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TEACHING AND LEARNING GEOMETRY WITH TECHNOLOGY

INTRODUCTION

In the past three decades, technological environments have been created that offer novel ways of carrying out geometrical activities in mathematics education. This chapter reviews research investigating the impact of these various technologies on the learning and teaching of geometry. The chapter begins by providing a general overview of the various theoretical approaches, on which such research is based. The second section focuses on specific technologies. The third section synthesizes results from research and organizes them within four categories: The nature of geometry mediated by technology, technology and the learning of geometry, the design of tasks, the use of geometry technology by teachers. The final section concludes the chapter by linking the results of past research to perspectives for the future. This review of research is based on research that developed in the International Group for the Psychology of Mathematics Education (PME) and/or was published in international journals on mathematics education or on computers in mathematics education.

GENERAL THEORETICAL ISSUES

This section addresses the theoretical approaches underlying various research studies on technology for the teaching and learning of geometry. It attempts to explain the development of technology by analyzing the background of this development: The epistemological nature of geometry, the problems with which the teaching of geometry was faced in the past decades, and the cognitive processes involved in geometry problem solving. We present various theoretical approaches that have been used in research for analyzing the impact of technology on learning and the interactions between students and technology. More recently, researchers have paid greater attention to the integration of technology into teaching and consequently different theoretical approaches have been employed.

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Epistemological perspective

The teaching of geometry can be understood only if geometry is considered as an activity with at least two aspects: On the one hand, it is the study of concepts and logical relations, which historically came from an extensive analysis of space, but later became a field of investigation and discussion of axiomatic foundations detached from any spatial experience. On the other hand, geometry refers to spatial concepts, procedures and relations used within society for various purposes, such as architecture, building, structuring settlements, villages, cities, designing packages of goods for storage and other purposes and activities (Strässer, 1996). Since ancient Greece, the dual nature of geometry has often been claimed and discussed: Was and is geometry dealing with what our senses perceive or with intellectual ideas? It was also stressed even by one of the founders of an axiomatic approach of geometry, namely Hilbert who claimed the coexistence of two tendencies, one toward abstraction and another toward intuitive understanding (Hilbert & Cohn Vossen, 1952, in the Preface).

Teaching and learning geometry

The teaching of geometry has always been an object of discussion in the past forty years and susceptible to dramatic changes with respect to the place given to the duality of empirical/theoretical. Particularly in several countries, it was deeply affected by the so-called reform of Modern Mathematics that mainly emphasized the formal part of geometry while avoiding recourse to diagrams. The argument was that geometry was especially difficult for students because of the use of diagrams; the combination of empirical evidence provided by diagrams and the teachers' demand to resort only to deductive thinking confused students.

The empirical/theoretical duality led to a problematic role of geometry in compulsory education curricula. Some educational researchers nevertheless viewed the origin of the problem in the absence of exploiting graphical representational registers associated with Geometry as part of the repertoire for expressing mathematical meanings, a shortcoming especially significant in geometry where they could be uniquely relevant. Freudenthal (1973) was one of the first researchers to raise the problem. He was followed by a growing number of voices, which pled for reintroducing diagrams in geometry teaching that eventually took place in the end of the seventies and beginning of the eighties. Even with this important role of graphical representations in the teaching of geometry, a conceptual analysis of this role was not yet carried out at that time. It is only at the end of the eighties and beginning of the nineties, that several theoretical approaches merged.

Duval (1988, 1998, 2000) distinguished three kinds of cognitive processes involved in a geometric activity: Visualization processes, construction processes by tools, and reasoning. Each of these processes fulfils a specific epistemological function but they are closely connected and "their synergy is cognitively necessary for proficiency in geometry" (Duval, 1998, p. 38). Duval also analyzed the role of visualization in the solution processes of a geometry problem and distinguished several approaches to a diagram in geometry: An immediate perceptual approach that may be an obstacle for the geometric interpretation of the diagram, an operative approach that is used for identifying sub-configurations useful for solving the problem and a discursive approach that is related to the statement describing the givens of the problem.

Another psychological approach that was almost simultaneously developed by Fischbein (1993; see also Mariotti, 1995), considers geometrical concepts as made of two components that cannot be dissociated: The figural one and the conceptual one, similar to two sides of a coin. This intrinsic link between figural and conceptual is not at all spontaneous and must be grounded in a long construction process by students.

The teaching of geometry is based on the use of two registers, the register of diagrams and the register of language. Language is a means of describing geometrical objects and relations using specific terminology while diagrams in 2D geometry play an ambiguous role. On the one hand they refer to theoretical objects whereas on the other hand they offer graphical –spatial properties, which can give rise to a perceptual activity from the individual (Parzysz, 1988; Strässer, 1991; Laborde, 1998). This ambiguous role of diagrams is completely implicit in the traditional teaching of geometry in which theoretical properties are assimilated into graphical ones (Berthelot & Salin, 1998). It is as if it were possible to read the properties of the theoretical object, which is represented by the diagram, by only looking at the diagram. One of the consequences is that students often assume that it is possible to construct a geometrical diagram using only visual cues, or to deduce a property empirically by checking on the diagram, as shown by several researchers (Chazan, 1993). When students are asked by the teacher to construct a diagram, the teacher expects them to use theoretical knowledge whereas students very often stay at the graphical level and try to satisfy only visual constraints.

The construction of the dual nature of geometrical concepts may be ignored by the teaching of geometry. This type of teaching obscures the distinction between the spatial and theoretical. Contrasting with this teaching practice, on the basis of their investigations, researchers and educators stressed the importance of the role of visualization in a geometry activity: Solving a geometry problem goes beyond the visual recognition of spatial relations. It is commonly assumed that the teaching of geometry should contribute to the learning of: (1) The distinction between spatial graphical relations and theoretical geometrical relations, (2) The movement between theoretical objects and their spatial representation, (3) The recognition of geometrical relations in a diagram, (4) The ability to imagine all possible diagrams attached to a geometrical object. The second kind of ability is particularly critical in the solving processes of students faced with geometry problems requiring exploration in which a cycle of interpreting, conjecturing, and proving may take place because of this flexibility between spatial representations and theoretical knowledge. Such assumptions about the teaching and learning of geometry have led some researchers to focus on the role of graphical representations provided by computer environments.

