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Accurate Energy Levels for the Anharmonic Oscillator and a Summable Series for the Double-Well Potential in Perturbation Theory

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We introduce a generalization of Wick-ordering which maps the anharmonic oscillator (AO) Hamiltonian for mass m and coupling λ exactly into a "Wick-ordered" Hamiltonian with an effective mass M which is a simple analytic function of λ and m. The effective coupling $\Lambda = \lambda/M^3$ is bounded. We transform the AO perturbation series in λ into one in Λ. This series may then be summed using Borel summation methods. We also introduce a new summation method for the AO series (which is a practical necessity to obtain accurate energy levels of the excited states). We obtain a numerical accuracy for $(E_{PT} - E_{exact})/E_{exact}$ of at least 10^{-7} (using 20 orders of perturbation theory) and 10^{-3} (using only 2 orders of perturbation theory) for *all* couplings and *all* energy levels of the anharmonic oscillator. The methods are applicable also to the double-well potential (DWP, the AO with a negative mass-squared). The only change is that now the effective coupling is unbounded as $\lambda \to 0$. The series in Λ is, however, still summable. The relative accuracy in the energy levels for 20 orders of perturbation theory varies from 10^{-7} for large coupling to 1% at $\lambda = 0.1$ and to 10% at $\lambda = .05$. We also present results for the sextic oscillator.

I. INTRODUCTION

Much of the information we have obtained from field theory has, of necessity, been obtained perturbatively. Perturbation theory has been an essential tool in the comparison of realistic physical theories to the results of experiment. There have been many studies of the nature of perturbation theory, with the hope of understanding both its limitations and of finding methods to best utilize the perturbative results we do have. This is particularly important in many problems for which the perturbation series is only an asymptotic series [1]. The divergence of the perturbation series for QED was first qualitatively discussed by Dyson [2]. We will consider the much simpler case of the anharmonic oscillator (AO).

The exact nature of the divergence (exponentially unbounded large-order behavior) of the AO perturbation series was first studied by Bender and Wu [3], using WKB methods. They also studied the divergence of the Wick-ordered theory [4]. These results have recently been derived by Lipatov [5] and by Brezin *et al* [6], using path integral methods: they can also study general scalar theories in quantum mechanics as

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well as renormalizable ϕ^{2M} theories. They explicitly find the large-order behavior of the perturbation series about zero coupling. Most theories show a factorial growth of the terms in the series.

The known large-order behavior of the field theory small-coupling (Rayleigh-Schrödinger) perturbation series has been useful in allowing very accurate resummation of the asymptotic series for the anharmonic oscillator. Practically it has allowed the accurate calculation of critical indices in statistical mechanics [7]. The typical divergence of the series as $E_n \sim n!$, where E_n is the coefficient of λ^n and λ is the coupling, suggests that we make a Borel transformation of the series (see Section V). The large-order behavior of the series then is used as a guide to the location of the nearest singularity of the resulting Borel transform. The energy (critical exponent,...) may then be written, by an appropriate transformation of the resulting integral, as an integral over a series convergent in the range of integration [7]. These methods give distinctly better results than the Padé method [8] or the Borel-Padé method [9], which make no (Padé) or little (Borel-Padé) use of the known asymptotic behavior of the series. In addition to the analyticity properties of the summed series and of the Borel transform of the series, there are exact transformations of the series which may be used to rearrange the series and to substantially improve its convergence. We introduce one such transformation, an effective mass method, which allows the accurate summation of the AO series for *all* coupling.

There is particular interest in theories for which these methods do not work: non-Borel-summable series. One of the sources of non-Borel-summable behavior is the occurrence of many perturbative vacua in the theory and of the related instanton solutions which tunnel between these vacua. In the physically interesting theory of QCD (non-Abelian gauge theory with fermions), these solutions result in a parameter θ, parameterizing the different vacua of the theory. This parameter occurs in the combination $\exp(i\theta - 8\pi^2/g^2)$ where g is the gauge coupling constant [10]. These cannot be seen in the usual perturbation expansion. The relation of these tunneling solutions to (Borel) summability has been studied in the simple quantum-mechanical model of the double-well potential [11]. The double-well potential (DWP) is the anharmonic oscillator with negative mass-squared (see Fig. 3). This system has the crucial property of multiple perturbative vacua and of an (analytically-known) instanton solution which goes between these vacua.

As explicitly found by Brezin *et al.* [12], the perturbation series for the ground state of the DWP, when expanded about one of the degenerate minima, may be calculated by the study of the contribution to the path integral of pairs of instanton solutions. The terms in the perturbation series diverge like $n!$ but do not oscillate in sign beyond the first term. This produces singularities of the Borel transform which occur on the positive real axis (on the integration contour). Attempts to sum this series have led to a study of the analytic structure of the DWP in the complex coupling-constant plane [13], as this is the crucial missing ingredient for the existence and uniqueness of the Borel transform of the series [14]. We show in this paper that there exists a summable perturbation series for the DWP in terms of an effective coupling. The success of this approach suggests that the DWP is analytic in the (open) right-half

plane, not quite enough for a unique Borel transform to be determined by the perturbation series.

We now discuss in more detail the organization of this paper.

The Hamiltonian for the AO is

$$\mathscr{H} = \frac{1}{2}\dot{x}^2 + \frac{m^2}{2}x^2 + \lambda x^4 \tag{1.1}$$

where m is the mass and λ is the coupling. The interpretation of the coefficient of the harmonic term x^2 as a mass comes from the one-dimensional field theory analogue of the above quantum mechanical problem (cf. Ref. [3, Appendix A]).

The Rayleigh-Schrödinger perturbation series in the coupling λ for the AO is known to be Borel summable [9]. One may thus sum the series, with a sufficient number of terms, to any desired accuracy for a given coupling. For couplings much greater than one, or for energy levels much higher than the first few, this is a practical impossibility. In both cases more of the terms are required than may be practically handled. It is also possible, for any fixed number of terms considered, to find couplings large enough that the relative error $(E_{pt} - E_{exact})/E_{exact}$ is arbitrarily large. It is often of interest in practical problems to know the result for large (or all) couplings. It is therefore of practical importance to discover better ways of using the perturbative information we have. In Section II we show how, by Wick-ordering the theory, we can transform the AO perturbation series into one which is equivalent but diverges less strongly than the unshifted (non-Wick-ordered) series. Further, the new expansion parameter (effective coupling constant) is bounded as the AO coupling constant diverges to infinity. Borel summation methods then allow us to sum the series, with the relative error being bounded for all couplings λ. This series, with coefficients which we will call $E_n{}^W$, may be simply generated from the AO perturbation series E_n by reexpansion of the series using the binomial theorem.

The Wick-ordered theory with mass M is equivalent to the AO with mass m related to M by

$$M^2 = m^2 + 6\lambda/M. \tag{1.2}$$

It is related to the DWP by simply letting $m^2 \to -m^2$ (see Eqn. 1.1),

$$M^2 = -m^2 + 6\lambda/M. \tag{1.3}$$

We show in Section II that the DWP energy levels for the Hamiltonian (1.1) with $m^2 \to -m^2$ is given by the perturbation series E^W evaluated at coupling λ and mass M determined by (1.3). In some sense we have expanded the DWP about the analytic continuation of its maximum, not about its minimum. We are careful never to match perturbation series except for ranges of coupling and mass in which they are both valid, and we at no point have imaginary masses or energies.

In Section III we show how, using path integral methods, a more general class of transformations of the perturbation series than Wick-ordering may be made. The

particular transformation we choose to focus on is the introduction of an effective mass into the theory which, for the Hamiltonian (1.1), satisfies

$$M^2 = m^2 + \beta\lambda/M \tag{1.4}$$

where β is an arbitrary parameter on which physical quantities cannot depend. In Section IV we discuss how β may be chosen to obtain accurate results from perturbation theory. The best β will depend on the energy level and on the number of terms in the series. We illustrate this method by finding accurate analytic approximations to the AO energy levels from first- and second-order perturbation theory.

Section V contains a non-rigorous discussion of the analyticity properties of the energy levels of E^W (or in general E^β) as a function of the effective coupling constant $\lambda_E = \lambda/M^3$. We are led to suggest that the DWP has a cut along the negative Imλ axis as a function of its coupling constant λ. We also suggest, using path integral methods, that the Borel transform of the series for E^β should have the same singularities in z as the AO has but weaker by a factor $e^{-\beta}$. It is this simplicity of the analytic structure of the Borel transform that makes the effective mass perturbation theory summable and therefore useful. Whether other effective parameters (eg. an effective coupling constant subtraction) are useful is an interesting open question.

The discussions of the DWP in Sections II and V would not be convincing if we were not able to show, at least numerically, that our claim that the transformed series converges to the correct energy eigenvalues is justified. It is possible to transform the series for the DWP about one minimum [12] into a "Borel summable" series. One way is to rotate the contour used in defining the gamma function introduced by the Borel transform. A second is to introduce $\Gamma(2n) \cdot \Gamma(-n + \frac{1}{2})$ instead of the usual $\Gamma(n)$. The methods converge but unfortunately to the wrong result, so one must be careful. We have done extensive studies of the AO perturbation series through 20 orders of perturbation theory, and for the ground state ($K = 0$) through $K = 10$ energy levels. In Section VI we present results for the anharmonic oscillator and for the double-well potential. We use both Borel methods (which are practically restricted to small K) and a summation method based on the parameter β introduced in Section III. For the DWP we compare to variational calculations of the energy levels and find excellent agreement.

