$.

- ***Use first theorem to determine whether all states are recurrent non-null or not.***
- ***In the latter case, use the second theorem to determine whether the states are transient or not.***

Example: Random walks.

$$P = \begin{pmatrix} 0 & 1 & & & \cdots \\ q & 0 & p & & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- All states can be reached from each other, and **thus the chain is irreducible**.
- A return to state 0 can occur only at steps numbered 2,4,6,... Therefore, state 0 is periodic with period $\delta = 2$.
- Since X is irreducible all states are periodic with period 2.
- Either all states are recurrent null, or all are recurrent non-null, or all the states are transient.

Check for a solution of $\nu = \nu P$.

$$\nu_0 = q\nu_1$$
$$\nu_1 = \nu_0 + q\nu_2$$
$$\nu_2 = p\nu_1 + q\nu_3$$
$$\nu_3 = p\nu_2 + q\nu_4$$

Hence,

$$\nu_1 = \tfrac{1}{q}\nu_0$$
$$\nu_2 = \tfrac{1}{q}\left(\tfrac{1}{q}\nu_0 - \nu_0\right) = \tfrac{p}{q^2}\nu_0$$
$$\nu_3 = \tfrac{1}{q}\left(\tfrac{p}{q^2} - \tfrac{p}{q}\right)\nu_0 = \tfrac{p^2}{q^3}\nu_0$$

Any solution is of the form

$$\nu_j = \frac{1}{q}\left(\frac{p}{q}\right)^{j-1}\nu_0, \qquad j = 1,2,...$$

If $p < q$, then $p/q < 1$ and

$$\sum_{j=0}^{\infty}\nu_j = \left(1 + \frac{1}{q}\sum_{j=1}^{\infty}\left(\frac{p}{q}\right)^{j-1}\right)\nu_0 = \frac{2q}{q-p}\nu_0$$

If we choose $\nu_0 = \frac{q-p}{2q}$ then $\sum \nu_j = 1$ and

$$\nu(j) = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{p}{q}\right), & j = 0 \\ \dfrac{1}{2q}\left(1 - \dfrac{p}{q}\right)\left(\dfrac{p}{q}\right)^{j-1}, & j \geq 1 \end{cases}$$

In this case all states are **recurrent non null**

If $p > q$ either all states are recurrent null or all states are transient. Consider the matrix (excluding state 0)

$$Q = \begin{pmatrix} 0 & p & & & \cdots \\ q & 0 & p & & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The equation $h = Qh$ gives ($h_i = h(i)$)

$$h_{i+1} = \left[\left(\frac{q}{p}\right)^i + \left(\frac{q}{p}\right)^{i-1} + \cdots + \frac{q}{p} + 1\right]h_1$$

- If $p = q$ then $h_i = ih_1$ for all $i \geq 1$ and the only way to have $0 \leq h_i \leq 1$ for all i is by choosing $h_1 = 0$ which implies $h_i = 0$ that is all states are **recurrent null**.
- If $p > q$, then choosing $h_1 = 1 - (q/p)$, we get

$$h_i = 1 - \left(\frac{q}{p}\right)^i$$

which also satisfies $0 \leq h_i \leq 1$. In this case all states are **transient**.

Calculation of R and F

♣ $R(i,j) = E_i[N_j]$ Expected number of visits to state j.
♣ $F(i,j) =$ The probability of **ever** reaching state j starting at i.

j **Recurrent state:** $F(j,j) = 1 \Rightarrow R(j,j) = \infty$

$$R(i,j) = F(i,j)R(j,j) \quad R(i,j) = \begin{cases} 0 & F(i,j) = 0 \\ +\infty & F(i,j) > 0 \end{cases}$$

j **Transient /** i **Recurrent state:** $F(i,j) = 0 \Rightarrow R(i,j) = 0$

i,j **Transient**
Let $D = \{$ the transient states $\}$, $Q(i,j) = P(i,j)$, $S(i,j) = R(i,j)$, $i,j \in D$.