Use and role of technology for learning geometry

The contribution of technology in the teaching and learning of geometry is now mainly perceived as strongly linked with dynamically manipulable interactive graphical representations. However, the first appearance of digital media for the learning of Geometry, the Logo-based Turtle Geometry (Abelson & diSessa, 1981) came at a time when this functionality was not yet available. In the absence of dynamic manipulation, the graphical representational register was not given central attention. The main priority was the newly realized potential affordances of dynamic text editing, programming, and constructionism for mathematical meaning making (Papert, 1980; Noss & Hoyles, 1996). Graphical representations were part of the picture but appeared to be third in line of importance, after symbolic and turtle–associated, body-syntonic representations (Papert, 1980). In the mid-nineties, Papert and his team put forward the notion of constructionism in mathematical learning to signify that special aspect of constructivist learning which involved the activity of dynamic construction (Kafai & Resnick, 1996).

In the last thirty years, research has been mainly devoted to two kinds of technology providing graphical representations:

- Logo driven Turtle Geometry (T.G.) and its intrinsically linked philosophy of micro-world,
- Dynamic geometry environments ("DGE" –with varying and generally growing degrees of interactivity and direct manipulation).

Both kinds of technology are attached to a theoretical perspective on learning. Logo T.G. and its decedents were clearly answering to a precise view of the nature of learning attached to the idea of a micro-world. The main underlying principle was to provide programmability as a means for expressing and exploring mathematical ideas and the joint use of three representational registers: Symbolic programming, graphics, and a notional connection with body movement. Taking advantage of the huge progress of the graphic interface of computers, dynamic geometry environments arrived later on the scene; the first DGEs appeared in the eighties. The main underlying learning principle was to provide a family of diagrams as representing a set of geometrical objects and relations instead of a single static diagram. One of the motivations was to help students see the general aspects of a static diagram.

A constructivist perspective in a broad sense is generally adopted in research on the role and use of technology in the teaching of geometry: Learning is not taken as a simple process of the incorporation of prescribed and given knowledge, but rather as the individual's (re)construction of geometry. The interactions taking place between the learner and the machine are viewed as impacting this reconstruction. However, it is important to mention that additional theoretical perspectives have been used and/or developed taking into account: The structure of knowledge to be (re)constructed, and the environment of the learner –in particular, the social interactions in which learning takes place as well as institutional constraints coming from the institution "responsible" for learning (e.g. the school embedded within the larger school system).

Growing attention to the epistemology of the mathematical content whose learning is at stake has also developed over time. In the first years of technology in school learning was mainly considered as emerging only from the interactions between the student and the machine, rather than between the students and appropriate tasks to be done with the machine. The focus moved onto the teaching environment, and in particular to the role of the teacher and the social interactions he/she could organize in the classroom as well as the social norms developing in the context of the classroom. The following paragraphs present the general theories underpinning this move.

According to an important hypothesis generally shared among researchers in particular in PME, when interacting with technology for solving mathematical tasks, students' actions and strategies are shaped by technology. Noss and Hoyles proposed the Using, Discriminating, Generalizing and Synthesizing (U.D.G.S.) model to describe the conceptualization process of students interacting with technology (Hoyles & Noss, 1987). Students start by using the technology and progressively discriminate the mathematical relations and concepts underpinning the behaviour of the tools. This is followed by generalizations, which are local to the situations from which they emerged. Finally they move on to synthesizing their generalizations with different contexts and representational registers outside the specific technology used. Noss and Hoyles introduced the concept of "situated abstraction" to account for the constructions of the learner. Situated abstractions are invariants that are shaped by the specific situation in which the learner forges them. Although these invariants are situated, they simultaneously contain the seed of the general that could be valid in other contexts.

The instrumental perspective, developed independently by psychologists in the mid-nineties (Vérillon & Rabardel, 1995) shares the same idea of the role of the tool on the constructs of the user. It was also recently adopted by researchers in mathematics education to understand the strategies used by students when beginning to use software programs for solving mathematical tasks. A tool is not transparent. It affects the way the user solves the tasks and thinks. The instrument, according to the terms of Vérillon and Rabardel, denotes this psychological construct of the user:

> The instrument does not exist in itself, it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it with his activity.

The subject develops procedures and rules of actions when using the artefact and so constructs instrumentation schemes and simultaneously a representation of the properties of the tool (according to what Vérillon and Rabardel call instrumentalisation schemes). The first studies about instrumentation processes addressed the use of the CAS by students (Guin & Trouche, 1999). They were mainly focusing on the difficulties of students using technology and on the detours they sometimes follow in order to perform an activity with technology (Artigue, 2002). Today, research pays more attention to the mathematical knowledge involved in instrumental knowledge. Because technology used in mathematics embarks mathematics, mathematical knowledge is intrinsically linked to the knowledge about how to use the tool. Developing instrumental knowledge may also involve developing mathematical knowledge (Artigue, 2002; Lagrange, 1999; Laborde, 2003).

Two theoretical approaches were developed according to which learning with the use of technology develops. In the first one, tools, and in particular technologies, offer opportunities for learning. The subject is faced with constraints imposed by the artefact and new possibilities of actions, to identify, to understand and with which to cope. In terms of the theory of didactic situations (Brousseau, 1997), the tool is part of the "milieu". In the second approach, following a Vygotskian perspective, operations carried out with technology may be subject to an internalization process with the guidance of the teacher and interpersonal exchanges within the class in the form of collective discussions (Bartolini Bussi, 1998; Mariotti & Bartolini Bussi, 1998). The interventions of the teacher are essential for making possible the construction of a correspondence between mathematical knowledge and knowledge constructed from the interactions with the computer environment. According to the instrumentation theory, the meaning constructed by the student when using the artifact may differ from what is intended by the teacher. Consequently, the interventions of the teacher are critical to let the meanings evolve towards culturally shared meanings of mathematical knowledge.

The page limitation of a chapter does not offer the opportunity of doing justice to all theoretical approaches underpinning research studies. In particular, the Van Hiele theory dedicated to geometry learning was not mentioned although it has been used in a small number of studies, essentially as a tool for assessing the impact of technology on the possible progress in the hierarchy of levels according to which students conceptualize geometrical figures.

RESEARCH ON SPECIFIC TECHNOLOGIES

This section emphasizes aspects of research related to specific features of various technologies. Two main kinds of technology are distinguished: The Logo-driven Turtle Geometry technology and related microworlds and the Dynamic Geometry environments. Students' learning of particular geometry topics, use of diagrams in problem solving, proving, and justifying were investigated by taking into account the role of technology in those processes.

Research on logo-driven turtle geometry

Logo is a programming language intended to bring to the field of mathematics education the philosophy of programming to express meaning and to tinker with difficult problems which characterized its older sister, the LISP AI language (Sinclair & Moon, 1991). This intention was part of a broader, powerful and forerunning idea at the time, to make this kind of activity with technology accessible to young children. It originated from Papert in the late 1960s and became well known in the mathematics education community as a result of his

'Mindstorms' book (Papert, 1980). From a mathematical learning theory point of view, Papert's main intention was to move beyond Piaget's paradigm, which focused on the shortcomings of children's thinking in relation to adult formal thinking, by asking the question of what mathematical thinking children can do in situations where they can explore with mathematically rich computational tools such as Logo.