The methods we have introduced here may be easily generalized to other systems than the AO. In Section VII we study the sextic oscillator (SO, with potential $(m^2/2)$ $x^2 + \lambda x^6$), with both positive and negative m^2. The convergence of the summation methods is less rapid (the perturbation series coefficients grow like $(2n)!$; see Ref. [6]), but there are no surprises. This calculation could not have been done using the Wick-ordering method, as the x^4 counterterm to the λx^6 potential would have been difficult to handle.

In Section VII we emphasize the simple connection to field theoretic methods. We believe that much of the freedom available in the renormalization procedure remains to be exploited, particularly in the calculation of universal objects such as critical indices.

Appendix A includes some of the details of the calculation of the AO perturbation series. Appendix B shows how to extract the ground-state energy from the propagator in a one-dimensional field theory.

II. Wick-Ordering as a Resummation of the Perturbation Series

We first study the introduction of an effective mass (equivalent oscillator) into the theory in a simple example: that of Wick-ordering. This will result in a series for the AO energy levels that is similar to the non-Wick-ordered series (an asymptotic series which is summable) except it is in terms of an effective coupling which is bounded. We emphasize that this is, for the AO, merely a convenient trick. The theory need not be Wick-ordered, as the equivalent one-dimensional field theory has no divergences.

We wish to study the energy levels (or $\langle x^{2n} \rangle$, where the brackets mean expectation value in some state) of the Hamiltonian

$$\mathscr{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}m^2x^2 + \lambda x^4 \tag{2.1}$$

This may be rewritten as a one-dimensional field theory

$$\mathscr{H} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}m^2\phi^2 + \lambda\phi^4 \tag{2.2}$$

where $\phi(t)$ is a quantum field at time t and $\dot{\phi}(t)$ is its conjugate momentum [3]. The energy levels are poles of the Green functions of the theory. To Wick-order this theory, we expand ϕ and $\dot{\phi}$ in terms of annihilation and creation operators (for mass M)

$$\begin{aligned} \phi &= (ae^{-itM} + a^+e^{itM})/(2M)^{1/2} \\ \dot{\phi} &= (-iae^{-itM} + ia^+e^{itM})\, M^{1/2}/2^{1/2}. \end{aligned} \tag{2.3}$$

We can then, using $[a, a^+] = 1$, find the Wick-ordered Hamiltonian $:H:$, with mass M and Wick-ordered relative to mass M. $:H:$ is the Hamiltonian with all creation operators to the left of all annihilation operators. Then $:H:$ is related to a non-Wick-ordered theory by

$$:H: = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}M^2\phi^2 + \lambda\phi^4 - \frac{3\lambda}{M}\phi^2 + \frac{3\lambda}{4M^2} - \frac{1}{2}M. \tag{2.4}$$

If we define a mass m^2 by

$$m^2 = M^2 - 6\lambda/M, \tag{2.5}$$

then the Hamiltonian $\mathscr{H}$ of Eqn. (2.2) is related to $:H:$ by

$$\mathscr{H} = :H: + \frac{1}{2}M - \frac{3\lambda}{4M^2}. \tag{2.6}$$

To find the energy levels of $\mathscr{H}$, we need only find the levels of $:H:$, where M is determined by Eqn. (2.5). Most importantly, we may derive the perturbation series for $:H:$ from the known series for $\mathscr{H}$ by making simple algebraic manipulations on Eqn. (2.6). Denote the energy of the AO by $E(\lambda, m)$ and the energy of the Wick-ordered AO by $E^W(\lambda, M)$. Then since

$$E(\lambda, m) = m \sum_{N=0}^{\infty} E_N(\lambda/m^3)^N,$$
$$E^W(\lambda, M) = M \sum_{N=0}^{\infty} E_N{}^W(\lambda/M^3)^N, \tag{2.7}$$

the Wick-ordered series is simply

$$E^W(\lambda, M) = E - \frac{1}{2} M + \frac{3\lambda}{4M^2}$$
$$= M(1 - 6\lambda/M^3)^{1/2} \sum_{N=0}^{\infty} E_N \left[\frac{\lambda/M^3}{(1 - 6\lambda/M^3)^{3/2}}\right]^N - \frac{1}{2} M + \frac{3}{4} M(\lambda/M^3). \tag{2.8}$$

Using the binomial expansion ($\lambda_E = \lambda/M^3$)

$$(1 - 6\lambda_E)^{1/2-3N/2} = \sum_{j=0}^{\infty} \frac{(6\lambda_E)^j \, \Gamma(\frac{3}{2}N - \frac{1}{2} + j)}{j! \, \Gamma(\frac{3}{2}N - \frac{1}{2})}, \tag{2.9}$$

we can find the terms E_N^W [4]:

$$E_0{}^W = K, \tag{2.10a}$$
$$E_1{}^W = \tfrac{3}{2}K(K - 1), \tag{2.10b}$$
$$E_N{}^W = \sum_{j=0}^{N} \frac{E_{N-j} \, 6^j \Gamma(\frac{3}{2}N - \frac{1}{2}j - \frac{1}{2})}{j! \, \Gamma(\frac{3}{2}N - \frac{3}{2}j - \frac{1}{2})} \qquad (N \geqslant 2). \tag{2.10c}$$

K is the energy level being studied. The necessary E_N are recorded in Appendix A.

The effective mass equation (2.5) is a cubic equation in canonical form. We record its solution for a general mass shift β ($\beta = 6$ for Wick-ordering). If we set the mass $m^2 = 1$, we must solve

$$M^2 = 1 + \beta\lambda/M. \tag{2.5'}$$

For $\gamma^2 > 4/27$, $\gamma = \beta\lambda$, the real root is

$$M = [(\gamma + (\gamma^2 - 4/27)^{1/2})^{1/3} + (\gamma - (\gamma^2 - 4/27)^{1/2})^{1/3}]/2^{1/3}. \tag{2.11a}$$

For $\gamma^2 < 4/27$ the root which equals the above root at $\gamma^2 = 4/27$ is

$$\begin{aligned} M &= \frac{2}{3^{1/2}} \cos(\phi/3), \\ \phi &= \cos^{-1}(\gamma\, 3^{3/2}/2), \\ 0 \leqslant \phi &\leqslant \pi/2 \qquad \text{for} \quad \gamma > 0. \end{aligned} \tag{2.11b}$$

The effective coupling $\lambda_E = \lambda/M^3$ is therefore a simple function of λ and of m. Clearly from (2.5′) $M > 1$ and as $\lambda \to \infty$, we find $M \to \infty$. However, $\lambda_E = \lambda/M^3 \to 1/\beta$—i.e., λ_E is bounded between 0 and $1/\beta$ for the AO.

For negative mass-squared ($m^2 = -1$) we must solve the equation

$$M^2 = -1 + \beta\lambda/M \tag{2.5″}$$

which for $\beta, \lambda > 0$ has the solution

$$M = [((\gamma^2 + 4/27)^{1/2} + \gamma)^{1/3} - ((\gamma^2 + 4/27)^{1/2} - \gamma)^{1/3}]/2^{1/3}. \tag{2.12}$$

This approaches $\lambda^{1/3}\beta^{1/3}$ as $\lambda \to \infty$, and 0 as $\lambda \to 0$, so that the effective coupling λ_E goes to infinity as $\lambda \to 0$. In Figure 1 we graph the effective coupling as a function of the coupling λ in the Hamiltonian (Eqn. 2.1) for the AO ($m^2 = 1$) and for the DWP ($m^2 = -1$). For Figure 1, and for the rest of this section, we fix $\beta = 6$(Wick-ordering).

To convince the reader of the usefulness of this transformation of the series, we will apply it to the AO and the DWP. We combine Eqns. 2.8 and 2.10 (using only the first term of the perturbation series)

$$E^K = M(K + \tfrac{1}{2} - \tfrac{3}{4}\lambda_E + \tfrac{3}{2}K(K-1)\,\lambda_E + \cdots) \tag{2.13}$$

where M is determined by (2.11a) with $\beta = 6$. The ground state ($K = 0$) thus has the energy

$$E^0 = M(\tfrac{1}{2} - \tfrac{3}{4}\lambda_E)\,. \tag{2.14}$$

The ground state energy is given by this formula to an accuracy of .02 for all couplings λ. However, the energies of the higher states are less accurately given by (2.13); the accuracy (for $\lambda \to \infty$) is only .05 for $K = 1$ and .90 for $K = 10$. We may of course, use more terms in the perturbation series (PS). Because the series is asymptotic (see Sect. V), the results we obtain are much worse in large order (and for the higher energy levels, in low order also). We must therefore sum the series.