Then $$P = \begin{pmatrix} K & 0 \\ L & Q \end{pmatrix} \Rightarrow P^m = \begin{pmatrix} K^m & 0 \\ L_m & Q^m \end{pmatrix}$$

Hence, $$R = \sum_{m=0}^{\infty} P^m = \begin{pmatrix} \sum K^m & 0 \\ \sum L_m & \sum Q^m \end{pmatrix} \Rightarrow S = \sum_{m=0}^{\infty} Q^m = I + Q + Q^2 + \cdots$$

Computation of S

$$S = I + Q + Q^2 + \cdots \Rightarrow$$
$$SQ = QS = Q + Q^2 + \cdots = S - I \Rightarrow$$
$$(I-Q)S = I, \quad S(I-Q) = I$$

Proposition: If there are finitely many transient states $S = (I-Q)^{-1}$

♣ When the set D of transient states is infinite, it is possible to have more than one solution to the system.

Theorem: S is the minimal solution of $(I-Q)Y = I$, $Y \geq 0$

Theorem: S is the unique solution of $(I-Q)Y = I$ if and only if the only bounded solution of $h = Qh$ is $h = 0$, or equivalently

$$h = Qh, 0 \leq h \leq 1 \Longleftrightarrow h = 0$$

Example: Let X a MC with state space $E = \{1,2,3,4,5,6,7,8\}$

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 & | & & & | & & & \\ 0. & 0.6 & 0.4 & | & & & | & & & \\ 0.5 & 0.5 & 0. & | & & & | & & & \\ - & - & - & | & - & - & | & - & - & - \\ & & & | & 0. & 1. & | & & & \\ & & & | & 0.8 & 0.2 & | & & & \\ - & - & - & | & - & - & | & - & - & - \\ 0. & 0. & 0. & | & & & | & 0.4 & 0.6 & 0. \\ 0.4 & 0.4 & 0. & | & & & | & 0. & 0. & 0.2 \\ 0.1 & 0. & 0.3 & | & & & | & 0.6 & 0. & 0. \end{pmatrix}$$

- $\{1,2,3\}$ are recurrent positive aperiodic.
- $\{4,5\}$ are recurrent positive aperiodic.
- $\{6,7,8\}$ are transient (reaching states 1,2,3 only)

$$Q = \begin{pmatrix} 0.4 & 0.6 & 0. \\ 0. & 0. & 0.2 \\ 0.6 & 0. & 0. \end{pmatrix} \Rightarrow S = (I-Q)^{-1} = \begin{pmatrix} 0.6 & -0.6 & 0. \\ 0. & 1. & -0.2 \\ -0.6 & 0. & 1. \end{pmatrix}^{-1}$$

j recurrent, can be reached from i

j recurrent, cannot be reached from i

j transient, i recurrent

j, i transient

j recurrent | j transient

i recurrent (first two row blocks), i transient (last row block)

$$R = \begin{pmatrix} \infty & \infty & \infty & | & 0 & 0 & | & 0 & 0 & 0 \\ \infty & \infty & \infty & | & 0 & 0 & | & 0 & 0 & 0 \\ \infty & \infty & \infty & | & 0 & 0 & | & 0 & 0 & 0 \\ - & - & - & | & - & - & | & - & - & - \\ 0 & 0 & 0 & | & \infty & \infty & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & \infty & \infty & | & 0 & 0 & 0 \\ - & - & - & | & - & - & | & - & - & - \\ \infty & \infty & \infty & | & 0 & 0 & | & \frac{125}{66} & \frac{75}{66} & \frac{15}{66} \\ \infty & \infty & \infty & | & 0 & 0 & | & \frac{15}{66} & \frac{75}{66} & \frac{15}{66} \\ \infty & \infty & \infty & | & 0 & 0 & | & \frac{75}{66} & \frac{45}{66} & \frac{75}{66} \end{pmatrix}$$

S

Computation of $F(i,j)$

♣ i,j **recurrent** belonging to the same irreducible closed set

$$F(i,j)=1$$

♣ i,j **recurrent** belonging to different irreducible closed sets

$$F(i,j)=0$$

♣ i,j **transient** Then $R(i,j)<\infty$ and

$$F(j,j)=1-\frac{1}{R(j,j)},\qquad F(i,j)=\frac{R(i,j)}{R(j,j)}$$

♣ i **transient**, j **recurrent** ????

Lemma: If C is irreducible closed set of recurrent states, then for any transient state i:

$$F(i,j)=F(i,k)$$

for all j, k $\in C$.

Proof: For $j,k\in C\Rightarrow F(j,k)=F(k,j)=1$. Thus, once the chain reaches any one of the states of C, it also visits all the other states. Hence, $F(i,j)=F(i,k)$ is the probability of entering the set C from i.