The connection between Logo and Geometry is through a specific subset of the former, which was termed "Turtle Geometry" (T.G.). T.G. is a computational environment where the user gives commands to a computational entity called 'the turtle' which has a position and a heading. Position changes create a linear graphical output on the screen and the turtle icon changes its position and heading as an immediate result of such commands. Turtle commands are Logo primitives and consequently, Logo programs can be written which drive the turtle to construct geometrical figures. Abelson and diSessa (1981) gave the mathematical foundations of Turtle Geometry and agreed with Papert that its geometrical nature was based on a different geometrical system to those usually associated with the learning of geometry in school curricula, i.e. the Euclidean or analytic (e.g. Cartesian): T.G. is based on differential (intrinsic) Geometry. Most T.G. related research, however, is based on the figural products created by the turtle, on the connections students make between the formal programming/mathematical code and the graphical output and on how children link experiences of their body movements to the behaviour of the turtle, termed 'body-syntonicity' by Papert in 1980.

Various Logo T.G. have been developed across the world and one can say that there are presently more than 100 such digital environments. The ones which gave rise to investigations in mathematics education including those in the PME group, listed alphabetically, include: Boxer (a programmable computational medium with the Logo-like language 'Boxer' including T.G., diSessa & Lay, 1986), Elica Logo (including 3d T.G., Boychev, 1999), Imagine Logo (Kalas, 2001), Microworlds Pro (Silverman, 1999), NetLogo (a parallel Logo programming language with a very large number of turtles, Wilensky, 1999), E-slate-based Turtleworlds (based on USB Logo but including dynamic variation tools offering dynamic manipulation of procedure variable values, Kynigos, 2001) and USB Berkeley Logo (Harvey, 2005).

Logo based micro-worlds. A central aspect of learning with Logo is the activity of mathematical exploration with micro-worlds. The term was originally borrowed from artificial intelligence (A.I.). Its meaning evolved within the mathematics education community and was shaped by Papert. He described it as a self-contained world where students can "learn to transfer habits of exploration from their personal lives to the formal domain of scientific construction" (Papert, 1980, p. 177).

Micro-worlds are computational environments embedding a coherent set of mathematical concepts and relations designed so that with an appropriate set of tasks and pedagogy, students can engage in exploration and construction activity rich in the generation of mathematical meaning. T.G. itself has been considered as a micro-world within Logo but later research involved learning with the use of micro-worlds embedding a much narrower set of mathematical concepts, escorted by more focused theories on learning and pedagogy (Noss & Hoyles, 1996; Edwards, 1988; Clements & Sarama, 1997; Sarama & Clements, 2002).

In PME research, a number of studies involved the design and use of geometrical micro-worlds. Hoyles and Noss (1987) used a parallelogram micro-world in developing their U.D.G.S. theory. This consisted of permutations of specially designed Logo procedures with independent variable values for turns and position changes, so that students would investigate relations between angular and linear features so that the procedure would construct a parallelogram. Their focus was on the process of students' formalization of intuitive descriptions (Hoyles & Noss, 1988). Edwards built a transformation geometry micro-world addressing the issue of the representational aspect of micro-worlds (Edwards, 1988, 1990) and then later used a micro-world generating star figures to study the process by which the students discriminated the underlying mathematical properties (Edwards, 1994). Kynigos took a more transversal approach by building a series of micro-worlds to study how students make links between Differential (intrinsic), Euclidean and Cartesian systems (Kynigos, 1988, 1989, 1991). Hoyles, Noss and Sutherland (1989) studied how students come up against pre-conceptions of additive or doubling strategies when working with a ration and proportion micro-world and were then joined by Sutherland in evaluating what students gained with respect to the strategies they applied in their investigations (Hoyles, Noss & Sutherland, 1991).

Learning processes at the core of research. T.G. appeared at a time when the 'problem solving' movement was thriving, at least in the U.S., and its exploratory nature along with the somewhat obscure connection with school curricula facilitated the views that T.G. was a tool for building learning strategies. In particular, learning how to learn. The geometrical concepts were there, but in the background of many researchers' attention.

In PME research, there was thus an emphasis on generative, focused theory-building methodologies. Many researchers perceived learning with Logo as a new kind of learning process, with the consequence that the paradigm of qualitative, illuminative research methods (i.e. adopting the role of 'naïve observer' for the researcher) seemed appropriate (Hoyles & Noss, 1987). The main research emphasis seemed to be 'what kind of learning goes on' rather than 'what kind of geometrical learning'. The focus was on student learning processes and some aspects of this process emerging from Logo geometry environments were recorded. Such aspects included:

– 12 year old students' use of a drawing cognitive scheme and their resistance to dissociate from the procedural visible aspects of their Logo work (Hillel, Kieran & Gurtner, 1989);

– 12 year old students' initial use of an intrinsic schema and their progressive dissociation to a composite schema including the use of absolute positioning on the plane (Kynigos, 1989);
– Students' process of using geometrical ideas, then discriminating the ones that matter for the task at hand, followed by generalization and then synthesis with other contexts (U.D.G.S., Hoyles & Noss, 1987);
– Students' situated abstractions, i.e. abstractions derived directly from the specific context of defining and changing geometrical figures and objects (Hoyles & Noss, 1988 –the construct was rigorously elaborated later in Noss & Hoyles, 1996).

In other studies, the focus was on the nature of interactions between students and the computer that included the use of: Multiple representations, feedback, and editing and constructing (Hoyles et al., 1991; Edwards, 1990; Kynigos & Psycharis, 2003). Also, in a number of studies the focus was on the issues related to the design of mathematically focused computational environments (micro-worlds) and discussion on their affordances (Hoyles et al., 1989; Edwards, 1988; Kynigos & Psycharis, 2003; Sacristán, 2001). Finally, there were some studies adopting a different strand which at the time was considered as a more mainstream approach to research, using standardized tests or experimental methods. In two cases for example, the focus was on using the Van Hiele levels to test students' abilities to identify geometrical figures (Olive & Lankenau, 1987; Scally, 1987). For a much larger study in this framework, see also Clements & Battista (1992).

