Applications to Queueing Theory

Introduction to Stochastic Processes (Erhan Cinlar)
Ch. 6.5, 6.6

Applications to Queueing Theory: M/G/1 Queue

M/G/1:

Arrival Process: **M**emoryless (Poisson arrival or exponential (geometric) interarrivals

Service Process: ***G***enerally-distributed service times

Number of servers: **1**

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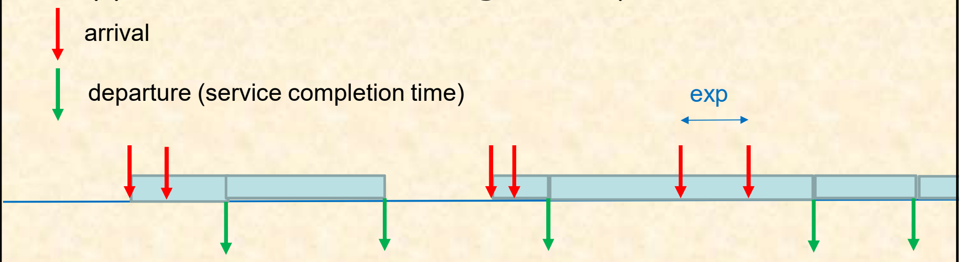

X(t): Number of customers in the system (queue and under service)

Consider a specific subset of times { t_e } only. That means that we embed X(t) on times { t_e } . We do not look at X(t) at times other than in { t_e }.

X(t_e) is the process X(t) embedded at times { t_e }.

X(t) is not a MC. Why?

If { t_e } = { times of customer departure }, then X(t_e) is a MC. Why?

Applications to Queueing Theory: M/G/1 Queue

$N_t(\omega)$: number of arrivals during the time interval $[0,t]$.

$Z_1(\omega), Z_2(\omega), \ldots$: service times of customers who depart first, second, ...

$Y_t(\omega)$: number of customers in the system (waiting or being served at time t)

Assumptions:

- ♣ $N = \{N_t; t \geq 0\} \sim P(a)$
- ♣ $Z_1, Z_2, \ldots$ i.i.d. $\sim \phi$

- Consider the future of Y from a time T of a departure onward.
- Define X_n as the number of customers in the system just after the instant of the n^{th} departure. (X_n is a SP embedded at departure times)

Theorem: X is a MC with the transition matrix

$$P = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 & \cdots \\ q_0 & q_1 & q_2 & q_3 & \cdots \\ & q_0 & q_1 & q_2 & \cdots \\ & & q_0 & q_1 & \cdots \\ & & & q_0 & \cdots \\ & & & & \ddots \end{pmatrix}, \quad q_k = \int_0^\infty \frac{e^{-at}(at)^k}{k!} d\phi(t), \quad k = 0,1,\ldots$$

Distribution of arrivals over a service time

Proof: We need to show $P\{X_{n+1}=j \mid X_0,\ldots,X_n\} = P\{X_{n+1}=j \mid X_n\}$

$$P\{X_{n+1}=j \mid X_n=i\} = \begin{cases} q_j & i=0, j\geq 0 \\ q_{j+1-i} & i>0, j\geq i-1 \\ 0 & \text{otherwise} \end{cases}$$

- Let T the time of the n^{th} departure.
- Let $Z = Z_{n+1}$ the service time of the $n+1$ customer.

Then, $X_{n+1} = \begin{cases} X_n + (N_{T+Z} - N_T) - 1, & X_n > 0 \\ N_{S+Z} - N_S, & X_n = 0 \end{cases}$ (S:arrival time of the n+1 customer)

Using Poisson properties: $P\{N_{T+Z} - N_T = k \mid X_0,\ldots X_n; T\} = P\{N_Z = k\}$

Distribution of arrivals over a service time

$$q_k = P\{N_Z = k\} = E\left[P\{N_Z = k \mid Z\}\right] = E\left[\frac{e^{-aZ}(aZ)^k}{k!}\right] = \int_0^\infty \frac{e^{-at}(at)^k}{k!}\, d\phi(t)$$