We have summed the PS of the Wick-ordered theory for 25 orders. We used the Borel transformation and mapping methods of LeGuillou and Zinn-Justin [7]. The results for the ground state and the first two excited states of the AO and the DWP are shown in Table I. The DWP is calculated using (2.5), (2.6), and the series (2.7) for the Wick-ordered energy levels. The higher excited states are not considered because

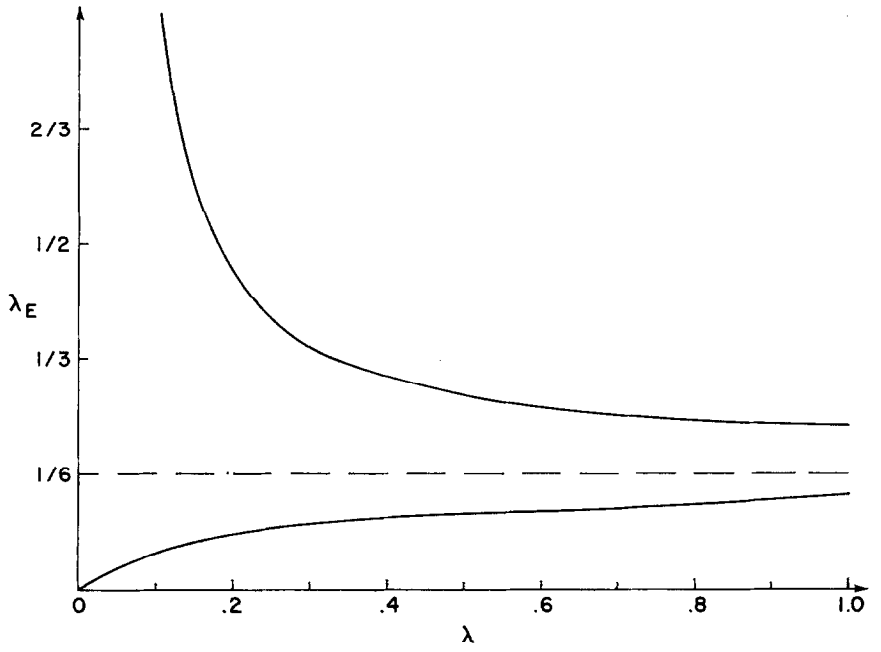

FIG. 1. Effective coupling $\lambda_E = \lambda/M^3$ as a function of the Hamiltonian coupling λ, for the Wick-ordered oscillator. M is determined by Equation (2.5′) for the anharmonic oscillator (lower curve) and by Equation (2.5″) for the DWP (upper curve). The lower curve fits $\lambda = \lambda_E/(1 - 6\lambda_E)^{3/2}$ and the upper curve fits $\lambda = \lambda_E/(6\lambda_E - 1)^{3/2}$.

summation methods work very poorly on these levels. The asymptotic size of the nth term in the series for energy level K is [4].

$$E_n^K \sim -\frac{(-)^n\, 12^K\, 6^{1/2}}{K!\, \pi^{3/2}}\, 3^n \Gamma(n + K + \tfrac{1}{2}) \tag{2.15}$$

The growth of the coefficient of the gamma function is, for moderate K, much faster than the growth of the energy of the level for large λ and K [15]:

$$E^K/\lambda^{1/3} \sim 1.3765 \cdot (K + \tfrac{1}{2})^{4/3} \tag{2.16}$$

In Section III we introduce a more general effective mass shift. We introduce a parameter into the theory which is irrelevant in the sense that the energy levels cannot depend on the value of the parameter. In Section IV we show how this parameter can be used to optimize the energy estimate of the series in a way that works for all energy levels. It will also give us a polynomial (rather than an integral) to evaluate for each new coupling.

The Wick-ordering result for the first (order λ) estimate of the ground state energy, Eqn. (2.14), has been found previously by several authors. Chang [16] has used it

as a guide to calculations in ϕ^4 theories in two and four dimensions. We believe that our methods are more systematic and can be used to improve on his mean-field calculation of the effective potential. He does not show how to extend his calculation beyond leading order, as he believes his results to be essentially non-perturbative. Bender *et al.* [17] introduced the truncated Green function approximation. In leading

TABLE I

Results of Summing the First 25 Terms of the Wick-Ordered Series Using a Borel Transformation and the Mapping of LeGuillou and Zinn-Justin [7][a]

	Anharmonic Oscillator		
λ	E_0	E_1	E_2
1	.803770651234 (1)	2.7378922679 (1)	5.1792917 (1)
10^3	6.6942208505 (2)	23.97220602 (4)	47.01735 (3)
10^6	66.800064630 (2)	239.3679844 (4)	469.6846 (3)
	Double-Well Potential		
λ	E_0	E_1	E_2
.0391	.7 (1)		
.0494	.63 (2)		
.0988	.473	.762 (4)	1.9 (5)
.198	.398	1.01	2.15

[a] For higher energy levels the series converges less well. For the double-well potential the variational estimates of the energy levels are listed in Table V. Estimated errors are in parentheses.

order it is easy to show diagrammatically that their lowest order is the Wick-ordered theory in tree approximation, the first term of Eqn. (2.13) Their second-order approximation sums up all propagator graphs through four loops (and many higher-order graphs). Their value for the single-particle pole ($K = 1$ quantum level) is then within .7% of the exact value for all couplings. It is therefore surprising that by using only two orders of perturbation theory (only two-loop diagrams in Feynman diagram language) we obtain an accuracy of .05% for all couplings for the $K = 1$ level. Their method is a low-energy approximation, as is Wick-ordering, and the methods we introduce in Section IV will give much more accurate energies for higher excited states. Further, the extension to higher orders is quite easy using the effective mass method, while the truncated Green function method requires the solution of coupled integral equations.

III. Path-Integral Generalization

We may look at Wick-ordering as just one of a large class of possible transformations (resummations) of the perturbation series. It is natural to look at the path-integral formulation of quantum mechanics to study transformations which leave the theory invariant. We refer the reader to Abers and Lee [18] for a concise review of path integral methods. For a Hamiltonian of the form

$$\mathscr{H} = \tfrac{1}{2}p^2 + V(q), \tag{3.1}$$

we may define the Lagrangian

$$L = \tfrac{1}{2}\dot{q}^2 - V(q) \tag{3.2}$$

and the action

$$S = \int L(q, \dot{q})\, dt. \tag{3.3}$$

The ground-state expectation value of a time-ordered product of coordinates (which in field theory are the Green functions) may be evaluated from $W[J]$, where $J(t)$ is an external source and

$$W[J] \propto \lim_{\substack{T' \to -i\infty \\ T \to i\infty}} \int [dq] \exp\left[i \int_T^{T'} (L(q, \dot{q}) + J(t)\, q(t))\, dt \right] \tag{3.4}$$

The energy levels (except for the ground-state) are found as the location of poles of the Green functions. We can, of course, study the Green functions as well as their poles in the following. The ground-state energy has not been lost; we show in Appendix B how it may be recovered from the propagator.

The Lagrangian $L(q, \dot{q})$ is for the AO

$$L(q, \dot{q}) = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}m^2q^2 - \lambda q^4. \tag{3.5}$$

When inserted in Eq. (3.4), it defines the theory. Consider the operation of adding and subtracting the quantity $\frac{1}{2}\beta\lambda q^2/M$ to L:

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}(m^2 + \beta\lambda/M)\, q^2 - \lambda q^4 + \frac{1}{2}\frac{\beta\lambda}{M} q^2. \tag{3.6}$$

This will clearly not affect the Green functions. If we define the mass M by

$$M^2 = m^2 + \frac{\beta\lambda}{M}, \tag{3.7}$$

the Lagrangian which generates the *PS* is

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}M^2q^2 - \lambda q^4 + \frac{1}{2}\frac{\beta\lambda}{M}q^2. \tag{3.8}$$

The *PS* for the energy is, by dimensional analysis (or more generally by scaling [8]), of the form

$$E(\lambda, M) = M\sum_{n=0}^{\infty} E_n^{\beta}(\lambda/M^3)^n. \tag{3.9}$$

This should be compared to the series for the unshifted theory

$$E(\lambda, m) = m\sum_{n=0}^{\infty} E_n(\lambda/m^3)^n. \tag{3.10}$$

These energies are equal. If we write $\lambda_E = \lambda/M^3$ and solve Eqn. (3.7) for m

$$m = M(1 - \beta\lambda_E)^{1/2}, \tag{3.11}$$

we may generate the series E_n^{β} from the known series E_n. This is done by matching the series in λ_E, as we did in Section II:

$$\sum_{n=0}^{\infty} E_n^{\beta}\lambda_E^n = (1 - \beta\lambda_E)^{1/2}\sum_{n=0}^{\infty}\frac{E_n\lambda_E^n}{(1 - \beta\lambda_E)^{3n/2}}. \tag{3.12}$$

For $\beta = 6$ this reproduces the Wick-ordering result (the first two terms in the series are defined differently). The solution for the effective mass $M(\lambda, m)$ is recorded in Section II.

If we let $m^2 \to -m^2$ (the double-well potential), the only change in the above is that

$$M^2 = -m^2 + \beta\lambda/M. \tag{3.7'}$$

The series for $E(\lambda, M)$ is now known (by expanding [Eqn. (3.12)]), so we may calculate

$$E_{\text{DWP}}(\lambda, m^2) = E(\lambda, M). \tag{3.13}$$

We at no point need to explicitly expand the path integral about its maximum. We simply use the now known series for $E(\lambda, M)$. The effective mass M is, however, no longer bounded below by m, and the effective coupling λ_E goes from $1/\beta$ (at $\lambda = \infty$) to ∞ (at $\lambda = 0$). In the next section we use the freedom of choice of the parameter β to find excellent low-order estimates of the AO energy levels.