Research on geometrical thinking with T.G. was highly influenced by two phenomena which that were related to the time in which the technology appeared. One was the power of the newly-born idea of deep structural access of ordinary people (even children) to a technology which was up to that point only used by computer scientists (see diSessa, 2000; Eisenberg, 1995 for an elaboration of the idea in later times, and Kynigos, 2004; diSessa, 1997 for this idea applied to teachers). This led to an emphasis on the learning process which was perceived as new in nature and was enhanced by the problem solving movement which focused on learning strategies rather than mathematical content at hand. It also led to a connection between research on geometrical learning with technology and the itinerary of Logo related learning research in general which in itself has been subject to big changes in different parts of the world (Kynigos, 2002; Papert, 2002). The second influence came from the constructivist learning movement, which in the mathematics education community appeared in the early eighties and initially adopted a rather individualist perspective on learning. Research on geometrical learning with T.G. did address geometrical concepts, communication between students, the design of tasks and the influence of the teacher. However, although these aspects were part of several research studies including PME research, they were not in focus, nor were they part of a more general research attention to these issues which came later along with the advent of DGE. technology.

Moreover, T.G. research focused on the idea of meaningful formalism (diSessa, 2000; Kynigos & Psycharis, 2003; Hoyles & Noss, 1988). With respect to mathematical formalism there is a strong view that, although it may be a powerful representational register for mathematicians, it can be rather meaningless for students (Dubinsky, 2000), i.e. an imposed code to tackle meaningless routines. Furthermore, the advent of DGE has provided access to mathematical ideas by allowing the bypassing of formal representation and access to dynamic graphing which is particularly important for the learning of geometry. This does not however necessarily mean that formalism can only be useful to established mathematicians who can convey abstract mathematical meaning through its use. Just as digital technology provides means to by-pass formalism, it may also provide the means to transform the way formalism is put to use by students. Technologies affording programmability and symbolic expression in conjunction with the representational repertoires of DGE could thus be considered in geometrical learning research set within developing theoretical frameworks (Clements & Sarama, 1995; Clements & Battista 1994; Sherin, 2002; Kynigos & Psycharis, 2003; Kynigos & Argyris, 2004).

Research on Dynamic Geometry Environments (DGE)

In DGE, diagrams result from sequences of primitives expressed in geometrical terms chosen by the user. When an element of such a diagram is dragged with the mouse, the diagram is modified while all the geometric relations used in its construction are preserved. These artificial realities can be compared to entities of the real world. It is as if diagrams react to the manipulations of the user by following the laws of geometry, just like material objects react by following the laws of physics. A crucial feature of these realities is their quasi-independence from the user once they have been created. When the user drags one element of the diagram, it is modified according to the geometry of its constructions rather than according to the wishes of the user. This is not the case in paper-and-pencil diagrams that can be slightly distorted by students in order to meet their expectations. In addition to the drag mode, dynamic geometry environments offer specific features such as macro-constructions, trace, and locus, which differ from paper and pencil tools (Strässer, 2002).

Computer diagrams are also external objects whose behaviour and feedback requires interpretation by the students. Geometry is one means, among others, of interpreting this behaviour. In the design of DGE, spatial invariants in the moving diagrams represent geometrical invariants and these geometry micro-worlds may offer a strong link between spatial graphical and geometrical aspects. Inspired by the theory of variation in the tradition of phenomenographic research approach, Leung (2003) suggests the idea of simultaneity is a promising agent to help bridge the gap between experimental and theoretical Mathematics, or the transition between the processes of conjecturing and formalizing. Simultaneity is intrinsically related to discernment in the theory of variation:

> In DGE, it is possible to define a way of seeing (discernment) in terms of actually seeing invariant critical features (a visual demarcation or focusing) under a continuous variation of certain components of a configuration.

Various DGEs have been developed across the world and one can say that there are presently around 70 such environments. However most of them are clones of original DGEs, which are no more than ten. The DGEs which gave rise to investigations in mathematics education and especially in the PME group, listed alphabetically, include: Cabri-géomètre (Laborde, Baulac & Bellemain, 1988; Laborde & Bellemain, 1995; Laborde, 1999), GEOLOG (Holland, 2002), Geometer's Sketchpad (Jackiw, 1991), Geometry Inventor (Brock, Cappo, Carmon, Erdös, Kamay, Kaplan & Rosi, 2003), Geometric Supposer (Schwartz, Yerushalmy & Shternberg, 1985, 2000), and Thales (Kadunz & Kautschitsch, 1993).

Dynamic Geometry Environments and the move from the spatial to the theoretical construction tasks

Several researchers investigated how students solve construction tasks in DGEs. Since the construction must be preserved by the drag mode, a construction by eye or by manual adjustment fails. Students must do the construction by using geometrical objects and relations offered by the environment (perpendicular, parallel, circle, ...). It has often been reported that beginners have difficulty in constructing diagrams in a DGE that is resistant to the drag mode (i.e. preserves relationships upon dragging) and resort to construction strategies by eye (e.g., Noss, Hoyles, Healy & Hoelzl, 1994). An indicator of the difficulty students experience in relating the spatial to the theoretical is also given by their difficulties in interpreting the behaviours of a diagram or of elements of a diagram under the drag mode. Soury-Lavergne (1998) shows how the immobility of a point in Cabri-geometry was not related to its geometrical independence from the dragged points. For the student there were two separate worlds, the mechanical world of the computer diagram (in our terms, the spatial) and the theoretical (or geometrical).

The distinction that we made between spatial-graphical and geometrical is expressed in various terms in several papers: For example, in Noss et al. (1994) this distinction is called empirical/theoretical.

The notion of dependency and functional relationship

Geometrical objects, which are linked by geometrical relationships, can be viewed as dependent. The drag mode can be used to externalize this theoretical dependency in the diagram. If students have not constructed a relation between the spatial-graphical level and the theoretical level, they may not recognize or understand the dependency relationship in a DGE. As a consequence, it is not surprising that several papers (mainly from the United Kingdom) focus on the construction of this notion of dependency by students interacting with a DGE (Hoyles, 1998; Jones, 1996; Pratt & Ainley, 1996). Several functionalities in a DGE (Jones, 1996) can serve as tools for externalizing the notion of dependency. Among these functionalities are the following: The drag mode, in which a dependent point cannot be dragged directly; the delete function (when an object is deleted all dependent objects are also deleted and a warning message is displayed), and the redefinition of an object (redefining an object in Cabri is changing its dependency relations with other objects).