Let

$$P=\begin{pmatrix} P_1 & & & & \\ & P_2 & & & \\ & & P_3 & & \\ & & & \ddots & \\ Q_1 & Q_2 & Q_3 & & Q \end{pmatrix}$$

Lump all states of C_j together to make one absorbing state:

$$\hat{P}=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ b_1 & b_2 & b_3 & \cdots & b_m & Q \end{pmatrix},\quad b_j(i)=\sum_{k\in C_j}P(i,k), i\in D$$

same quantity

The probability of ever reaching the absorbing state j from the transient state i by the chain with the transition matrix $\hat{P}$ is the same as that of ever reaching C_j from i.

$$\hat{P}=\begin{pmatrix} I & 0 \\ B & Q \end{pmatrix},\qquad B=\begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix},\qquad B(i,j)=\sum_{k\in C_j}P(i,k),\quad i\in D$$

$$\hat{P}^n=\begin{pmatrix} I & 0 \\ B_n & Q^n \end{pmatrix},\qquad B_n=(I+Q+Q^2+\cdots+Q^{n-1})B$$

$B_n(i,j)$ is the probability that starting from i, the chain enters the recurrent class C_j by step n

Define:

$$G = \lim_{n\to\infty} B_n = \left(\sum_{k=0}^{\infty} Q^k\right) B = SB$$

♣ $G(i,j)$ is the probability of ever reaching the set C_j from the transient state i: ($F(i,j)$)

Proposition: Let Q the matrix obtained from P by deleting all the rows and columns corresponding to the recurrent states, and let B be defined as previously, for each transient i and recurrent class C_j.

- Compute S
- Compute $G = SB$
- $G(i,j) = F(i,k)$, $\forall k \in C_j$.

• If there is only one recurrent class and finitely many transient states, then things are different.
In this case, it can be proved that:

$$G = 1 \Rightarrow F(i,j) = 1, \qquad \forall j \in C$$

Example: Let X a MC with state space $E = \{1,2,3,4,5,6,7,8\}$

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 & | & & & | & & & \\ 0. & 0.6 & 0.4 & | & & & | & & & \\ 0.5 & 0.5 & 0. & | & & & | & & & \\ - & - & - & | & - & - & | & - & - & - \\ & & & | & 0. & 1. & | & & & \\ & & & | & 0.8 & 0.2 & | & & & \\ - & - & - & | & - & - & | & - & - & - \\ 0. & 0. & 0. & | & & & | & 0.4 & 0.6 & 0. \\ 0.4 & 0.4 & 0. & | & & & | & 0. & 0. & 0.2 \\ 0.1 & 0. & 0.3 & | & & & | & 0.6 & 0. & 0. \end{pmatrix}$$

i, j recurrent belonging to the same irreducible closed set

i, j recurrent belonging to different irreducible closed sets

j recurrent | j transient

i recurrent / i transient

$$F = \left(\begin{array}{ccc|cc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 0 & 0 & 0.472 & 1. & 0.20 \\ 1 & 1 & 1 & 0 & 0 & 0.12 & 0.12 & 0.20 \\ 1 & 1 & 1 & 0 & 0 & 0.60 & 0.60 & 0.12 \end{array}\right)$$

j transient, i recurrent

j, i transient

$$F(j,j) = 1 - \frac{1}{R(j,j)},$$

$$F(i,j) = \frac{R(i,j)}{R(j,j)}$$

one (reachable) recurrent class and finitely many transient states

Example:

$$P = \begin{pmatrix} 0.5 & 0.5 & & & & & \\ 0.8 & 0.2 & & & & & \\ & & 0. & 0.4 & 0.6 & & \\ & & 1. & 0. & 0. & & \\ & & 1 & 0. & 0. & & \\ 0.1 & 0. & 0.2 & 0.2 & 0.1 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0. & 0.1 & 0.2 & 0.4 \end{pmatrix} \Rightarrow \hat{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{pmatrix}$$

Thus,

$$S = (I-Q)^{-1} = \begin{pmatrix} 0.7 & -0.1 \\ -0.2 & 0.6 \end{pmatrix}^{-1} = \begin{pmatrix} 1.50 & 0.25 \\ 0.50 & 1.75 \end{pmatrix}$$

$$G = S \cdot B = \begin{pmatrix} 1.50 & 0.25 \\ 0.50 & 1.75 \end{pmatrix} \begin{pmatrix} 0.1 & 0.5 \\ 0.2 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ & & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & & \\ 0.2 & 0.2 & 0.8 & 0.8 & 0.8 & \frac{1}{3} & \frac{1}{7} \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.6 & \frac{1}{3} & \frac{3}{7} \end{pmatrix}$$