There are many other possible transformations we have not explored. We could add and subtract terms of the form $\alpha\Lambda^2/m^3q^4$ or $\beta\Lambda^2/M^4q^2$. We could do two trans-

formations at once to find $\Lambda = \lambda + \alpha\Lambda^2/M^3$ and $M^2 = m^2 + \beta\Lambda/M$. We do not know if the resulting series may be summed.

Formal transformations of the path integral to simplify practical calculations (though not to transform the series) have been introduced previously. The form closest to ours is used by Nickel.[19] He defines the transformation

$$m^2 = M^2 - \Sigma(0, M) \tag{3.14}$$

where $\Sigma(0, M)$ is the self-energy part of the single-particle propagator evaluated at mass M. The lowest-order approximation is then Wick-ordering, but M is transformed further in higher orders. Since $\Sigma(0, M)$ is an asymptotic series itself, this equivalence produces a series for the physical mass (location of the single-particle pole) which is asymptotic and must be summed. The method is a useful calculational device and may hint at how to extend our series transformation technique.

IV. Low-Order Calculations

We have shown that we may introduce a parameter β into the perturbation series on which the energies and wave-functions of the summed theory do not depend. We can hope that, for particular choices of β, the *PS* may diverge less rapidly or the effective coupling may be smaller. We could then find the energy levels more accurately. We found in Section II that for $\beta = 6$ (Wick-ordering) the AO series may be resummed to give accurate results for all couplings (Table I).

The choice of the variable β is also crucial to the choice of series summation method that we use. If the series has not yet reached its asymptotic behavior (in particular, the oscillation of sign), then the summation methods will not work. Numerically, we must subtract off the terms in the series that are not properly asymptotic and apply summation methods to those that are left. A large value of β postpones the oscillation in sign. A value of β around the order of perturbation theory studied is adequate to ensure that none of the usual summation methods work. We shall show that it is possible to choose β such that summation methods are not necessary. We will obtain a series in an effective coupling, which converges to the AO energy levels for arbitrary coupling.

We illustrate how we choose β by studying the first two orders of perturbation theory. We will find both accurate approximations to the AO energy levels and a method for fixing β that is valid to all orders of *PT* which we we have studied. We have chosen for simplicity to study only constant β, which depends on the order of perturbation theory and the energy level considered but not on the coupling. Because the energy is independent of β, it is natural to require that the derivative of $E_n^K(\lambda, \beta)$ with respect to β be zero. Here K is the energy level studied and n is the order of perturbation theory. This β could depend on λ, the coupling. Because the energy estimate is already expected to be accurate for small couplings, we have chosen to require that

$$\frac{d\tilde{E}_n^K(\beta)}{d\beta} = 0 \tag{4.1a}$$

where

$$\lim_{\lambda\to\infty} E_n^K(\lambda, \beta)/\lambda^{1/3} = \tilde{E}_n^K(\beta). \tag{4.1b}$$

For the first order of perturbation theory, we have

$$E_1^K(\lambda) = (E_0^K + \lambda E_1^K). \tag{4.2}$$

E_0^K and E_1^K are listed in Appendix A, and we have set the mass $m = 1$. The shifted series is defined by Eqn. (2.10c) with 6 replaced by β. To this order,

$$E_1(\lambda, \beta) = M\left(E_0 + \frac{\lambda}{M^3}\left(E_1 - \frac{1}{2}E_0\beta\right)\right) \tag{4.3}$$

where M is determined by Eqn. (2.11). In the limit $\lambda \to \infty$ the effective mass M is simply $M = \beta^{1/3}\lambda^{1/3}$. The scaled energy, Eqn. (4.1b), is then

$$\tilde{E}_1(\beta) = \beta^{1/3}E_0/2 + \beta^{-2/3}E_1\,. \tag{4.4}$$

Equation (4.1a) fixes β:

$$\beta = 4E_1/E_0 = 6(2K^2 + 2K + 1)/(2K + 1). \tag{4.5}$$

Note that for the ground state ($K = 0$), we find $\beta = 6$. For the ground state, to leading order in *PT*, Wick-ordering is the best we can do. For $K \neq 0$ we can do better. The energy is to this order

$$E_1(\lambda) = M\left(K + \frac{1}{2} - \frac{3}{4}\frac{\lambda}{M^3}(2K^2 + 2K + 1)\right) \tag{4.6}$$

where M is determined from Eqn. (2.11) with β from Eqn. (4.5). This simple formula fits the energy levels of the AO to 2% for the ground state, $1\frac{1}{2}$% for $K = 1$, and 1% for $K > 1$ for all couplings. As we show below, the agreement may be systematically improved using *PT*.

For two orders of perturbation theory the analogue of Eqn. (4.4) is

$$\tilde{E}_2(\beta) = \beta^{1/3}(\tfrac{3}{8}E_0 + 2E_1/\beta + E_2/\beta^2). \tag{4.7}$$

This has an extremum for

$$\beta = (8E_0^2 + 2 \pm (22/3E_0^4 - 143/6E_0^2 + 4)^{1/2})/E_0 \tag{4.8}$$

where $E_0 = K + \frac{1}{2}$. For the ground state and first excited state, β is imaginary and it is natural to take the real part,

$$\beta = (8E_0^2 + 2)/E_0 , K = 0, 1. \tag{4.9a}$$

For $K \geqslant 2$ we must choose among the roots of Eqn. (4.8). We have found that the best choice of β is the (real part of the) root at which the second derivate of $\tilde{E}(\beta)$ is of the same sign as the last term of the series, in this case negative because E_2 is negative. When there are many roots (in nth order PT there are n roots), we choose the one where the second derivative of $\tilde{E}(\beta)$ is smallest in magnitude. We therefore choose the minus sign in Eqn. (4.8),

$$\beta = (8E_0^2 + 2 - (22/3E_0^4 - 143/6E_0^2 + 4)^{1/2})/E_0 , K \geqslant 2. \tag{4.9b}$$

The conditions on the derivatives of $\tilde{E}(\beta)$ reflect the β-independence of the energy. With this β the series (3.9) through second order gives an analytic approximation to the energy levels of the AO which is accurate to .005 for all energy levels (and to .0001 for $K = 0$, .001 for $K > 6$) for all couplings.

The accuracy of the shifted series (3.9) is strongly affected by the exact choice of β. To two orders, the above criteria fix β to within .5 % of its best value. It is important to at least that accuracy in higher orders.

The effective oscillator method of Hioe *et al.* [20], used to study the AO for large K and small λ, has a similar starting point to ours. If we were to exactly cancel the first-order term in perturbation theory ($\beta = 2E_1/E_0 \simeq 3(K + \frac{1}{2})$ in the large-K limit, for the AO), we would reproduce the "renormalized" oscillator and the PS of their Section V. The PS in our approach is asymptotic and may, of course, be summed. They write their series in terms of what we would call $\beta\lambda_E$. They observe that the [0, 1] Padé approximant for the energy minus the equivalent oscillator energy is closer to the known value than the [1, 0] approximant. While the diagonal Padé approximants to the series will, in fact, converge to the correct answer, the series for the inverse (which they use) is not convergent.

V. General Properties of the Shifted Perturbation Series

We first wish to study the Borel transform of the new perturbation series. The location of the singularities of the Borel transform will tell us the leading asymptotic behavior of the series and will also tell us which summation methods will work best [7]. We follow 't Hooft's analysis [10] of how to find the singularities of the Borel transform from the path-integral formulation of the quantum theory. The Borel transform $F(z)$ of a Green function $G(\lambda)$ is defined by

$$G(\lambda) = \int_0^\infty F(z)\, e^{-z/\lambda}\, dz \tag{5.1}$$

Our field theory is in one dimension, so the Green function (or more generally the vacuum-to-vacuum amplitude in the presence of an external field J) may be written in the Euclidean form [18]

$$G(\lambda) = \int [\mathscr{D}\phi] \exp\{-A\} \tag{5.2}$$

where the action

$$A = \int dt \, (\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi^2(1 - \beta\lambda) + \lambda\phi^4) \tag{5.3}$$

We have taken the mass M in the Lagrangian (3.8) equal to 1. We rescale the fields by $\phi \to \bar{\phi}/(\lambda)^{1/2}$ and find a new action

$$\begin{aligned} A &= -\frac{\beta}{2}\int dt \, \bar{\phi}^2 + \bar{A}/\lambda \\ \bar{A} &= \int dt \left(\frac{\dot{\bar{\phi}}^2}{2} + \frac{1}{2}\bar{\phi}^2 + \bar{\phi}^4\right) \end{aligned} \tag{5.4}$$

Equation (5.2) in terms of $\bar{A}$ is, dropping normalization factors,

$$G(\lambda) = \int [\mathscr{D}\bar{\phi}] \exp\left\{\frac{\beta}{2}\int dt \, \bar{\phi}^2\right\} \exp\{-\bar{A}/\lambda\} \tag{5.5}$$

The simple form of the β-dependence in (5.5), and the consequent simple form of the change in the location of the Borel transform singularities, is a result of only doing a mass-shift in the effective Lagrangian. Comparing (5.5) to the definition of the Borel transform (5.1) we see that we may take

$$F(z) = \int [\mathscr{D}\bar{\phi}] \exp\left\{\frac{\beta}{2}\int dt \, \bar{\phi}^2\right\} \delta(z - \bar{A}[\bar{\phi}]) \tag{5.6}$$

We can determine $F(z)$ by finding all solutions of $\bar{A}[\bar{\phi}] = z$, which we call $\bar{\phi}_i(z)$. The integral is then evaluated to be

$$\sum_i \left(\frac{\delta\bar{A}(\bar{\phi})}{\delta\bar{\phi}}\right)^{-1}_{\bar{\phi}=\bar{\phi}_i(z)} \exp\left\{\frac{\beta}{2}\int dt \, \bar{\phi}_i^2\right\} \tag{5.7}$$

The singularities in the z-plane are therefore at solutions of $\delta\bar{A}(\bar{\phi})/\delta\bar{\phi} = 0$, the classical field equation of the system. At such a solution $\bar{z} = \bar{A}(\bar{\phi})$ and there is a square root branch point at $z = \bar{z}$ [10]. These are just the usual singularities of the anharmonic oscillator. The exponential terms can have additional singularities only if $\exp(\beta/2 \int dt \, \bar{\phi}^2)$ has singularities in z at solutions of $z = \bar{A}(\bar{\phi})$. These cannot be simply the shrinking of the integration contour to a point, and appear unlikely.