All papers mention that the notion of dependency is difficult for students and not understood initially. By interacting with a DGE and a teacher students may construct this notion of dependency, as expressed by two pairs of pupils in Pratt and Ainley (1996): "They are all real shapes because you can move them without deforming the shape", and "The initial objects are the ones on which everything else depends". It is also clear from these papers that such a constructed notion of dependency is situated in the context of the DGE. It contains some generality but in terms of the context or of the tool (an example of what Hoyles and Noss call "situated abstraction"). "So because it depends on it, it moves" (a student in Jones, 1996). However simply considering that two objects are dependent because one moves when the other one moves does not mean that students are able to analyze the dependency existing among objects.

The use of drag mode. Certainly, the drag mode is a key element of DGEs. The mathematical counterpart of the drag mode is variation. Experts can immediately recognize variation in the dragging of elements of a diagram. But the learners being more at a spatial graphical level may just view the drag mode as a mode moving and changing the shape of a diagram, or as a mechanical motion of solid objects.

From the beginning of the use of Dynamic Geometry Environments, it has been observed that the students did not spontaneously use the drag mode. Bellemain and Capponi (1992) claim that a new contract must be negotiated in the classroom and that it takes time for students to enter this contract. They mention that all but one pair of students called the teacher to check that their diagram was correct. It has also been observed that when students use the drag mode, they do not use it on a wide zone but on a small surface as if they were afraid to destroy their construction (Rolet, 1996). Sinclair (2003) observed that 12th graders, although initially intrigued by the ability to drag points, usually stopped dragging after a short time and concentrated on interpreting a static figure. Some of them inadvertently created a special case by dragging, then generalized from this static but unsuitable figure. Talmon and Yerushalmy (2004) asked ninth grade students and mathematics education graduate students to predict the dynamic behaviour of points that were part of a geometric construction they had executed using a DGE (The Geometer's Sketchpad 3 and The Geometric Supposer for Windows) according to a given procedure, and to explain their predictions. The study reveals that users often grasp a reverse hierarchy in which dragging an object affects its parent. The authors suggest that this reversed hierarchy may be caused by terms and knowledge built into paper-and-pencil geometry. All these observations can be interpreted in terms of instrumentation theory: The instrumental genesis of the drag mode is a long process, and students construct several schemes of utilization that are influenced by former tools and may differ from the expected use by the designers of the environments. As already claimed by Strässer (1992), dragging offers a mediation between drawing and figure and can only be used as such at the cost of an explicit introduction and analysis organized by the teacher.

Dragging for conjecturing in an exploratory approach

Investigations about the way dragging is used by students in solving problems was carried out by Italian researchers (Arzarello, Micheletti, Olivero & Robutti, 1998; Arzarello, 2000; Olivero, 2002). By means of a very fine analysis of the use of dragging, these researchers established a categorization of different kinds of dragging, in particular (Olivero, 2002, p.98):

- *Wandering dragging* is moving the points on the screen randomly in order to discover configurations,
- *Guided dragging* is done with the intention to obtain a particular shape,
- *Lieu muet dragging* is moving a point with the constraint of keeping a particular property satisfied at the initial state, the variable point follows a hidden path even without being aware of this.

Olivero observed that "wandering dragging" and especially "guided dragging" were mainly used by students whereas "lieu muet dragging" was only sometimes used and not by all students. An example of "lieu muet dragging" is provided by a girl, Tiziana, investigating at what conditions the quadrilateral HKLM built by the perpendicular bisectors of the sides of a quadrilateral ABCD is a point (Figure 1).

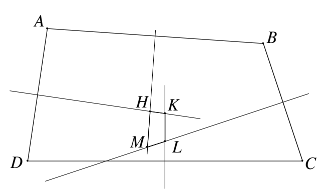

Figure 1. Quadrilateral HKML.

Working together, Tiziana and Bartolomeo used wandering dragging to discover that HKLM is a point when ABCD is a rectangle (Figure 2). Bartolomeo wanted to drag the vertices of ABCD in order to obtain a specific quadrilateral HKLM, a parallelogram, a rhombus, a trapezium (guided dragging). Tiziana did not share this approach and tried to drag point B of the rectangle in order to keep HKLM as a point (an example of lieu muet dragging).

Other researchers stressed the key role of dragging in forming a mathematical conjecture (Healy, 2000; Hölzl, 2001; Hollebrands, 2002; Leung & Lopez-Real, 2000) and even proving by contradiction (Leung & Lopez-Real, 2002). Hölzl (2001) distinguished two ways of using the mediating functions of the drag mode, a test mode on the one hand, and a search mode on the other. All his observations led him to conclude that the second use of the drag mode is not a short term affair but results from a "learning process that is characterized by different layers of conceptions".

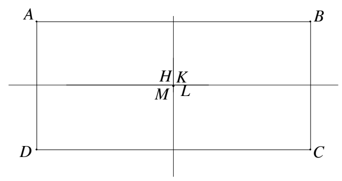

Figure 2. HKML reduced to a point.

Dragging for adjusting in construction tasks

As mentioned above, students encounter difficulties in using DGE to construct "robust" diagrams, which keep their properties in the drag mode. However it was observed that students refined their successive constructions partly made by adjusting (see for instance Jones, 1998, pp. 79-82). Students elaborate a sequence of successive constructions involving more and more geometrical properties. A construction obtained by adjustment enables the students to recognize properties and to mobilize them in a further construction giving less room to visual adjusting. Constructions done by adjusting are not only part of the solving process but they scaffold the path to a definite robust construction. They play an important role in moving from a purely visual solution using adjustments to a solution entirely based on theoretical solutions but achieved by dragging. These constructions are culturally not accepted. Since the time of the Greeks, geometry rejected constructions based on motion and restricted the allowed constructions to those created with straight edge and compass. Hölzl (1996) also observed what he called a "drag and link approach" in students' strategies for solving construction tasks in Cabri. Students relax one condition to do the construction and then drag to satisfy the last condition. They obtain a diagram visually correct and want to secure it by using the redefinition facility of Cabri. However most of the time it does not work because of hidden dependencies.

Robust versus soft constructions

Later Healy (2000) introduced the distinction 'soft versus robust constructions' to give account of constructions that students could change by dragging in order to satisfy a condition. She discovered through observation that, rather than constructions preserved under dragging, students preferred to investigate constructions, "in which one of the chosen properties is purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student". Healy introduced in that paper the distinction "soft/robust" and decided to call the latter constructions soft constructions and the former ones robust constructions. She illustrated the distinction between robust and soft approaches by means of the example of two pairs of students investigating whether the conditions, two congruent sides and a congruent angle, determined one triangle or not. Students using a soft construction immediately found a point for which the third side was not congruent and rejected the condition Side-Side-Angle. Healy (2000) comments how the two kinds of construction are complementary: The general emerges in the exploration of soft constructions and can be checked by using robust constructions as in the case of Tiziana reported above. The cycle 'soft then robust' seems to be a driving force behind students' generalization processes.