Recurrent states and Limiting probabilities

♣ Consider only an irreducible set of states.
Theorem: Suppose X is irreducible and aperiodic. Then all states are recurrent non-null if and only if

$$\pi(j) = \sum_{i \in E} \pi(i) P(i,j),\ j \in E, \quad \sum_{j \in E} \pi(j) = 1$$

has a solution π. If there exists a solution π, then it is strictly positive, there are no other solutions, and we have

$$\pi(j) = \lim_{n \to \infty} P^n(i,j), \forall i, j \in E$$

Corollary: If X in an irreducible aperiodic MC with finitely many states (no-null states, no transient states), then

$$\pi \cdot P = \pi, \quad \pi \cdot 1 = 1$$

has a unique solution. The solution π is strictly positive, and $\pi(j) = \lim_{n \to \infty} P^n(i,j)$, $\forall i, j$.

♣ A **probability** distribution π which satisfies $\pi = \pi \cdot P$, is called an **invariant** distribution for X.

♣ If π is the initial distribution of X, that is, $P\{X_0 = j\} = \pi(j), \quad j \in E$

then $P\{X_n = j\} = \sum_i \pi(i) P^n(i,j) = \pi(j)$, for any $n \in E$

<u>Proof:</u> $\pi = \pi \cdot P = \pi \cdot P^2 = \cdots = \pi \cdot P^n$

<u>Algorithm:</u> for finding $\lim_{n \to \infty} P^n(i,j)$

- Consider the irreducible closed set containing j
- Solve for $\pi(j)$. Thus, we find $\lim_{n \to \infty} P^n(j,j)$
- For every i (not necessarily in E)

$$\lim_{n \to \infty} P^n(i,j) = F(i,j) \lim_{n \to \infty} P^n(j,j)$$

Compute $F(i,j)$ first. Then, find $\lim_{n \to \infty} P^n(i,j)$

Example:

$$E=\{1, 2, 3\},\ P=\begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0. & 0.4 \\ 0. & 0.4 & 0. \end{pmatrix}$$

$$\pi P=\pi \quad \Rightarrow \quad \begin{array}{lllllll} \pi(1) & = & \pi(1)0.3 & + & \pi(2)0.6 & & \\ \pi(2) & = & \pi(1)0.5 & + & & + & \pi(3)0.4 \\ \pi(3) & = & \pi(1)0.2 & + & \pi(2)0.4 & + & \pi(3)0.6 \end{array}$$

$$\pi 1=1$$

System's Solution:

$$\pi=\begin{pmatrix} \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \end{pmatrix} \Rightarrow P^{\infty}=\lim_{n\to\infty} P^{n}(i,j)=\begin{pmatrix} \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \\ \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \\ \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \end{pmatrix}$$

Example:

$$E=\{1, 2, 3, 4, 5, 6, 7\},\quad P=\begin{pmatrix} 0.2 & 0.8 & & & & & \\ 0.7 & 0.3 & & & & & \\ & & 0.3 & 0.5 & 0.2 & & \\ & & 0.6 & 0. & 0.4 & & \\ & & 0. & 0.4 & 0.6 & & \\ 0. & 0.1 & 0.1 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0. & 0.1 & 0.2 & 0.4 \end{pmatrix}$$

$$P_1=\begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix} \Rightarrow \pi_1=\begin{pmatrix} \frac{7}{15} & \frac{8}{15} \end{pmatrix},\quad P_2=\begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0. & 0.4 \\ 0. & 0.4 & 0.6 \end{pmatrix} \Rightarrow \pi_2=\begin{pmatrix} \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \end{pmatrix}$$

Compute $F(i,j)$ first. Then, find $\lim_{n\to\infty} P^{n}(i,j)$ (for the transient at the bottom)

- Compute S
- Compute $G=SB$
- $G(i,j)=F(i,k)$, $\forall k \in C_j$.

$$\begin{bmatrix} F(6,1) & \cdots & F(6,5) \\ F(7,1) & \cdots & F(7,5) \end{bmatrix}=\begin{bmatrix} 0.2 & 0.2 & 0.8 & 0.8 & 0.8 \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.6 \end{bmatrix}$$

