THE BOREL SUM OF THE DOUBLE-WELL PERTURBATION SERIES AND THE ZINN-JUSTIN CONJECTURE

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The Borel sum of the perturbation expansion of any eigenvalue of the double-well oscillator $p^2 + x^2 + 2gx^3 + g^2x^4$, $g > 0$, is proved to represent a complex eigenvalue of the non self-adjoint operator $e^{i\pi/3}(p^2 + e^{-2i\pi/3}x^2 + 2ge^{-5i\pi/6}x^3 + g^2e^{-i\pi}x^4)$. This result makes the Zinn-Justin conjecture equivalent to the coincidence between the $l = 0$ eigenvalues of the two-dimensional oscillator $\frac{1}{2}(|p|^2 + |x|^2) + g^2|x|^4$ and those of the complex double well $p^2 + x^2 + 2ge^{-i\pi/2}x^3 + g^2e^{-i\pi}x^4$.

It is well known that the simplest example of vacuum instability, where perturbation theory fails to recover the solution through the Borel summability, is the standard double-well anharmonic oscillator

$$H(g) = p^2 + x^2 + 2gx^3 + g^2x^4 , \quad g > 0 . \tag{1}$$

This problem has received much attention in recent times (see, e.g. the overview by Simon [1]; subsequent investigations include refs. [2,3]) also in connection with the so-called Zinn-Justin conjecture, to be recalled later.

The results of the Saclay group [4,5] on the large-n behaviour of the coefficients A_n^i of the Rayleigh–Schrödinger perturbation expansion (hereafter RSPE) of any eigenvalue $E_i(g)$ of (1) clearly indicate that its Borel sum, if any, has to be a complex quantity for g real. Therefore it cannot represent an eigenvalue of the self-adjoint operator $H(g)$ and its relation to the problem has to be clarified.

The main idea in trying to identify the Borel sum of the RSPE of (1) is first to generate through complex scaling a non self-adjoint, "single-well" problem having the same RSPE, and then to prove the stability of its eigenvalues with respect to the unperturbed ones, together with the estimates on the remainder of the RSPE in some sector of the complex g plane needed to establish analyticity and Borel summability.

We will indeed see that the operator H_1 defined as the realization in $L^2(R)$ of the differential expression (here $\alpha = \pi/3 + 2\theta/3$),

$$\begin{aligned} H_1(|g|, \theta) &= e^{i\alpha}(p^2 + e^{-2i\alpha}x^2 \\ &\quad + 2|g|\, e^{-5i\pi/6 - 2i\theta/3}x^3 + |g|^2\, e^{-i\pi}x^4) , \\ &\quad -\tfrac{5}{4}\pi < \theta = \arg g < \tfrac{1}{4}\pi , \end{aligned} \tag{2}$$

admits for each eigenvalue $\lambda_i(|g|, \theta)$, $i = 0, 1, ...,$ a Borel summable RSPE coinciding with the RSPE of the eigenvalue $E_i(g)$ of $H(g)$. The eigenvalues of $H_1(\cdot)$ are in turn related to the original problem much in the same way as resonances are in standard dilation analytic problems. For a discussion on this point see ref. [6], where the same result was obtained for any triple-well oscillator of the type

$$H_m(g) = p^2 + x^2 - 2g^{2m}x^{2m+2} + g^{4m}x^{4m+2} , \quad m > 1 , \quad m \in Z . \tag{3}$$

$H_m(g)$ reduces indeed to (1) for $m = 1/2$ (our results

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hold for any half-integer $m > 0$: we omit the details); however the present arguments are completely different because of an essential technical difficulty to be described later.

We now come to the Zinn-Justin conjecture which is based on the numerical computation of the RSPE coefficients [7] and can be stated as follows: let $\Sigma_{n=0}^{\infty} A_n^i g^{2n}$ be the RSPE of the eigenvalue $E_i(g)$ of (1), and $\Sigma_{n=0}^{\infty} B_n^i g^{2n}$ the RSPE of the eigenvalue $\mu_i(g)$ of the $l = 0$ two-dimensional anharmonic oscillator $K(g) = \frac{1}{2}(p^2 + |x|^2) + g^2|x|^4$. Then:

$$A_n^i = (-1)^n B_n^i \quad (i, n) = 0, 1, 2, \dots . \tag{4}$$

Avron and Seiler [8] proved (4) for $n = 0, 1, \dots, 9$ and all i.