Proving and justifying processes. There is a continuing discussion about the question whether the "authority" of the computer leads to a greater resistance to proving on the part of the learner or if adequately chosen and presented proof problems within a computer "milieu" further the need for proofs by the learner (not only) of geometry. Among arguments in favour of the use of DGE, the need in DGE to carry out explicit construction methods based on theoretical properties could lead to consider them as good environments for introducing formal proof. Among arguments given about students' not seeing the need to construct proofs due to the authority of the computer, the facility of computer programs to provide measurements is often mentioned. The role of measurement in DGE was investigated in proving activities (Kakihana, Shimizu & Nohda, 1996; Vadcard, 1999; Flanagan, 2001; Hollebrands, 2002). Studies generally conclude that measurement is not restricted to empirical arguments but is also used in deductive arguments. The study of proving processes carried out in DGE shed light on the explanatory power of proof. Whereas proof is often considered as a means of deciding about the truth of statements, it becomes a means of explanation of phenomena observed on the computer screen that are striking or surprising (De Villiers, 1991; Chazan, 1993; Hanna, 1998). The greater integration of DGE into teaching allows for opportunities to design instructional activities, even sometimes over a long-term period aimed at introducing or fostering deductive reasoning and proof (Sánchez & Sacristán, 2003). Four papers show the diversity and novelty of ways offered by DGE to promote understanding of the need for and the roles of proof:

– Students must give explanations for the fact that a drawing remains a specified quadrilateral in the drag mode (Jones, 2000),
– A teaching experiment is designed to enable students to produce deductive justifications of the correctness of their constructions. In a fine analysis the growing of the quality of the justifications is documented. The teacher plays a critical role in guiding the discussion and ensuring that the justification rules are correct (Marrades & Gutiérrez, 2000),
– By means of an adequate sequence of tasks in a DGE, the need for proof is created through a cognitive conflict that generates in students an intellectual

curiosity about why an unexpected property is true (Hadas, Hershkowitz & Schwarz, 2000)
– A system of axioms and theorems is constructed by students themselves as a system of commands introduced in the software, which has no geometric relations implemented at the beginning of the teaching sequence. Proof is the means for justifying that the new command will provide the expected outcome (Mariotti, 2000).

In two papers (Marrades & Gutiérrez, and Mariotti) proof fulfils a twofold role: Establishing the validity of a construction for each individual and convincing the other students to accept the construction process.

GENERAL TRENDS OF RESEARCH ON THE USE OF TECHNOLOGY FOR TEACHING AND LEARNING GEOMETRY

Research on the use of technology in geometry learning and teaching is multifaceted and based on several theoretical frameworks presented earlier in this chapter. The geometrical topics studied by researchers are very heterogeneous. Topics include basic traditional Euclidean concepts such as triangles, quadrilaterals (in particular parallelograms Hoyles and Noss, 1987, 1988), geometric transformations (Edwards, 1988, 1990; Gallou-Dumiel, 1989; Jahn, 2000; Bellemain & Capponi, 1992; Hollebrands, 2003), polyhedra (Pallascio, 1987), angles (Zack, 1988; Parmentier, 1989; Magina & Hoyles, 1991; Kieran, 1986) measurements of areas, and ratio and proportion (Hoyles et al., 1989; Hoyles et al., 1991). Other topics are less typical like fractals as chaos-game investigation (Sereno, 1994), as a context to study infinity (Sacristán, 2001), curvature (Kynigos & Psycharis, 2003), inscribed star figures (Edwards, 1994), or in precalculus the variations of functions (Arcavi & Hadas, 2000; Falcade, Laborde & Mariotti, 2004; Furinghetti, Morselli & Paola, 2005). Some transversal topics were also addressed like construction activity and of course proof that is a recurring theme in research. Traditional Euclidean concepts have received more attention by researchers than other concepts such as geometric transformations. There is a scarcity of research on using technology in the teaching and learning of loci (Jahn, 2002) and little on students' use of the macro facility available with many DGE (Jones, 2002; Kadunz, 2002). Research on multiple linked representations often mentioned in algebra or calculus have appeared only recently, even though several DGE offer the possibility of constructing graphical representations dynamically linked with geometric diagrams.

In most countries technology is not yet fully adopted by teachers. As a consequence there is very little research that has been done on geometry curricula that start from scratch with technology. Pratt and Ainley (1996) investigated how primary school children in England without explicit geometric teaching create their own geometric constructions with Logo and Cabri. Most research investigated the impact of technology on geometry learning for students already introduced to geometrical concepts. An example of introducing a formal approach by means of a DGE is given in Mariotti (2001) who reports on a long term teaching which taking advantage of the flexibility of Cabri-geometry started the teaching with an empty menu and introduced a command only after it was discussed, according to specific statements selected as axioms. Then, in the sequence of the activities, the other elements of the microworld were added, according to new constructions and in parallel with corresponding theorems. Inspired by the multidimensional analysis of Lagrange, Artigue, Laborde and Trouche (2003), the research trends are presented below according to the following four dimensions: (1) Epistemological and semiotic dimension: The nature of geometry mediated by technology, (2) Cognitive dimension: Technology supporting learning, (3) Situational dimension: The role of the design of the tasks on learning, and (4) Teacher dimension: The role of the teacher.

Nature of geometry mediated by technology

The objects offered by technology on the computer or calculator screen are representations of theoretical objects, which behave by following a computerized (hopefully mathematical) model underlying the software program. The representation process may introduce some differences between the theoretical behaviour and the actual behaviour on the computer screen. Based on the analogy with the process of didactical transposition, this transformation of knowledge due to its technological mediation is called “computational transposition” (Balacheff, 1993, in French: “transposition informatique”). For example, drawing a “circle” in Logo involves a differential (intrinsic) perception of curve, i.e. the construction of a polygon with a large number of sides (more than 30) and a small (e.g. 1 degree) constant turtle turn from side to side. Goldenberg and Cuoco (1998) have pointed to some differences between Euclidean Geometry and DGE Geometry. For example, the behaviour of a point on a segment is a direct result of a design decision and is taken as a postulate upon which other DGE theorems are based. Such a postulate does not exist in Euclidean Geometry. Do students make distinctions between behaviour that results as a consequence of the tool design and behaviour that is a direct result of mathematics? Scher (2001) found that students in fact did not make those types of distinctions and rather they considered the behaviours of each type to be of equal importance. Ways to address this dilemma include the careful design of tasks and the milieu that is not restricted to technology and a focus on the critical role of the teacher (§3.4).