Thus,

$$P^{\infty} = \lim_{n\to\infty} P^n = \begin{pmatrix} \frac{7}{15} & \frac{8}{15} & & & & & \\ \frac{7}{15} & \frac{8}{15} & & & & & \\ & & \frac{6}{23} & \frac{7}{23} & \frac{10}{23} & & \\ & & \frac{6}{23} & \frac{7}{23} & \frac{10}{23} & & \\ & & \frac{6}{23} & \frac{7}{23} & \frac{10}{23} & & \\ \frac{1.4}{15} & \frac{1.6}{15} & \frac{4.8}{23} & \frac{5.6}{23} & \frac{8}{23} & 0. & 0. \\ \frac{2.8}{15} & \frac{3.2}{15} & \frac{3.6}{23} & \frac{4.2}{23} & \frac{6}{23} & 0. & 0. \end{pmatrix}$$

(for the transient at the bottom)

$$\lim_{n\to\infty} P^n(i,j) = F(i,j) \lim_{n\to\infty} P^n(j,j)$$

Example:

Random walks: $P = \begin{pmatrix} q & p & & & \cdots \\ q & 0 & p & & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ (X irreducible aperiodic (since state 0 is aperiodic))

$$\left.\begin{array}{lcl} \pi_0 & = & \pi_0 q + \pi_1 q \\ \pi_1 & = & \pi_0 p + \pi_2 q \\ \pi_2 & = & \pi_1 p + \pi_3 q \\ \vdots & & \vdots \end{array}\right\} \overset{\pi_0 = 1}{\Longrightarrow} \left.\begin{array}{lcl} \pi_1 & = & \frac{p}{q} \\ \pi_2 & = & \left(\frac{p}{q} - p\right)/q = \frac{p^2}{q^2} \\ \pi_3 & = & \left(\frac{p^2}{q^2} - \frac{p^2}{q}\right)/q = \frac{p^3}{q^3} \\ \vdots & & \vdots \end{array}\right\} \overset{\pi_0 = 1}{\Longrightarrow} \pi = \begin{pmatrix} 1 & \frac{p}{q} & \frac{p^2}{q^2} & \cdots \end{pmatrix}$$

♣ If $p \ge q$: no solution of $\pi = \pi \cdot P$, $\pi \cdot 1 = 1$

♣ If $p < q$: $\lim_{n\to\infty} P^n(i,j) = \left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^j$

Example: Remaining lifetime

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\left.\begin{array}{rcl} \pi_0 & = & \pi_0 p_1 + \pi_1 \\ \pi_1 & = & \pi_0 p_2 + \quad + \pi_2 \\ \pi_2 & = & \pi_0 p_3 + \quad + \pi_3 \\ \vdots & & \vdots \end{array}\right\} \overset{\pi_0 = 1}{\Longrightarrow} \begin{array}{rcl} \nu_0 & = & 1 \\ \nu_1 & = & 1 - p_1 \\ \nu_2 & = & 1 - p_1 - p_2 \\ \vdots & & \vdots \end{array}$$

Thus,

$$\sum_{j=0}^{\infty} \nu_j = (p_1 + p_2 + p_3 + \cdots) + (p_2 + p_3 + \cdots) + (p_3 + \cdots) + \cdots$$
$$= p_1 + 2p_2 + 3p_3 + \cdots = m$$

♣ $m = E[Z_n]$ is the expected lifetime.

♣ If $m = \infty$ then all states are recurrent null and $\lim_{n\to\infty} P^n(i,j) = 0$

Interpretation of Limiting Probabilities

Proposition: Let j be an aperiodic recurrent non-null state, and let $m(j)$ be the expected time between two returns to j. Then,

$$\pi(j) = \lim_{n\to\infty} P^n(j,j) = \frac{1}{m(j)}$$

The limiting probability $\pi(j)$ of being in state j is equal to the **rate** at which j is visited.

Proposition: Let j be an aperiodic recurrent non-null and let $\pi(j)$ defined as previously. Then, for almost all $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} 1_j(X_m(\omega)) = \pi(j)$$

♣. If f is a bounded function on E, then

$$\sum_{m=0}^{n} f(X_m) = \sum_{j\in E} f(j) \sum_{m=0}^{n} 1_j(X_m)$$

Corollary: X irreducible recurrent MC, with limiting probability π. Then, for any bounded function f on E:

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} f(X_m) = \pi \cdot f, \qquad \pi \cdot f = \sum_{j\in E} \pi(j) f(j)$$