Now the RSPE for $\mu_i(g)$ is Borel summable [9]. If (4) is true, it coincides with the RSPE for the eigenvalue $\lambda_i(|g|, -\pi/2)$ of $H_1(|g|, -\pi/2)$: hence by the present result the Zinn-Justin conjecture yields:

$$\lambda_i(|g|, -\pi/2) = \mu_i(g)\,, \quad i = 0, 1, \dots , \tag{5}$$

because the eigenvalues are uniquely determined by their RSPE through the Borel summability. If (4), and hence (5), holds, the limit $|g| \to \infty$ provides a particularly simple consistency check, and will be discussed at the end of the paper, where a numerical computation will also be reported supporting the validity of the Zinn-Justin conjecture in the form (5).

We now formalize the results outlined above.

Proposition 1. Let $g = |g|\,e^{i\theta}$, $-\frac{5}{4}\pi < \theta < \frac{1}{4}\pi$, $|g| > 0$, and let $\dot{H}_1(|g|, \theta)$ be the differential operator in L^2 defined as:

$$\dot{H}_1(\cdot) = e^{i\alpha}(p^2 + e^{-2i\alpha}x^2 + 2|g|\,e^{-5i\pi/6 - 2i\theta/3}\,x^3 + |g|^2\,e^{-i\pi}x^4)\,,$$

$$D(H_1(\cdot)) = C^2 \cap L^2\,. \tag{6}$$

Then $H_1(\cdot)$ is closable. Its closure $H_1(\cdot)$ has discrete spectrum.

Remarks. (i) This statements extends theorem 1 of ref. [6], but through a different argument, since the numerical range of $\dot{H}_1(|g|, \theta)$ is the whole complex plane, so that its closability has to be proved.

(ii) It will appear below that $H_1(|g|, \theta)$ is the realization in L^2 through complex scaling of the eigenvalue problem $(p^2 + x^2 + 2gx^3 + g^2x^4)\,\psi(x, g) = \lambda(g) \times \psi(x, g)$, $g \in \mathrm{R}$, looking for eigenfunctions vanishing along the direction $\arg(\pm x) = \pm\pi/6$ as $|x| \to \infty$. In this respect it resembles a single-well problem, and further insight is provided by the Herbst–Simon example [10] $\tilde{H}(g) = p^2 + x^2(gx + 1)^2 - 2gx - 1$: the function $\psi(x, g) = \exp(-x^2/2 - gx^3/3)$, $g > 0$, vanishes as $|x| \to \infty$ only if $\arg(\pm x) = \pm\pi/6$ and solves $\tilde{H}(g)\,\psi(x, g) = 0$ for all $g \geqslant 0$: hence the eigenvalue 0 of $\tilde{H}(g)$ is stable only if the L^2 conditions are imposed along the complex direction $\arg(\pm x) = \pm\pi/6$.

Proof. Set, for z complex:

$$P(x, |g|, \theta, z) = e^{-2i\alpha}x^2 + 2|g|\,e^{-5i\pi/6 - 2i\theta/3}x^3 + |g|^2\,e^{-i\pi}x^4 - e^{-i\alpha}z\,. \tag{7}$$

Note that for z fixed $P(\cdot) \neq 0$ for all θ if $z = |z| \times e^{i(\alpha+\pi)}$, $|z|$ large enough. By well-known WKB estimates (see, e.g. Sibuya [11], par. 13,19) there are two solutions $v_1(x, |g|, \theta, z)$ of the ODE $\dot{H}_1(|g|, \theta)\,v = zv$ which are entire functions of x and have the following asymptotic behaviours (here we take the positive branch of the square root)

$$v_{1,2}(x, |g|, \theta, z) \sim P(x, \cdot)^{-1/4} \exp\left[-\int_{x_0}^{x} P(y, \cdot)^{1/2}\,dy\right], \tag{8}$$