To find the location of the leading singularity of the Borel transform we define ϕ_c as a solution to the classical field equation

$$\ddot{\phi}_c = \phi_c + 4\phi_c^3 \tag{5.8}$$

This has the solution

$$\phi_c = i/\cosh(t)\,(2)^{1/2} \tag{5.9}$$

Note that this solution is an instanton — it is clearly localized in the Euclidean time variable t. The singularity $\bar{z}$ is at

$$\begin{aligned}\bar{z} = \bar{A}(\phi_c) &= \int_{-\infty}^{\infty} dt \left(\frac{\dot{\phi}_c^2}{2} + \frac{\phi_c^2}{2} + \phi_c^4\right) \\ &= -1/3.\end{aligned} \tag{5.10}$$

The leading contribution to the singularity of the Borel transform is therefore the usual result [7] multiplied by Eq. (5.7):

$$\exp\left\{\frac{\beta}{2}\int_{-\infty}^{\infty} dt\ \phi_c^2(t)\right\} = \exp\{-\beta/2\}. \tag{5.11}$$

The leading large-order behavior of perturbation theory is determined by this singularity. For the AO the large-order behavior of the shifted theory is in leading order the same as that of the unshifted theory multiplied by exp $(-\beta/2)$. This result is obtained in a more direct way by Bender and Wu [4]. From their method we easily find that the X^{2M} oscillator, $M > 2$, has a large-order behavior unchanged by the shifting of the mass. The simplest derivation of the leading large-order behavior is found by simply treating the mass-shift term as a (renormalization) counterterm [6].

We wish to comment on the analyticity properties of the energy levels $E(\lambda)$ of the DWP. We have shown in Section III that the energy of a double-well oscillator $E(\lambda, -m^2)$ may be exactly calculated as the energy of the effective Hamiltonian

$$\mathscr{H} = \frac{1}{2}\,\dot{q}^2 + \frac{1}{2}\,M^2q^2 + \lambda q^4 - \frac{\beta\lambda}{2M}\,q^2 \tag{5.12}$$

where $M^2 = -m^2 + \beta\lambda/M$ and β is an arbitrary real number. The ability to rewrite the series in this way, in terms of the effective coupling $\lambda_E = \lambda/M^3$, leads us to discuss the analyticity properties of the energy expressed in terms of λ_E. As we show below, these appear to be simple, and can help us understand the more complicated analytic structure when expressed as a function of λ. From the analysis of this section, supported by the numerical studies of Section VI, we believe that the DWP has a Borel-summable asymptotic series in λ_E. This suggests that the energy $E(\lambda_E, M)$ is analytic in λ_E at least in the range $-\pi/2 - \epsilon < \arg \lambda_E < \pi/2 + \epsilon$ (and probably in the cut λ_E plane).

Some results on the analyticity of the DWP energy levels are easily obtained using (5.12). All the results on regular perturbations are true (Simon [8]). The energy $\tilde{E} = E + \beta\lambda^2/16m^2$ is Herglotz, and thus has no poles or essential singularities [8]. If we assume that $E(\lambda_E)$ is in fact analytic in the cut λ_E plane [21], the cut maps by Eq. (2.5″) into the negative Imλ axis from Im $\lambda = 0$ to $-2/\beta$ and back to zero. There is also a pole contribution $-1/16\lambda$ that shifts the zero of energy to the minimum of the well. This is consistent with the known analyticity structure of the DWP. Crutchfield [13] finds a region of analyticity consistent with a circle in the λ plane centered on the real axis and tangent to the imaginary axis (thus not including the Im λ axis).

VI. High-Order Calculations (AO and DWP)

We have extended the calculations discussed in Section IV to 20 orders of perturbation theory. This was both to study the convergence of the mass-shifted series and to study the choice of the shift term β. Our main object was to study the convergence in the DWP regime—to demonstrate both that the method worked and that it could be

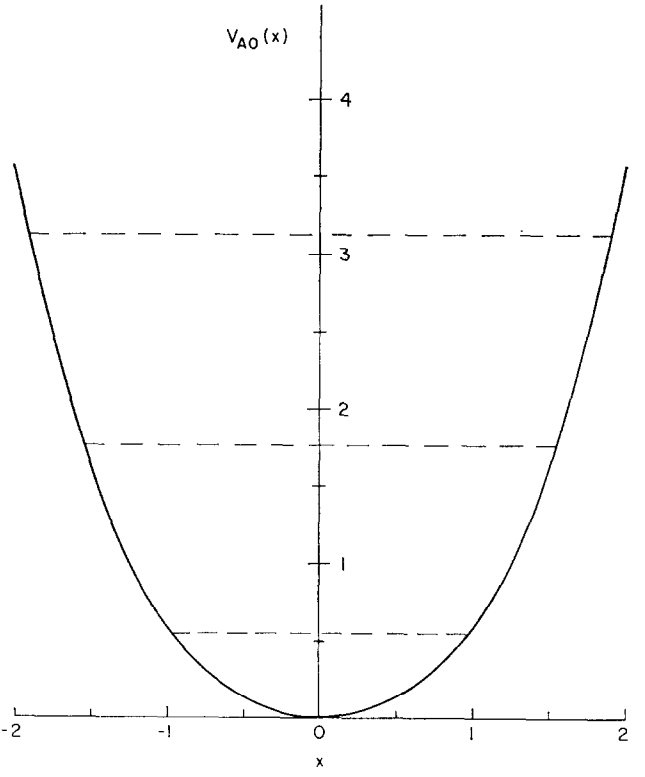

FIG. 2. Anharmonic oscillator for mass $m^2 = 1$ and coupling $\lambda = 0.1$. The dashed lines are the energy eigenvalues for the three lowest energy levels.

TABLE II

Energy Levels of the Anharmonic Oscillator, Using 20 Terms of the Perturbation Series[a]

λ	E_0	E_1	E_2	E_3
.01	.507256204524602	1.53564827829680	2.59084579619070	3.67109494222579
.1	.5591463271835 (3)	1.769502643949 (1)	3.138624308493 (3)	4.628882808 (15)
1.0	.8037706513 (7)	2.737892268 (1)	5.1792916813 (12)	7.942404 (2)
10.	1.504972409 (7)	5.321608256 (6)	10.347055532 (8)	16.09015 (1)
100.	3.13138417 (2)	11.18725425 (3)	21.90689797 (2)	34.18252 (3)
$E_\infty/\lambda^{1/3}$	.667986260 (5)	2.393644016 (6)	4.696795344 (8)	7.335730 (6)

λ	E_4	E_5	E_6	E_7
.01	4.77491311865551	5.90102667411262	7.04832687908357	8.21583781042628
.1	6.220300900011 (5)	7.89976722786 (1)	9.65783999216 (6)	11.48731557946 (9)
1.0	10.963583096 (2)	14.203139102 (2)	17.634049115 (8)	21.236435489 (8)
10.	22.40875130 (1)	29.21148484 (1)	36.43690897 (6)	44.04010131 (6)
100.	47.70720593 (3)	62.28123793 (3)	77.7707706 (2)	94.07804870 (9)
$E_\infty/\lambda^{1/3}$	10.24430846 (1)	13.379336542 (6)	16.71188963 (4)	20.22084947 (3)

λ	E_8	E_9	E_{10}
.01	9.40269230644141	10.6081135651095	11.8314007940998
.1	13.38247480747 (6)	15.3386420395 (1)	17.3519076357 (2)
1.0	24.994936412 (8)	28.89725112 (1)	32.93326304 (2)
10.	51.98651104 (4)	60.24857559 (6)	68.80369517 (6)
100.	111.12795974 (8)	128.8606293 (1)	147.2269942 (2)
$E_\infty/\lambda^{1/3}$	23.88999364 (2)	27.70639345 (6)	31.65945647 (4)

[a] The numbers in parentheses are estimates of the accuracy of the final digit obtained solely from the apparent rate of convergence of the series (consistency of the predictions using different numbers of terms in the series). The series were summed using the methods of Section VI.

extended (in theory at least) to arbitrarily small DWP coupling. We believe that the only limit on our accuracy here was the intrinsic accuracy of double-precision (14-place) arithmetic on the computer. There are large cancellations in the asymptotic series, and this emphasizes the numerical errors in the calculation of β. The choice of 20 orders (which may be optimistic) was motivated by more extensive studies of the ground-state energy approximations and by the similar limit found by Simon [8].