Technology and the learning of geometry

Software environments like Logo T.G. or DGE are considered as favouring learning as they require actions from the students to achieve a goal and in the process, students “learn by coordinating and reflecting upon the form of their interactions” (Hoyles, 1995, p. 202). Hoyles and Noss developed the concept of “situated abstraction” to account for the development of the conceptual framework developed by students in such interactions with a computer environment: Situated abstractions are both general and situated in the environment in which they develop. The development of situated abstractions in the eyes of Noss and Hoyles is certainly related to the exploratory nature of the environments. Logo T.G. or DGE allow students to explore screen constructions and offer a way of accessing the mathematical characteristics of the underlying geometry. In such processes, software tools become extensions of the own thinking of the students (Mason, 1992). The 'computational scaffolding' (i.e. the support system available in the setting (Hoyles *ibid.*)) contributes to the process of constructing situated abstractions.

> The software tools exploited by the students provide them with the hooks they need on which to hang their developing ideas.

The examples given above on the use of the drag mode and on soft constructions illustrate very well the idea of 'computational scaffolding' (see §2.2.2). The concept of situated abstraction nevertheless also points to the importance of the necessity of a transfer from the computer environment to the world outside the computer (Olive & Lankenau, 1987; Zack, 1988; Parmentier, 1989; Scally, 1987).

The exploratory nature of these environments amplifies the search processes of students solving a task and brings under a spotlight their understandings. Hoyles contends that the software constrains students' actions in novel ways and forces the researcher or the teacher to notice a student's point of view, which could have not been noticed in a paper and pencil environment. It offers a "window" on the students' conceptions and learning (Noss & Hoyles, 1996). The constraints and new possibilities with regard to paper and pencil technology are often considered as shaping the students' strategies and thus their ways of thinking. They can encourage new ways of conceptualizing mathematical ideas. By means of several examples, Resnick (1995) shows how Star Logo used with 5000 turtles leads to solving classical geometric problems with a statistical approach, giving a new meaning to geometrical configurations.

The design of tasks

Some researchers also stress that the choice of the tasks in relation to the affordances of the technological geometry environment may be critical for the development of the students' understandings. A relevant combination of tools made available to students and of problem situations is generally considered as a good "milieu" (in the sense of the theory of didactic situations) for the emergence of new knowledge (see for example Kordaki & Potari, 2002 about the use of a micro-world for area measurement offering several tools and feedback).

Arzarello, Olivero, Paola and Robutti (2002) argue that task design and teacher moderation play very important parts in encouraging students to press on beyond perceptual impression and empirical verification in DGE. Pratt and Davison (2003) conclude from an investigation on the use of the Interactive White Board (IWB) with a dynamic geometry software that the visual and kinaesthetic affordances of the IWB are insufficient to encourage the fusion of conceptual and visual aspects of children's figural concepts when these affordances are embodied in tasks that simply focus on the visual transformation of geometric figures. They claim that the kinaesthetic affordances of the IWB need to be embodied in tasks based on the utilities of contrasting definitions that draw attention to the conceptual aspect. Sinclair (2003) draws the same type of conclusion about the use of pre-constructed dynamic diagrams: The design of the accompanying material has the potential to support or impede the development of exploration strategies and geometric thinking skills.

The role of technology in students' solving processes is multiple: The tools offered by the environment allow students' strategies that are not possible in paper and pencil environment, the meaning of the task is provided by the environment, the environment offers feedback to the students' actions.

Laborde (2001) distinguishes four kinds of tasks used by teachers with DGE:

– Tasks in which the environment facilitates the material actions but does not change the task for the students, for example, producing figures and measuring their elements.
– Tasks in which the environment facilitates students' exploration and analysis, for example, identifying relations within a figure through dragging
– Tasks that have a paper and pencil counterpart but can be solved differently in the environment, for example a construction task may be solved in DGE by using a geometric transformation or the sum of vectors.
– Tasks that cannot be posed without the mediation of the environment, for example, reconstructing a dynamic diagram through experimenting with it to identify its properties.

In the first two types, tasks are facilitated, rather than changed, by the mediation of DGE. In the last two types, tasks are changed in some way by the mediation of DGE, either because the solving strategies differ from what they usually were, or because they simply are not possible outside DGE. In the example of the last type, the meaning of the task comes from the possibility of dragging.

The second type of task may be used as a research tool for investigating students' ideas. It acts as a window on students' conceptions and understandings, as visible in Arzarello et al.'s (2002) research reported above. The last two types may be used in teaching as a tool for fostering learning. For example, DGE may foster the use of geometric transformations as construction tools for providing geometric relationships between objects (third type of tasks). The last type may also be the source of a different perspective on mathematics. The task of identifying properties in a dynamic diagram requires a back-and-forth process made of guesses based on visualization and checks on the diagram possibly involving deductions drawn from what has been observed. The nature of mathematical activity is changed and becomes a modelling activity –what may deeply differ from the kind of mathematics the teacher wants to develop.

The role of feedback

Technology offers feedback to the actions of the user. The role of feedback was stressed by research on micro-worlds. It was also stressed by research on DGE, when students check their constructions through the drag mode or check their conjectures using various tools (e.g., measuring and constructing). Such feedback can be used to create the need for searching for another solution in case provided feedback gives evidence of the incorrectness or inadequacy of the solution. In the sequence of tasks (mentioned above; designed by Hadas et al., 2000), a cognitive conflict was created because students developed expectations, which turned out to be wrong when they checked them in the dynamic geometry environment. This interplay of conjectures and checks, of certainty and uncertainty was made possible by the explorative power and checking facilities offered by the DG environment.

Feedback can be the source of refinements in students' answers. Leron and Hazzan (1998) describe a strategy by successive refinements provoked by feedback generated by the software. Hillel, Kieran and Gurtner (1989) reported similar results with middle-school students working in a Logo computer environment. Within a Logo programming environment, students enter commands and then they can visually observe and interpret the results of the code and make modifications to their commands. Edwards (1992) found students working with a computer micro-world for geometric transformations, who refined their understanding of transformations based on the visual feedback they received from the computer as they engaged in a matching game. The software incorporating knowledge and reacting in a way consistent with theory impacts on the student's learning trajectory in the solving process. Here, one can recognize the philosophy underpinning the notion of micro-world discussed in §2.1.1.

The use of geometry technology by teachers

Since the very beginning, research carried out on the use of technology focused on the students and their solving processes in technology based tasks. Some researchers pointed out, that interactions with technology, even in carefully designed tasks, could not lead to learning by themselves and stressed the need for teacher interventions (about the notion of angle in Hoyles & Sutherland, 1990, about the notion of reflection in Gallou-Dumiel, 1989). In the mid-nineties, the way teachers integrated technology into their teaching practice started to become an object of investigation. Noss and Hoyles (1996, ch. 8) related the teacher practice to their attitude with regard to technology and learning in a case study of some teachers following a university case about the use of micro-worlds in mathematics.