as $x \to +\infty$ for v_1, $x \to -\infty$ for v_2, respectively. These behaviours hold both for $x \pm\infty$ respectively, z fixed, and $|z| \to \infty$, $z = |z|\,e^{i(\alpha+\pi)}$, x fixed, uniformly in θ, $-\frac{5}{4}\pi + \eta \leqslant \theta \leqslant \frac{1}{4}\pi - \eta$, $\eta > 0$. $v_1(\cdot)$ and $v_2(\cdot)$ are linearly independent. To see this, rescale $x \to |g|^{1/3} \times e^{-i\pi/6}$ so that $[\dot{H}_1(\cdot) - z]\,v$ transforms into the equation

$$(p^2 + g^{-4/3}x^2 + 2g^{-2/3}x^2 + x^4)\,v = g^{-2/3}\,zv\,,$$

then apply theorem 26.4 of Sibuya yielding the nonvanishing of the wronskian of its two solutions approaching zero as $|x| \to \infty$ in the sectors $|\arg x| \leqslant \frac{1}{6}\pi$ and $|\arg x - \frac{2}{3}\pi| \leqslant \frac{1}{6}\pi$, respectively, and finally rescale back to conclude that $W(|g|, \theta, z) \neq 0$, $W(\cdot)$ the wronskian of $v_1(\cdot)$, $v_2(\cdot)$, z large enough. For z fixed the behaviours (8) can be made explicit with respect to x (ref. [11], par. 6,21,24)

$$v_{1,2}(x, |g|; \theta, z) \sim |x|^{-1/2}(e^{-i\alpha}|g|x + 1)^{-1/2} \times \exp(|g|\,e^{-i\pi/2}x^3/3 - e^{-i\alpha}x^2/2)\,, \tag{9}$$

as $x \to +\infty$ for v_1, $x \to -\infty$ for v_2, respectively. Further-more (ref. [11], par. 21) there are constants $c_{1,2}(|g|, \theta, z)$ such that:

$$v_{1,2}(x, |g|; \theta, z) \sim c_{1,2}(\cdot)|x|^{-1/2}(e^{-i\alpha}|g|x + 1)^{-1/2} \times \exp(|g|\, e^{-i\pi/2}x^3/3 + e^{-i\alpha}x^2/2)\,, \tag{10}$$

as $x \to +\infty$ for v_1, $x \to -\infty$ for v_2, respectively. Hence, if we define the integral kernel

$$\begin{aligned} G(x, y; |g|, \theta, z) &= W(|g|, \theta, z)^{-1} v_1(x, \cdot)\, v_2(y, \cdot)\,, \quad y \leqslant x\,, \\ &= W(|g|, \theta, z)^{-1} v_2(x, \cdot)\, v_1(y, \cdot)\,, \quad x \leqslant y\,, \end{aligned} \tag{11}$$

an easy computation shows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x, y; \cdot)|^2 \, dx\, dy < +\infty$$

uniformly with respect to θ. Hence $G(\cdot)$ is a Hilbert–Schmidt kernel and generates a compact operator $T(|g|, \theta, z)$ in L^2. Since the ODE $[\dot{H}_1(\cdot) - z]\, u = 0$ admits for some $z \in C$ two linearly independent solutions, one of which is L^2 at $+\infty$ and the other L^2 at $-\infty$, standard ODE arguments (see, e.g. Hellwig [12], par. V.2) show that $\dot{H}_1(\cdot) - z$ is invertible, maps $C^{(2)} \cap L^2$ into $C^{(0)} \cap L^2$ one-to-one, and $[\dot{H}_1(\cdot) - z]^{-1}$ agrees with $T(\cdot, z)$ on $C^{(0)} \cap L^2$. Since $C^{(0)} \cap L^2$ is dense in L^2, and $T(\cdot)$ is bounded, the closure of its restriction to $C^{(0)} \cap L^2$ exists and is $T(\cdot)$ itself. Now the inverse of the closure is the closure of the inverse: hence $\dot{H}_1(\cdot) - z$, and thus $\dot{H}_1(\cdot)$, are closable, and $[H_1(|g|, \theta) - z]^{-1} = T(|g|, \theta, z)$. This proves proposition 1.