- $i=0$ $\quad P\{X_{n+1}=j \mid X_n=0\} = P\{N_{S+Z} - N_S = j\} = P\{N_Z = j\} = q_j$
- $i>0$ $\quad P\{X_{n+1}=j \mid X_n=i\} = P\{N_{T+Z} - N_T = j+1-i\}$

$$= P\{N_Z = j+1-i\} = \begin{cases} q_{j+1-i}, & j \geq i-1 \\ 0, & j < i-1 \end{cases}$$

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The MC X is irreducible and aperiodic. If

$$r = E[N_Z] = aE[Z] = ab$$

Mean number of arrivals over a mean service time

Then (intuitively based on queue evolution/growth, also rigorously proven)

- If $r > 1$ all states are transient
- If $r < 1$ all states are recurrent non-null.
- If $r = 1$ all states are recurrent null

Notation:

$r_k = 1 - q_0 - \cdots - q_k$ (prob arrivals over a service time exceed k; summing them we get r, next)

$r = r_0 + r_1 + \cdots = (q_1 + q_2 + q_3 + \cdots) + (q_2 + q_3 + \cdots) + (q_3 + \cdots) + \cdots$

$= q_1 + 2q_2 + 3q_3 + \cdots$ (this is the definition of the mean r of the distr of arrivals over a service time)

Proposition: The chain X is recurrent non-null aperiodic if and only if $r < 1$.

Proof: We need to show that

$$\pi = \pi \cdot P, \qquad \pi \cdot 1 = 1$$

$$\left.\begin{aligned} \pi_0 &= \pi_0 q_0 + \pi_1 q_0 \\ \pi_1 &= \pi_0 q_1 + \pi_1 q_1 + \pi_2 q_0 \\ \pi_2 &= \pi_0 q_2 + \pi_1 q_2 + \pi_2 q_1 + \pi_3 q_0 \\ &\vdots \end{aligned}\right\} \Rightarrow \begin{aligned} \pi_1 q_0 &= \pi_0 r_0 \\ \pi_2 q_0 &= \pi_0 r_1 + \pi_1 r_1 \\ \pi_3 q_0 &= \pi_0 r_2 \pi_1 r_2 + \pi_2 r_1 \\ &\vdots \end{aligned}$$

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Summing all equations ($q_0 = 1 - r_0$, $r = r_0 + r_1 + r_2 + \cdots$)

$$(1-r_0)\cdot\sum_{j=1}^{\infty}\pi_j = \pi_0 r + (r - r_0)\sum_{j=1}^{\infty}\pi_j$$

If $r < 1$, then we obtain $\sum_{j=1}^{\infty}\pi_j = \frac{r}{1-r}\pi_0 \quad\Rightarrow\quad \sum_{j=0}^{\infty}\pi_j = \frac{1}{1-r}\pi_0$

The condition $\pi\cdot 1 = 1$ is satisfied with $\pi_0 = 1 - r$

Theorem: The limits $\pi(j) = \lim_{n\to\infty} P^n(i,j)$ exist $\forall j \in E$ and are independent of the initial state i.

- If $r \geq 1$, then $\pi(j) = 0, \quad \forall j$.
- If $r < 1$, then

$$\begin{aligned}
\pi(0) &= 1-r \\
\pi(1) &= (1-r)\frac{r_0}{q_0} \\
&\vdots \\
\pi(j+1) &= (1-r)\sum_{k=1}^{j}\left(\frac{1}{q_0}\right)^{k+1}\sum_{\mathbf{a}\in S_{jk}} r_{a_1} r_{a_2}\cdots r_{a_k}
\end{aligned}$$

where S_{jk} is the set of all k-tuples $\mathbf{a} = (a_1,\ldots,a_k)$ of integers $a_i \geq 1$ with $a_1 + \cdots a_k = j$

Applications to Queueing Theory: G/M/1 Queue

G/M/1:

<u>Arrival Process</u>: **G**enerally-distributed arrival times

<u>Service Process:</u> **M**emoryless (exponential (geometric) service times)