TABLE III

Coefficients of the Shifted Perturbation Series (to Order 20) for the Ground-State Energy Level of the Anharmonic Oscillator[a]

0.5000000000000000	D	00
−0.6523534322394895	D	01
−0.3370869857187187	D	02
−0.3048686635875011	D	03
−0.3064233001381720	D	04
−0.3072648903641905	D	05
−0.2904081280474071	D	06
−0.2428494513666112	D	07
−0.1548612132433587	D	08
−0.2216637357312011	D	08
0.1508138983335937	D	10
0.3475838196620312	D	11
0.5371021199564374	D	12
0.6735744906815999	D	13
0.6997971786427200	D	14
0.5562799670927359	D	15
0.1928995594960896	D	16
−0.3969867511129701	D	17
−0.1230034221431521	D	19
−0.1874198474433704	D	20
−0.4051846902688923	D	21

[a] The mass M is determined by Eqn. (3.7) with the parameter $\beta = 29.09413728957958$. The resulting polynomial fits the ground-state energy to at least one part in 10^8 for all couplings.

We record the energies of the AO, for a representative set of couplings and levels, in Table II. We show the lower energy levels for the AO with $\lambda = 0.1$ in Figure II. There are several accurate compilations of non-perturbative numerical results on the AO energy levels to compare to. The most extensive are tables to 9-figure accuracy by

TABLE IV

Energy Levels of the Double-Well Potential, Using 20 Terms of the Perturbation Series[a]

λ	E_0	E_1	E_2	E_3
.1	.4709 (1)	.76776 (3)	1.63485 (3)	2.574 (1)
1.0	.57728042 (4)	2.08305212 (5)	4.25357128 (4)	6.76802 (2)
10.	1.37781685 (2)	4.99566652 (3)	9.89474235 (2)	15.522245 (2)
100.	3.07010341 (2)	11.03370601 (4)	21.69467967 (4)	33.91675 (3)

λ	E_4	E_5	E_6	E_7
.1	3.683582 (3)	4.913652 (3)	6.24893 (1)	7.676658 (4)
1.0	9.56409026 (3)	12.59462094 (3)	15.8288358 (1)	19.24438069 (8)
10.	21.73654008 (2)	28.44223644 (2)	35.5763503 (1)	43.0928089 (1)
100.	47.39292007 (2)	61.92191104 (2)	77.36906099 (20)	93.6360806 (2)

λ	E_8	E_9	E_{10}
.1	9.187467 (2)	10.774008 (2)	12.430344 (2)
1.0	22.82417577 (8)	26.5546660 (1)	30.4247840 (1)
10.	50.95626302 (8)	59.13856734 (10)	67.6166804 (1)
100.	110.6474869 (1)	128.3431349 (1)	146.6737564 (2)

[a] The series were summed using the methods of Section VI.

Hioe and Montroll [15]. Less extensive, but in some cases more precise, values have been tabulated by Biswas *et al.* [22]. We agree in almost all cases with these authors to the accuracy we quote [23]. In some cases, particularly for $\lambda < 1$, our results are more precise (we are inherently perturbative). There are other methods for finding accurate estimates of the AO energy eigenvalues [24].

This agreement gives us faith that our method is correct and out error estimates are reasonable. Much more extensive tables could easily have been generated, since once the polynomial is generated further values are found simply by a polynomial evaluation. A reasonable estimate of the uncertainty in the quoted numbers would be several times the quoted error (which was found by estimating the rate of convergence of the series as more terms were taken). The series will sometimes appear to converge to a value for several orders of perturbation theory and will then jump to a new value, nearby but clearly closer to the known value.

The effective mass parameter β was chosen as described in Section IV. We can

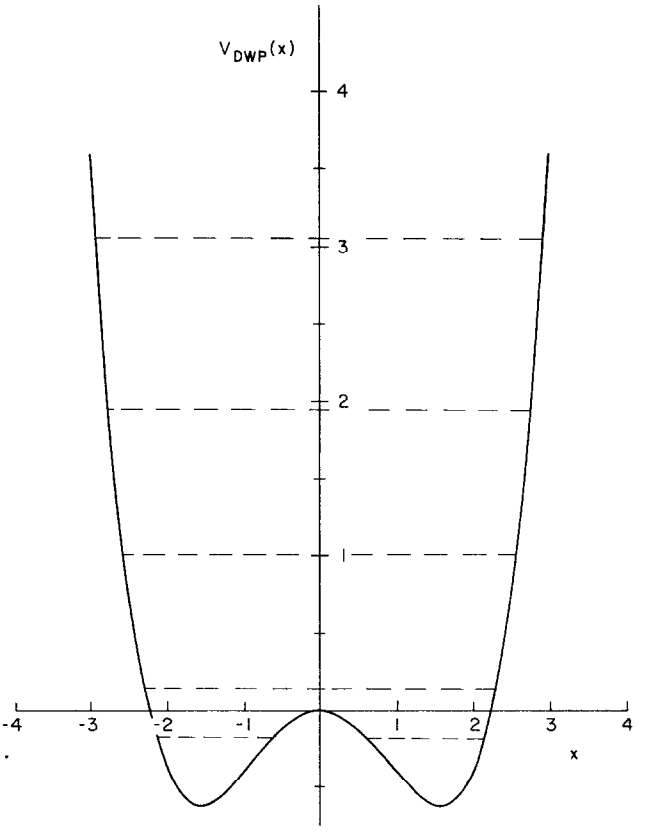

FIG. 3. Double-well potential for mass $m^2 = 1$ (anharmonic oscillator with mass $m^2 = -1$) and coupling $\lambda = 0.1$. The dashed lines are the energy eigenvalues of the five lowest levels, calculated using the perturbation series (Table IV).

estimate its value by simply requiring that the lowest-order term, times β^n, be of the same size as the nth order term. This estimate is correct within a factor of three and shows that the effect of β is to at least partly cancel out the higher-order terms. A sample polynomial which this method generates is the one for the ground state of the AO in 20th order, which we present in Table III.

TABLE V

Comparison of the Series Prediction of the Energy Levels of the Double-Well Potential, Using 20 Terms of the Perturbation Series, to a Variational Calculation [25][a]

λ	E_0	var	E_1	var
.0272		.67616		.67790
.0391		.6517	.61 (15)	.6703
.0494	.55 (6)	.6192	.67 (1)	.6730
.0988	.473	.473	.765	.773
.198	.398	.398	1.01	1.01
λ	E_2	var	E_3	var
.0272		1.83		1.92
.0391	.33 (3)	1.60	1.82 (6)	1.93
.0494	1.39	1.51	2.01	2.02
.0988	1.63	1.63	2.56	2.56
.198	2.15	2.15	3.46	3.46
λ	E_4	var	E_5	var
.0272	3.7 (2)	2.63	2.9 (1)	3.12
.0391	2.65	2.63	3.36	3.36
.0494	2.79	2.79	3.65	3.66
.0988	3.66	3.66	4.89	4.90
.198	4.97	4.96	6.61	6.62
λ	E_6	var	E_7	var
.0272	3.81 (5)	3.82	4.54 (2)	4.55
.0391	4.21	4.21	5.12	5.13
.0494	4.61	4.60	5.64	5.63
.0988	6.22	6.20	7.64	7.63
.198	8.39	8.42	10.3	10.3
λ	E_8	var		
.0272	5.33	5.44		
.0391	6.10	6.11		
.0494	6.74	6.76		
.0988	9.14	9.13		
.198	12.3	12.7		

[a] If there is no entry in the table, the series showed no signs of convergence at that coupling.

The same methods we used for the AO were applied to the $m^2 \to -1$ case, the DWP. Here the effective coupling goes to infinity as the coupling goes to zero, so that poor convergence is found in that region. In Table IV we record the energy levels we found, with their estimated errors, for a range of couplings of the DWP. The energies are measured with respect to the bottom of the well. The lower energy levels are shown for $\lambda = 0.1$ in Figure III. To convince the reader that our series converges to the correct DWP energy eigenvalues, we have also compared our results to variational calculations of the energies [25]. The comparison is made in Table V. Their choice of couplings fortunately covers the region in which good convergence is obtained (λ large) to the small λ region where the summed series does not converge well. The agreement is excellent. There is no perturbative estimate of these energies from ordinary perturbation theory, except for very small coupling. The usual PS about a minimum of the potential is asymptotic but does not change in sign (and thus is not summable) [12].

We have found that more orders in perturbation theory extend (albeit only very slowly) the region of coupling in which a given accuracy is found. There appears (numerically) to be no theoretical block to obtaining an arbitrary accuracy using many higher orders of PT. The situation is comparable to that for the usual PS for the AO with large coupling. We believe that our series summation could, in general, be limited by a phase transition at some coupling. There is no phase transition possible in one dimension, and so we should be able to sum the series for arbitrarily small couplings.