Proposition 2. Let $g, \theta, H_1(|g|, \theta)$ be as above. Then:

(i) Any eigenvalue $E_i(0) = \lambda_i(0) = (2i + 1)$, $i = 0, 1, \ldots$, of the unperturbed operator $H_0(\theta) = e^{2i\alpha}(p^2 + e^{-4i\alpha}x^2)$ is table for $|g| > 0$ small, i.e. given $\epsilon(i) > 0$ there is $\delta(i) > 0$ such that $H_1(\cdot)$ has one and only one eigenvalue in any circle $|z - \lambda_i(0)| = \epsilon(i)$ if $|g| < \delta(i)$, $\epsilon(i)$, $\delta(i)$ independent of θ, $-\frac{5}{4}\pi + \eta \leqslant \theta \leqslant \frac{1}{4}\pi - \eta$, $\eta > 0$.

(ii) There is $B(i) > 0$ such that any eigenvalue $\lambda_i(|g|, \theta)$ is an analytic function of g in the sector $0 < |g| < B(i)$, $-\frac{5}{4}\pi < \theta < \frac{1}{4}\pi$.

(iii) The RSPE of any $\lambda_i(g)$ can be written as $\lambda_i(g) \sim \sum_{n=0}^{\infty} A_n^i g^{2n}$, is uniformly asymptotic to $\lambda_i(g)$ in any sector $-\frac{5}{2}\pi + \eta \leqslant 2\theta \leqslant \frac{1}{2}\pi - \eta$, and is Borel summable to $\lambda_i(g) \equiv \lambda_i(|g|, \theta)$ in any sector $-2\pi + \eta_1 \leqslant \arg(g^2) = 2\theta < -\eta_1$, $\eta_1 > 0$, $|g|^2 < B(i)^2$.

(iv) If $H(g)$ is the standard double-well operator family, self-adjoint for g real (see, e.g. Reed–Simon [13], XII.4) the RSPE of any eigenvalue $E_i(g)$ of $H(g)$ coincides with the RSPE of $\lambda_i(g)$.

Remarks. (1) If $\theta = -\pi/2$, i.e. $g = -i|g|$, the RSPE of $\lambda_i(g)$ is real by (iii). Thus by the Borel summability $\lambda_i(|g|, -\pi/2)$ is real for all i. Moreover if (4) holds the above RSPE coincides with the Borel summable series of the eigenvalue $\mu_i(g)$ of the $l = 0$ two-dimensional oscillator

$$K(g) = \tfrac{1}{2}(p^2 + |x|^2) + g^2|x|^4|_{l=0}\,, \quad g > 0\,.$$

Therefore $\lambda_i(|g|, -\pi/2) = \mu_i(|g|)$, i.e. (4) implies (5).

(2) Not only $\lambda_i(|g|, -\pi/2)$ is real, but $\lambda_i(|g|, \theta)$ is reflection symmetric about $-\pi/2$, $\lambda_i(|g|, \theta) = \lambda_i(|g|, -\pi - \theta)$. Setting indeed $\theta = \theta' - \pi/2$, we have

$$H_1(|g|, \theta) = e^{2i\theta'/3}(p^2 + e^{-4i\theta'/3}x^2 + 2|g|\, e^{i\pi/2 - 2i\theta'/3}x^3 + |g|^2 e^{-i\pi}x^4),$$

unitarily equivalent to

$$e^{2i\theta'/3}(p^2 + e^{-4i\theta'/3}x^2 + 2|g|\, e^{-i\pi/2 - 2i\theta'/3}x^3 + |g|^2 e^{-i\pi}x^4)$$

under $x \to -x$. Hence $\lambda_i(|g|, \theta' - \pi/2) = \overline{\lambda_i(|g|, -\theta' - \pi/2)}$, i.e. $\lambda_i(|g|, \theta) = \lambda_i(|g|, -\pi - \theta)$. When $\theta = 0$ or $\theta = \pi$, i.e. $\arg(g^2) = 0$ or 2π, λ_i is still determined by the RSPE through Borel summability in $-\pi + \eta < \theta < -\eta$ and analytic continuation up to $\eta = 0$. Hence we can say that the RSPE of any eigenvalue $E_i(g)$ if $H(g)$, $g > 0$, sums to $\lambda_i(|g|, 0)$ which is complex because otherwise the RSPE would be convergent. $\lambda_i(g)$ has thus a branch cut at $\mathrm{Im}\, g = 0$, i.e. $\arg(g^2) = 0$, and the discontinuity across the cut is by the reflection symmetry $2i\, \mathrm{Im}\, \lambda_i(|g|, 0)$. This function is exponentially small, i.e. it has zero asymptotic series, because the RSPE is real.