<u>Number of servers</u>: **1**

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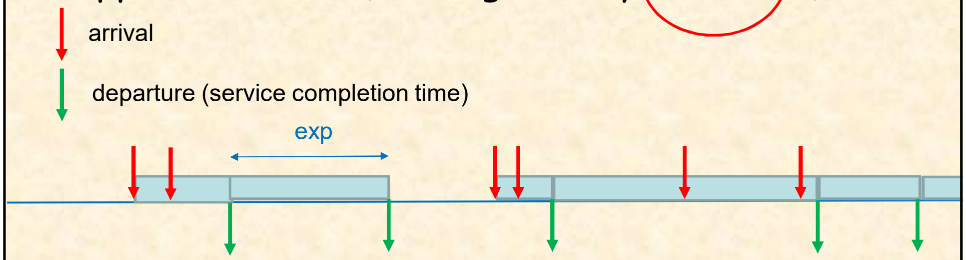

X(t): Number of customers in the system (queue and under service)

Consider a specific subset of times { t_e } only. That means that we embed X(t) on times { t_e } . We do not look at X(t) an times other than in { t_e }.

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If { t_e } = { times of customer **arrival** }, then X(t_e) is a MC.

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Exponentially distributed service times ∼ exp(a)
i.i.d. interarrival times ∼ ϕ .

In this case $q_n = \int_0^\infty \frac{e^{-at}(at)^n}{n!} d\phi(t)$ ← Distribution of services over an interarrival time

is the probability that the server completes exactly n services during an interarrival time (provided that there are that many customers).

Define: $r_n = q_{n+1} + q_{n+2} + \cdots$

$$r = \sum_{n=1}^{\infty} n q_n = r_0 + r_1 + r_2 + \cdots$$

r is the expected number of services which the server is capable of completing during an iterarrival time. It can be proved that

- $r \geq 1$ Server can keep up with arrivals (recurrent)
- $r < 1$ Queue size increases to infinity (transient)

If X_n^ε is the number of customers present in the system just before the time T_n of the n^{th} arrival, then

Theorem: $X^\varepsilon = \{X_n^\varepsilon; n \in N\}$ is a MC with $E = \{0,1,2,...\}$, $P^\varepsilon = \begin{pmatrix} r_0 & q_0 & & & \\ r_1 & q_1 & q_0 & & \\ r_2 & q_2 & q_1 & q_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

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Proof: Let M_{n+1} be the number of services completed during the $n+1^{\text{th}}$ interarrival time $[T_n, T_{n+1})$. Then,

$$X^{\varepsilon}_{n+1} = X^{\varepsilon}_n + 1 - M_{n+1}$$

But M_{n+1} is conditionally independent of the past history before T_n given the present number X^{ε}_n. If $Z = T_{n+1} - T_n$

$$P\left\{M_{n+1} = k \mid X^{\varepsilon}_n, Z\right\} = \begin{cases} \dfrac{e^{-aZ}(aZ)^k}{k!} & X^{\varepsilon}_n + 1 > k \\ \displaystyle\sum_{m=k}^{\infty} \dfrac{e^{-aZ}(aZ)^m}{m!} & X^{\varepsilon}_n + 1 = k \quad (*) \\ 0 & \text{otherwise} \end{cases}$$

Taking expectations with respect to Z, which is independent of X^{ε}_n, we obtain

$$P\left\{M_{n+1} = k \mid X^{\varepsilon}_n = i\right\} = \begin{cases} q_k & k \le i \\ r_{k-1} & k = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Equation $X^{\varepsilon}_{n+1} = X^{\varepsilon}_n + 1 - M_{n+1}$ and the previous one provide matrix P^{ε}

Applications to Queuing Theory: G/M/1 Queue

Theorem: X^{ε} is recurrent non-null if and only if $r > 1$. If $r > 1$,

$$\pi^{\varepsilon}(j) = \lim_{n\to\infty} P^{\varepsilon\, n}(i,j) = \lim_{n\to\infty} P^{\varepsilon}\{X^{\varepsilon}_n = j \mid X^{\varepsilon}_0 = i\}$$

and

$$\pi^{\varepsilon}(j) = (1-\beta)\beta^j, \qquad j = 0,1,2,...$$

where β is the unique number satisfying

$$\beta = q_0 + q_1\beta + q_2\beta^2 + \cdots$$

If $r \le 1$ then $\pi^{\varepsilon}(j) = 0$ for all j.