VII. Sextic Oscillator

To illustrate these methods further, we also treat the sextic oscillator (SAO), which has the potential

$$V(X) = \tfrac{1}{2}m^2x^2 + \lambda x^6. \tag{7.1}$$

The sextic oscillator perturbation series diverges much more rapidly than the x^4 series does. The nth term diverges for large n approximately as $(-)^n\,(2n)!$ [26]. We will be able to see how the convergence of the series is affected by this more rapid growth. The procedure found for choosing β, in Section IV, is adequate here also. The Wick-ordering method of Section II is not directly applicable here—the x^6 term generates an x^4 counterterm that does not appear in the unshifted potential.

The equation for the effective mass is now

$$M^2 = m^2 + \beta\lambda/M^2. \tag{7.2}$$

This is a simple quadratic in M^2 and has the solution (choosing the one with the correct $\lambda = 0$ limit)

$$M = (m^2 + (m^4 + 4\beta\lambda)^{1/2})^{1/2}/(2)^{1/2}. \tag{7.3}$$

The solution for the double-well case (SDWP) is simply obtained by letting $m^2 \to -m^2$. The equivalent shifted series is generated by the analogue of Eqn. (3.12), remembering that $\lambda_E = \lambda/M^4$:

$$E^\beta(\lambda_E) = (1 - \beta\lambda_E)^{1/2}\, E(\lambda_E/(1 - \beta\lambda_E)^2). \tag{7.4}$$

The leading term in the perturbation series for the energy level K is simply calculated (Appendix A) to be

$$\begin{aligned} E_0 &= K + \frac{1}{2}, \\ E_1 &= \frac{5}{2} E_0^3 + \frac{25}{8} E_0 . \end{aligned} \tag{7.5}$$

Following the same procedures as in Section IV, we find the extremum of the energy as $\lambda \to \infty$,

$$\tilde{E}_1^K(\beta) = \frac{E_0}{2} \beta^{1/4} + \left(\frac{5}{2} E_0^3 + \frac{25}{8} E_0\right) \beta^{-3/4} \tag{7.6}$$

occurs for $\beta = 15(E_0^2 + \frac{5}{4})$. With M defined by Eqn. (7.3) and this β, the energy of the sextic oscillator is

$$E(\lambda, m) = M(E_0 - 2\lambda E_1/M^4) \tag{7.7}$$

This formula is accurate to 7 % for the ground state, 5 % for $K = 1$, and at least 3 % for all other states. The next order correction gives at least 2 % accuracy for all energy levels (though for $K = 2$ and 3 the agreement is slightly worse than the first order correction—this is not true in higher orders).

We report in Table VI the results for the SAO for 15 orders of perturbation theory. The values reported are all consistent with other tabulations of the energy levels. [20] However, the error estimates for the ground state are about a factor of ten smaller than the difference from the known eigenvalues. The internal consistency error estimate can easily be misleading. Such false convergence also appears in lower orders of the AO energy calculations. There it disappeared in one large jump (compared to the estimated error) and converged to the correct value in higher orders. This points out a need to find other ways to estimate the approximate (or maximum) error that is being made. Such problems are also found using any other summation method [27].

The SDWP energy levels are found by simply letting $m^2 \to -m^2$ in the formulas of this section and shifting the energy so that the zero of the energy is at the bottom of the well:

$$E(\lambda, m) = E(\lambda, -m^2) + 1/3/(6\lambda)^{1/2}. \tag{7.8}$$

The results for selected couplings are shown in Table VII. These values also exhibit the (low order) false convergence noted above. Quoted errors, particularly for the ground-state energy, are too small.

TABLE VI

Energy Levels for the Sextic Oscillator, Using 15 Terms in the Perturbation Series

λ	E_0		E_1		E_2		E_3	
.01	.51544283	(5)	1.59544246	(9)	2.793818	(2)	4.131694	(1)
.1	.586926	(3)	1.950450	(6)	3.6908	(1)	5.77377	(3)
1.0	.80482	(2)	2.87488	(3)	5.7719	(5)	9.3250	(2)
10.	1.28190	(8)	4.75658	(9)	9.8069	(9)	16.0432	(6)
100.	2.1925	(1)	8.2542	(2)	17.180	(3)	28.2245	(8)
$E_\infty/\lambda^{1/4}$	.68041	(4)	2.58011	(8)	5.3949	(8)	8.8807	(2)

λ	E_4		E_5		E_6		E_7	
.01	5.60608	(1)	7.208735	(8)	8.931524	(2)	10.76731	(3)
.1	8.14751	(8)	10.7798	(1)	13.64647	(4)	16.7300	(6)
1.0	13.4149	(3)	17.9794	(7)	22.9701	(2)	28.354	(2)
10.	23.2437	(5)	31.296	(2)	40.1112	(4)	49.630	(4)
100.	40.9884	(8)	55.272	(3)	70.9140	(7)	87.81	(1)
$E_\infty/\lambda^{1/4}$	12.9111	(1)	17.4226	(10)	22.3642	(3)	27.703	(3)

λ	E_8		E_9		E_{10}	
.01	12.71004	(1)	14.75426	(3)	16.89534	(6)
.1	20.0176	(3)	23.4952	(5)	27.1533	(8)
1.0	34.110	(1)	40.208	(2)	46.630	(2)
10.	59.816	(2)	70.611	(4)	81.986	(5)
100.	105.896	(4)	125.065	(8)	145.266	(9)
$E_\infty/\lambda^{1/4}$	33.418	(1)	39.476	(3)	45.860	(2)

TABLE VII

Energy Levels for the Sextic Oscillator with Negative Mass-Squared (SDWP), Using 15 Terms in the Perturbation Series

λ	E_0		E_1		E_2		E_3	
.005	.26	(4)	1.13	(3)	2.6	(2)	2.40	(5)
.007	.63	(1)	.99	(1)	2.12	(7)	2.54	(2)
.01	.733	(4)	.950	(5)	1.95	(3)	2.69	(1)
.02	.6895	(9)	.970	(2)	1.974	(9)	3.075	(2)
.05	.5836	(3)	1.1140	(3)	2.339	(3)	3.8425	(10)
.1	.54340	(8)	1.3006	(2)	2.775	(2)	4.6171	(9)
1.0	.68088	(8)	2.4070	(1)	5.148	(1)	8.5676	(6)
10.	1.17906	(8)	4.4602	(1)	9.422	(1)	15.5837	(6)
100.	2.1241	(1)	8.0770	(2)	16.953	(2)	27.9555	(10)

λ	E_4		E_5		E_6		E_7	
.005	3.41	(4)	4.43	(5)	5.466	(4)	6.69	(6)
.007	3.555	(9)	4.69	(3)	5.912	(8)	7.28	(3)
.01	3.792	(3)	5.06	(2)	6.451	(5)	7.99	(3)
.02	4.436	(2)	6.000	(6)	7.722	(2)	9.605	(9)
.05	5.6245	(8)	7.650	(2)	9.881	(1)	12.309	(6)
.1	6.7791	(5)	9.224	(2)	11.9155	(6)	14.836	(5)
1.0	12.5389	(4)	16.998	(2)	21.8906	(3)	27.183	(4)
10.	22.7175	(4)	30.711	(2)	39.4707	(8)	48.938	(5)
100.	40.6819	(6)	54.932	(3)	70.5431	(9)	87.41	(1)

λ	E_8		E_9		E_{10}	
.005	8.17	(4)	9.62	(5)	11.14	(2)
.007	8.85	(2)	10.47	(3)	12.16	(3)
.01	9.699	(9)	11.49	(2)	13.379	(8)
.02	11.659	(5)	13.833	(8)	16.13	(1)
.05	14.930	(3)	17.712	(6)	20.649	(5)
.1	17.980	(2)	21.317	(3)	24.839	(4)
1.0	32.858	(2)	38.877	(3)	45.223	(3)
10.	59.078	(2)	69.829	(5)	81.161	(4)
100.	105.470	(3)	124.61	(1)	144.791	(8)

VIII. Conclusions

We believe that some variant of these methods should be applicable to field theories in higher dimensions than one. All the PT calculations in this paper could have been done using Feynman diagrams and bound-state equations. The complications of renormalization also allow great freedom in the choice of renormalization scheme employed. For particular choices of renormalization scale this has already been exploited to obtain more accurate perturbative results. This has been used in QCD [28], where one is led to choose a scale on the size of the relevant momenta in the problem considered. In our problem we chose a different β for different energy levles. There is much more freedom in the choice of finite renormalization for masses and couplings, that is similar to the freedom exploited here. The problem of the best renormalization scheme for a given problem has not been comprehensively studied.

It is surprising that we can find accurate values for the $\lambda \to \infty$ limit (which is just the zero-mass field theory) by introducing an effective mass into the theory. The calculation of numbers which are both renormalization independent and characteristic of the zero-mass theory—eg. critical indices in ϕ^{2M} theories in two and three dimensions—may also be aided by using variants of the methods of this paper. This is particularly so if some of the series have large non-asymptotic parts in low orders of PT that can be removed by a proper choice of effective mass.