After several years, it appeared that an analysis of the use of technology in classrooms cannot be carried out without taking into account the complexity of teaching and learning situations and the multiplicity of factors related to the use of technology in the classrooms. Teachers are key elements in this complexity. What changed in recent years is the focus on the teacher practice in ordinary classrooms using technology.

Ruthven, Hennessy and Deaney (2005) report on a multiple-case study of "archetypical current practice" in using DGE in secondary mathematics education in England. The authors (p. 155) found that

> the prime purpose of DGE use by teachers was evidencing geometric properties through dragging figures. Most commonly, this involved dragging to examine multiple examples or special cases.

But teachers very seldom used dragging to analyze dynamic variation. The authors also found most striking, "the common emphasis on mediating geometrical properties through numerical measures, with little direct geometrical analysis of situations in order to explain numerical patterns and theorize geometrical properties". This probably results from (a long before DGE prevailing) didactical norm anchored in the teacher practice in the UK. The teachers adapt the available tools to this norm. The authors also report how teachers may reduce the exploratory dimension of DGE in order to control students' exploration and to avoid students meeting situations that could obscure the underlying rule or could require explanations going beyond the narrow scope of the lesson, like for example explanations about rounding measurements in a lesson about inscribed angle in a circle.

The study of teacher practice when using technology revealed that teachers must cope with all the complexity of the management of a classroom: Instead of following textbooks the teacher must design worksheets (Monaghan, 2004), they must adapt the management of several kinds of time in their classroom and the relationship between, on the one hand old and new knowledge, and on the other hand paper and pencil techniques and Cabri techniques (Assude, 2005). Monaghan (*ibid*.) also showed how technology could affect the emergent goals of the teacher during the lesson.

According to a Vygotskian approach, some researchers investigated semiotic mediation processes organized by teachers making use of technology for mediating mathematical knowledge through the use of DGE. External operations are carried out by students faced with tasks completed in the environment and the teacher contributes to an internalization process by organizing social interactions and collective discussions in the classroom in which s/he intervenes in order to transform the meaning of what has been done on the computer into a meaning that could be related to the "official" mathematical meaning (see Mariotti, 2000, about the notion of geometric construction, and Mariotti, Laborde & Falcade, 2003, about the notion of graph of function with a DGE).

FROM THE PAST TOWARDS THE FUTURE

Geometry is often characterized by its recourse to diagrams and its special relationship with reasoning and proof, a specificity that we interpreted at the beginning of this chapter as due to the two-folded nature of geometry: Geometry resorts both to visualization and theory. The introduction of technology for the teaching and learning of geometry influenced both aspects of geometry in different ways, somehow complementing each other. According to the classification proposed by Hoyles and Noss (2003), Logo and micro-worlds belong to the "programming and micro-worlds" category whereas DGE belong to the "expressive tools" category. Logo somehow introduced the idea of micro-world and exploratory environment that goes beyond only geometry. Several researchers also consider Logo as a tool to forge links between students' actions and the corresponding symbolic representations they develop. Students must express actions in a symbolic language to produce diagrams on the computer screen. In dynamic geometry, students' actions deal directly with tools producing geometric objects and relations or consist in manipulating dynamic diagrams: Students move from action and visualization to a theoretical analysis of diagrams and possibly to the expression of conjectures and reasoning. It must be mentioned that some dynamic geometry environments may also be considered as micro-worlds. Conversely, T.G. micro-worlds can be used as expressive tools.

Research on the use of technology in geometry not only offered a window on students' mathematical conceptions of notions such as angle, quadrilaterals, transformations, but also showed that technology contributes to the construction of other views of these concepts. Research gave evidence of changes and progress in students conceptualization due to geometrical activities (such as construction activities or proof activities) making use of technology with the design of adequate tasks and pedagogical organization. Technology revealed how much the tools shape the mathematical activity and led researchers to revisit the epistemology of geometry.

Even if the various technologies differ in the access to geometrical concepts, some invariants can be drawn from the research studies and their development over time. The focus initially was on the learner and his/her interactions with technology, giving rise to theoretical reflections about learning processes in mathematics by means of technology. The focus moved to the design of adequate tasks in order to meet some learning aims and then to the role of the teacher. The integration of technology into the everyday teacher practice became the object of investigation. Finally, the role of the features of software and technology design were also questioned and investigated in order to better understand how the appropriation of the technological environment by students could interfere with the learning of mathematics and how the teacher organizes students' work for managing this interaction between appropriation of the tool and learning. By focusing on everyday teacher practice, the constraints of the teaching institutions come to the foreground in the analysis of the integration of technology: How does the teacher manage the use of technology in taking into account the curriculum, and the time constraints? Technology is also still developing at a high pace: At the moment, the integration of symbolic programming with DGE, the development of special software for spatial Geometry software in the sense of 3D-software and the integration of Geometry software (especially DGE) with Computer Algebra Systems (CAS) seem to be most noteworthy. Additional research issues could deal with the integration of algebra and geometry allowed by technology. The fast evolution of interfaces calls for two main research strands:

– From the perspective of learning how do the students' instrumentation processes develop and what are their links with growth of mathematical knowledge? An object of investigation could especially be the impact of the novel kind of 3D dynamic and direct manipulation, in particular from an embodied cognition approach focusing, for example, on the role of gestures in mathematical construction of knowledge. One can incidentally wonder why the impact of technology extensively used outside school (such as the gaming technology) has only very recently started to become an object of investigation in the research in mathematic education community;

– From the perspective of teaching, the integration processes of technologies by teachers into their everyday practice could be extended to new technologies. Such studies seem to be particularly relevant in this time of massive entry of a new generation of teachers. Teacher preparation offers a research domain on such issues of high social importance.

Since geometry teaching has been changing so frequently in most countries in the past decades and differs from one country to another one, future research could address the question whether technology would lead to the tendency of smoothing the differences among geometry curricula across the world.

Studying the role of technology in the teaching and learning geometry led research in mathematics education to take into account and question all the complexity of teaching and learning processes. There is a dialectical link between the development of theories and research on the use of technology in geometry and generally speaking in mathematics teaching. Technology gave the opportunity of making use of available theoretical approaches, but also acted as a catalyst for the growth of new theoretical approaches and concepts in research in mathematics education. This is why we believe that research on the use of technology in geometry teaching needs more contributors.

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