(3) The eigenvalues $\lambda_i(|g|, 0)$ appear to be the natural notion of resonance in this type of problems (for details, see ref. [6]). $\mathrm{Im}\, \lambda_i(|g|, 0)$ is of course related to the WKB penetration formula through the barrier between the two wells and to the instanton calculus (see specifically Harrell [14], Zinn-Justin [2], Bogomolny [15] and Simon [1] for the connection with the Bender–Wu formula and the asymptotics of the RSPE coefficients).

Proof. (iv) follows by a simple scaling argument given the analyticity of the unperturbed eigenfunctions. By well-known arguments of perturbation theory (see specifically Simon [16]) (i) and (ii) are true if $[H_1(|g|,\theta)-z]^{-1}$ tends to $[H_0(\theta)-z]^{-1}$ in norm as $|g|\to 0$ for some $z\in\mathrm{C}$, uniformly in θ, $-\frac{5}{4}\pi+\eta\leqslant\theta\leqslant\frac{1}{4}\pi-\eta$, $\eta>0$. If $G(x,y;0,z)$ denotes the Green function of $H_0(\theta)$, $z\neq\{2i+1\}_{i=0}^{\infty}$, i.e. the integral kernel of $[H_0(\theta)-z]^{-1}$, it is of course enough to prove

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|G(x,y;|g|,\theta,z)-G(x,y;0,\theta,z)|^2\,\mathrm{d}x\,\mathrm{d}y \underset{|g|\to 0}{\to} 0\,, \tag{12}$$

for some $z\in\mathrm{C}$, with the stated uniformity in θ. Now, once more by a result in Sibuya [11] (Ch. 3; actually some purely algebraic modifications are needed to account for the fact that here the perturbation is a polynomial instead of a monomial, we omit the details)

$$\lim_{|g|\to 0} v_{1,2}(x,|g|,\theta,z)=v_{1,2}(0,|g|,\theta,z)$$

uniformly on compacts in (θ,z), z, R, θ as above. Hence

$$\lim_{|g|\to 0} W(|g|,\theta,z)=W(0,\theta,z)\,,$$

with the stated uniformity and therefore $W(|g|,\theta,z)$ is bounded uniformly in $(|g|,\theta,z)$, $|g|$ small, θ as above, z away from $\{2i+1\}_{i=0}^{\infty}$. Now by (9) and (10) it is easy to check that given $\epsilon>0$ there is $M(\epsilon)>0$ independent of $(|g|,\theta)$ such that

$$\iint_{\mathrm{R}^2/M(\epsilon)}|G(x,y;|g|,\theta,z)|^2\,\mathrm{d}x\,\mathrm{d}y<\epsilon\,,$$

Λ being the square of side $M(\epsilon)$ centered in the origin. By the continuity of $v_{1,2}(x,|g|,\cdot)$ as $|g|\to 0$,

$$\iint_{\Lambda}|G(x,y;|g|,\cdot)-G(x,y,0,\cdot)|^2\,\mathrm{d}x\,\mathrm{d}y\to 0$$

as $|g|\to 0$. This proves (12) and hence (i) and (ii). To see (iii), first note that the RSPE contains only even powers of g because of the symmetry $x\to -x$, $x\to -g$. To prove the Borel summability, given the analyticity of $\lambda_i(g)$ by the Watson theorem (see e.g. ref. [13], XII.5) we have only to verify a strong asymptotic condition of the type

$$|R_N^i(|g|,\theta)|\equiv\left|\lambda_i(|g|,\theta)-\sum_{n=0}^{N-1}A_n^i|g|^{2n}\,\mathrm{e}^{2in\theta}\right| \leqslant AD^N N!\,|g|^{2N} \tag{13}$$

for some $A>0$, $D>0$ independent of θ, $-\frac{5}{4}\pi+\eta\leqslant\theta\leqslant\frac{1}{4}\pi-\eta$, $\eta>0$. By the norm convergence of $[H_1(|g|,\theta)-z]^{-1}$ to $[H_0(\theta)-z]^{-1}$, $[H_1(|g|,\theta)-z]^{-1}$ is bounded uniformly with respect to $(|g|,\theta,z)$, $|g|$ suitably small, θ as above, z on the circumference $|\lambda(0)-z|=r$, $r>0$ suitably small. Hence we can repeat word by word the argument of ref. [6], theorem 2, to establish (13). This proves proposition 2.