The use of transformations of the series, such as we have discussed in this paper, is very similar in spirit to the usual resummation methods. These methods (Padé, Borel) use specific analyticity information of the quantity approximated to improve the convergence of the series. We use formal transformations of the defining path integral to improve the convergence. The study of the best way to use perturbative results is clearly an important part of any practical calculation. We believe our method is one very effective way to use perturbative information.

APPENDIX A

The calculation of the terms in the perturbation series for the energy of any level of an arbitrary anharmonic oscillator may be reduced to the straight-forward iteration of an algebraic recursion relation [3]. The simplest form of these relations, which also gives us the coefficients for the series of $\langle X^{2N} \rangle$, has been derived by Swenson and Danforth [29]. This method is both very efficient and requires only a trivial change (the unperturbed energy) to treat higher quantum states. We refer the interested reader to their paper for a simple derivation of the recursion relation. We illustrate its use here.

For the potential

$$V(X) = \tfrac{1}{2}X^2 + \lambda X^{2M} \tag{A-1}$$

we define the expectation of an operator A in a quantum state K by $\langle A \rangle_K$. The expec-

tation value of X^{2N} has the perturbation series (we drop the index K in all further formulas)

$$\langle X^{2N} \rangle = \sum_{i=0}^{\infty} Q_i{}^N \lambda^i. \tag{A-2}$$

The eigenfunctions of $V(X)$ are eigenstates of parity, so $\langle X^N \rangle$ vanishes for N odd. Normalization of the exact wave function is $\langle 1 \rangle = 1$. This implies

$$Q_i{}^0 = \delta_{i0} \tag{A-3}$$

The Hellman-Feynman theorem implies

$$\frac{\partial E}{\partial \lambda} = \langle X^{2M} \rangle. \tag{A-4}$$

This relates the coefficients of the perturbation series for the energy $E = \sum_{i=0}^{\infty} \lambda^i E_i$ to those for $\langle X^{2M} \rangle$,

$$E_i = Q_{i-1}^M / i. \tag{A-5}$$

The desired recursion relation is

$$Q_L^{N+1} = \frac{2N+1}{N+1} \sum_{i+j=L} E_i Q_j{}^N + \frac{(2N+1)\,N(2N-1)}{4(N+1)} Q_L^{N-1} - \frac{(2N+1+M)}{N+1} Q_{L-1}^{N+M} \tag{A-6}$$

where $Q_{-1}^N = 0$. If we lay out a table for $Q_L{}^N$ with N increasing to the right and L increasing to the top, the left-hand column is given by Eqn. (A-3). For $L = 0$ we can solve for $Q_0{}^N$ for arbitrary N, in terms of the unperturbed energy E_0, by starting at $N = 1$ and moving to the right. Since the recursion relation depends only on terms in the line below (and at most $M - 1$ to the right) or on terms to the left but not above (not larger L), we can do the same for $L = 1$ as we did for $L = 0$: solve for $N = 1$ and continue to the right. Repeating this until we get to $L = N - 1$ allows us to find the perturbation series for the energy to order N. We need only a finite number of Q's on any line since the last term requires only up to $M - 1$ terms to the right. For example, for Nth order perturbation theory we will only need, for $L = 0$, $Q_0{}^1$ through Q_0^{NM-N}. All terms are polynomials in the unperturbed energy E_0.

In terms of the unperturbed energy $E_0 = K + \frac{1}{2}$, the first six terms in the perturbation series for the AO are

$$
\begin{aligned}
E_0 &= K + \frac{1}{2} \\
E_1 &= \frac{3}{2} E_0^2 + \frac{3}{8} \\
-E_2 &= \frac{17}{4} E_0^3 + \frac{67}{16} E_0 \\
E_3 &= \frac{375}{16} E_0^4 + \frac{1707}{32} E_0^2 + \frac{1539}{256} \\
-E_4 &= \frac{10689}{64} E_0^5 + \frac{89165}{128} E_0^3 + \frac{305141}{1024} E_0 \\
E_5 &= \frac{87549}{64} E_0^6 + \frac{587265}{64} E_0^4 + \frac{9317949}{1024} E_0^2 + \frac{1456569}{2048} \\
-E_6 &= \frac{3132399}{256} E_0^7 + \frac{124269873}{1024} E_0^5 + \frac{912774217}{4096} E_0^3 + \frac{1056412343}{16384} E_0 .
\end{aligned}
\tag{A-7}
$$

This sort of calculation is trivial using MACSYMA and may be easily performed to 20th order without approximation.

APPENDIX B: Calculation of the Ground-State Energy in Field Theory

The calculation of the perturbation series used in this paper could have been done using the usual Feynman diagram methods of field theory [3] rather than the quantum mechanical methods of Appendix A. The poles of the Green functions are the excited states of the theory and are located at $E_n - E_0$, where E_0 is the ground-state energy. It is surprising that, knowing the propagator of a one-dimensional theory, we can easily reconstruct the ground-state energy. We give a simple derivation of this fact using path-integral methods [18]. The vacuum-to-vacuum amplitude for a one-dimensional quantum system is

$$
\begin{aligned}
\langle 0 \mid 0 \rangle &= \int [dq] \exp\left\{ i \int_{-T/2}^{T/2} dt\, L(q, \dot{q}) \right\} \\
&= \exp(-iE_0 T)
\end{aligned}
\tag{B-1}
$$

Here T is the difference between final and initial times, and for T large all other contributions are exponentially negligible. The generator of connected Green functions in the presence of a source J, $Z(J)$, is calculated from the following path integral (where we use (B-1) for the normalizing integral and $Z(0)$ is defined to be zero):

$$
iZ(J) = \ln \left\{ \int [dq] \exp\left\{ i \int dt\, (L(q, \dot{q}) + Jq) \right\} \right\} - iE_0 T. \tag{B-2}
$$

All time integrals are from $-T/2$ to $T/2$, with $T \to -i\infty$ at the end of the calculation.

The effective Lagrangian for a theory with potential $V(q) = \frac{1}{2}m^2q^2 + \lambda q^{2N}$ is

$$L(q, \dot{q}) = \tfrac{1}{2}\dot{q}^2 - V(q). \tag{B-3}$$

We may take the derivative of (B-2) with respect to either the mass m or the coupling λ to get (for $J = 0$):

$$\frac{1}{m}\frac{dE_0}{dm} = -\left\langle \int dt\, q^2(t) \right\rangle_{J=0} \Big/ T, \tag{B-4a}$$

$$\frac{dE_0}{d\lambda} = -\left\langle \int dt\, q^{2N}(t) \right\rangle_{J=0} \Big/ T. \tag{B-4b}$$

Defining the two-point (and n-point) functions as appropriate functional derivatives with respect to J of $Z(J)$, and integrating over the relative coordinates of the resulting Green functions, we obtain, for example,

$$\left\langle \int dt\, q^2(t) \right\rangle_{J=0} \Big/ T = -\int G(p)\frac{dp}{2\pi}, \tag{B-5a}$$

$$\left\langle \int dt\, q^4(t) \right\rangle_{J=0} \Big/ T = -\int G(p, q-p, r-q, -r)\frac{dp}{2\pi}\frac{dq}{2\pi}\frac{dr}{2\pi}. \tag{B-5b}$$

The momentum integrals are from $-\infty$ to ∞. Combining (B-4) and (B-5), we find

$$m\frac{dE_0}{dm} = \int m^2 G(p)\frac{dp}{2\pi} \tag{B-6}$$

and a similar equation for $dE_0/d\lambda$. By applying $\int dX[\delta/\delta J(X)]$ to the equation of motion

$$\int [dq]\frac{\delta}{\delta q(X)}\exp\left[i\int (\mathscr{L}(q) + Jq)\, dt\right] = 0, \tag{B-7}$$

we obtain the identity (for $J = 0$)

$$\int dX \langle q(X)(-\partial^2 - m^2)\, q(X)\rangle + iT\,\delta(0) = 2N\lambda \int dX \langle q^{2N}(X)\rangle. \tag{B-8}$$

Using (B-4b) and Fourier transforming, this becomes

$$\int \frac{i\,dp}{2\pi}(G_0^{-1}(p)\, G(p) - 1) = 2N\lambda\frac{dE_0}{d\lambda} \tag{B-9}$$

where $G_0(p) = i/(p^2 - m^2)$ is the bare propagator. Carl Bender suggested Eqn. (B-9) to us as an empirically true and useful formula for the AO ($N = 2$). Dimensional analysis on $E_0(m, \lambda)$ requires

$$m\frac{dE_0}{dm} + (N+1)\,\lambda\frac{dE_0}{d\lambda} = E_0. \tag{B-10}$$

We may combine (B-6), (B-9) and (B-10) to obtain a form for the ground-state energy as determined by the (one-particle-irreducible) single-particle propagator $G(p)$:

$$E_0 = \int \frac{dp}{2\pi} \left[m^2(Gp) + i \frac{N+1}{2N} (G_0^{-1}(p)\, G(p) - 1) \right]. \tag{B-11}$$

The energy E_0 is measured from the minimum of the potential (for $G(p) = G_0(p)$, $E_0 = m/2$). Using (1.2) and the propagator $G(p) = i/(p^2 - M^2)$, we reproduce the result for the ground-state energy found using the Wick-ordered oscillator [Eqn. (2.14)]. A potential with more terms (N_1, N_2, etc.) would require higher Green functions to find E_0 [see eg. Eqn. (B-5b)].

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