Proof: X^{ε} is recurrent non-null if and only if

$$\nu = \nu \cdot P^{\varepsilon}, \qquad \nu \cdot 1 = 1$$

has a solution.

$$\begin{array}{rclcccccccc} \nu_0 & = & & & q_1\nu_0 & + & q_2\nu_0 & + & q_3\nu_0 & + & \cdots \\ & & & & & + & q_2\nu_1 & + & q_3\nu_1 & + & \cdots \\ & & & & & & & + & q_3\nu_2 & + & \cdots \\ \nu_1 & = & q_0\nu_0 & + & q_1\nu_1 & + & q_2\nu_2 & + & q_3\nu_3 & + & \cdots \\ \nu_2 & = & q_0\nu_1 & + & q_1\nu_2 & + & q_2\nu_3 & + & q_3\nu_4 & + & \cdots \end{array}$$

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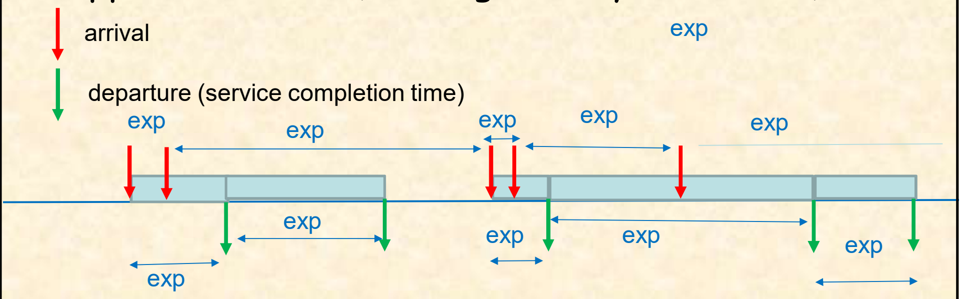

X(t): Number of customers in the system (queue and under service)

X(t) is a MC. Why?

Special case M/M/1

We can consider this queue as a special case of M/G/1 or G/M/1. In the sequel we use G/M/1. Now the interarrival distribution is given by:

$$\phi(t) = 1 - e^{-\lambda t}, \qquad t \geq 0$$

To compute the limiting distribution of X^{ε} (queue size just before the n^{th} arrival, we find first β, where

$$\beta = \sum_{k=0}^{\infty} q_k \beta^k = \sum_{k=0}^{\infty} \beta^k \int_0^{\infty} \frac{e^{-at}(at)^k}{k!} \lambda e^{-\lambda t} dt = \int_0^{\infty} e^{-at(1-\beta)} \lambda e^{-\lambda t} dt = \frac{\lambda}{\lambda + a - a\beta}$$

The previous equation becomes $\beta = \dfrac{\lambda}{\lambda + a - a\beta}$ or $(1-\beta)(\lambda - ab) = 0$

with solutions $\beta = 1$ and $\boxed{\beta = \frac{\lambda}{a}}$. When $r = \frac{a}{\lambda} > 1$, the smallest solution is $\beta = \dfrac{\lambda}{a}$

So we have $$\lim_{n\to\infty} P\{X_n^{\varepsilon} = j\} = \left(1 - \frac{\lambda}{a}\right)\left(\frac{\lambda}{a}\right)^j, \qquad j = 0,1,\ldots$$

It turns out that $\lim_{t\to\infty} P\{Y_t = j\} = \left(1 - \frac{\lambda}{a}\right)\left(\frac{\lambda}{a}\right)^j$, $j = 0,1,\ldots$ for the queue size Y_t at time t.

and $\lim_{n\to\infty} P\{X_n = j\} = \left(1 - \frac{\lambda}{a}\right)\left(\frac{\lambda}{a}\right)^j$, $j = 0,1,\ldots$ for the queue size X_n just after the n^{th} departure.