Let us now consider the $|g|\to\infty$ limit of $H_1(|g|,0)$. Performing first the rescaling $x\to|g|^{-1/3}x$ and then the unitary translation $x\to x+\beta$ we can instead consider the operator $|g|^{2/3}\,\mathrm{e}^{\mathrm{i}\pi/3}H_2^{\beta}(|g|,0)$ where

$$H_2^{\beta}(|g|,0)=[p^2+|g|^{-4/3}\,\mathrm{e}^{-2\pi\mathrm{i}/3}(x+\beta)^2 + 2|g|^{-2/3}\,\mathrm{e}^{-5\mathrm{i}\pi/6}(x+\beta)^3+\mathrm{e}^{-\mathrm{i}\pi}(x+\beta)^4]\,, \tag{14}$$

realized as in proposition 1. The above differential expression makes of course sense also β complex. Proceeding as in proposition 1 it is not difficult to see that, since the asymptotic behaviours (9) and (10) are β-independent for $-\frac{1}{4}|g|^{-2/3}<\mathrm{Im}\,\beta$, the resolvent $[H_2^{\beta}(\cdot)-z]^{-1}$ exists for some $z\in\mathrm{C}$ and is represented by an integral kernel which is a $\mathrm{L}^2(\mathrm{R}^2)$-valued holomorphic function of β for $-|g|^{-2/3}<\mathrm{Im}\,\beta$. Hence $H_2^{\beta}(|g|,0)$ is by definition a translation analytic family of operators in L^2 (for this notion, see Avron–Herbst [17]) so that its eigenvalues do not depend on β, $\mathrm{Im}\,\beta>-|g|^{-2/3}$.

On the other hand for $\mathrm{Im}\,\beta>0$ (the choice of sign(Im β) is immaterial: we take $\mathrm{Im}\,\beta>0$ to be consistent with $\mathrm{Im}\,\beta>-\frac{1}{4}|g|^{-2/3}$ in $H_2^{\beta}(|g|,0)$ at the limit $|g|\to\infty$) we can explicitly realize the differential expression $p^2+\mathrm{e}^{-\mathrm{i}\pi}(x+\mathrm{i}\beta)^4$ as the discrete-spectrum operator H_{∞}^{β} defined as the inverse of the integral operator of kernel

$$\begin{aligned} G_{\infty}^{\beta}(x,y)&=\pi^{-1}H_{1/3}^{(1)}(\tfrac{1}{3}(x+\mathrm{i}\beta)^3)\,H_{1/3}^{(2)}(\tfrac{1}{3}(y+\mathrm{i}\beta)^3)\,, & y\leqslant x\,,\\ &=\pi^{-1}H_{1/3}^{(2)}(\tfrac{1}{3}(x+\mathrm{i}\beta)^3)\,H_{1/3}^{(1)}(\tfrac{1}{3}(y+\mathrm{i}\beta)^3)\,, & x\leqslant y\,. \end{aligned} \tag{15}$$

$H_\nu^{(1),(2)}$ are the Hankel functions of order $\nu = 1/3$, which is easily seen to be Hilbert–Schmidt. If $G_2^\beta(x, y; |g|)$ denotes the Green function of $H_2^\beta(\cdot)$, i.e. the integral kernel of $[H_2(|g|, -\frac{1}{6}\pi) - z]^{-1}|_{z=0}$, by a known continuity result on solutions of ODE with polynomial coefficients (see, e.g. ref. [11] or ref. [16]; note that here the vanishing parameter multiplies the subdominant terms) and the same $\epsilon, M(\epsilon)$ argument of proposition 2 we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G_2^\beta(x,y;|g|) - G_\infty^\beta(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \to 0$$

as $|g| \to \infty$. Therefore the eigenvalues of $H_2^\beta(\cdot)$ converge to the eigenvalues of H_2^∞, which are thus β-independent. (Note that the present realization of $p^2 + \mathrm{e}^{-\mathrm{i}\pi}x^4$ through a complex translation differs from the realization by analytic continuation $\lambda(p^2 + \lambda^{-3}x^4)$ through the complex scaling $\lambda = \mathrm{e}^{\mathrm{i}\pi/3}$ starting from the self-adjoint realization of $p^2 + x^4$ in L^2.)

Rescaling $x \to |g|^{-1/3}x$ in $K(g) = \frac{1}{2}(p^2 + |x|^2) + g^2|x|^4$ we can consider $K_2(g) = \frac{1}{2}p^2 + \frac{1}{2}|g|^{-4/3}|x|^2 + |x|^4$ since $K_1(g) = |g|^{2/3}K_2(g)$ has the same eigenvalues of $K(g)$. It is known [16] that $K_2(g)$ converges in norm resolvent sense to $K_2(\infty) = \frac{1}{2}p^2 + |x|^4$ as $|g| \to \infty$. If (4) holds, we have $\lambda_i(|g|, 0) = \mu_i(g)$ for all i. Hence:

Proposition 3. Let the Zinn–Justin conjecture to be true. Then the operators $K_2(\infty)|_{l=0} = \frac{1}{2}p^2 + |x|^4|_{l=0}$ and $H_2^\beta(\infty) = p^2 + \mathrm{e}^{-\mathrm{i}\pi}(x+\beta)^4$, $\mathrm{Im}\,\beta > 0$, have the same eigenvalues.

The above form of the conjecture is the most convenient to submit to a numerical test. By the rescaling $x \to 2^{1/6}x$ the eigenvalues of $K_2(\infty)|_{l=0}$ are $2^{-2/3} \times \mu_i(\infty)$, $\mu_i(\infty)$ the eigenvalues of $p^2 + |x|^4|_{l=0}$, computed up to 28 exact figures in ref. [18]. Setting $\beta = \gamma/2$, $\gamma > 0$, $\delta = \gamma^{-3/2}$, and rescaling $x \to \gamma^{1/2}x$ the eigenvalues $\lambda_i(\infty)$ of $H_2^\beta(\infty)$ are those of

$$H_2^\delta(\infty) = \delta^{-2/3}[p^2 + \tfrac{3}{2}x^2 - \delta^2x^4 - \tfrac{1}{16}\delta^{-2} - 2\mathrm{i}\delta x^3 - (2\mathrm{i}\delta)^{-1}x]\,. \qquad (16)$$

Therefore we must have $2^{2/3}\lambda_i(\infty) = \mu_i(\infty)$. In table 1 we list the "first" 7 eigenvalues $2^{2/3}\lambda_i(\infty)$, computed through the standard Rayleigh–Ritz procedure diagonalizing the 40×40 matrix $\|\langle e_i, H_2^\delta(\infty)e_k\rangle\|_{i,k=1,\dots,40}$, $\{e_i\}_{i=1}^\infty$ the Hermite functions, $\delta = 0.1458$, against the first 7 eigenvalues $\mu_i(\infty)$. The imaginary part of each computed $\lambda_i(\infty)$ is always smaller than 10^{-11} and hence omitted. We see that $2^{2/3}\lambda_i(\infty)$ and $\mu_i(\infty)$ agree at least up to the 10th figure.

Table 1

N	$2^{2/3}\lambda_N(\infty)$ computed	$\mu_i(\infty)$ exact
0	$0.234482907285 \times 10^1$	$0.234482907274 \times 10^1$
1	$0.952978138419 \times 10^1$	$0.952978138401 \times 10^1$
2	$0.187351955048 \times 10^2$	$0.187351955047 \times 10^2$
3	$0.293015482292 \times 10^2$	$0.293015482287 \times 10^2$
4	$0.409419183539 \times 10^2$	$0.409419183538 \times 10^2$
5	$0.534863353215 \times 10^2$	$0.534863353214 \times 10^2$
6	$0.668197546416 \times 10^2$	$0.668197546415 \times 10^2$

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