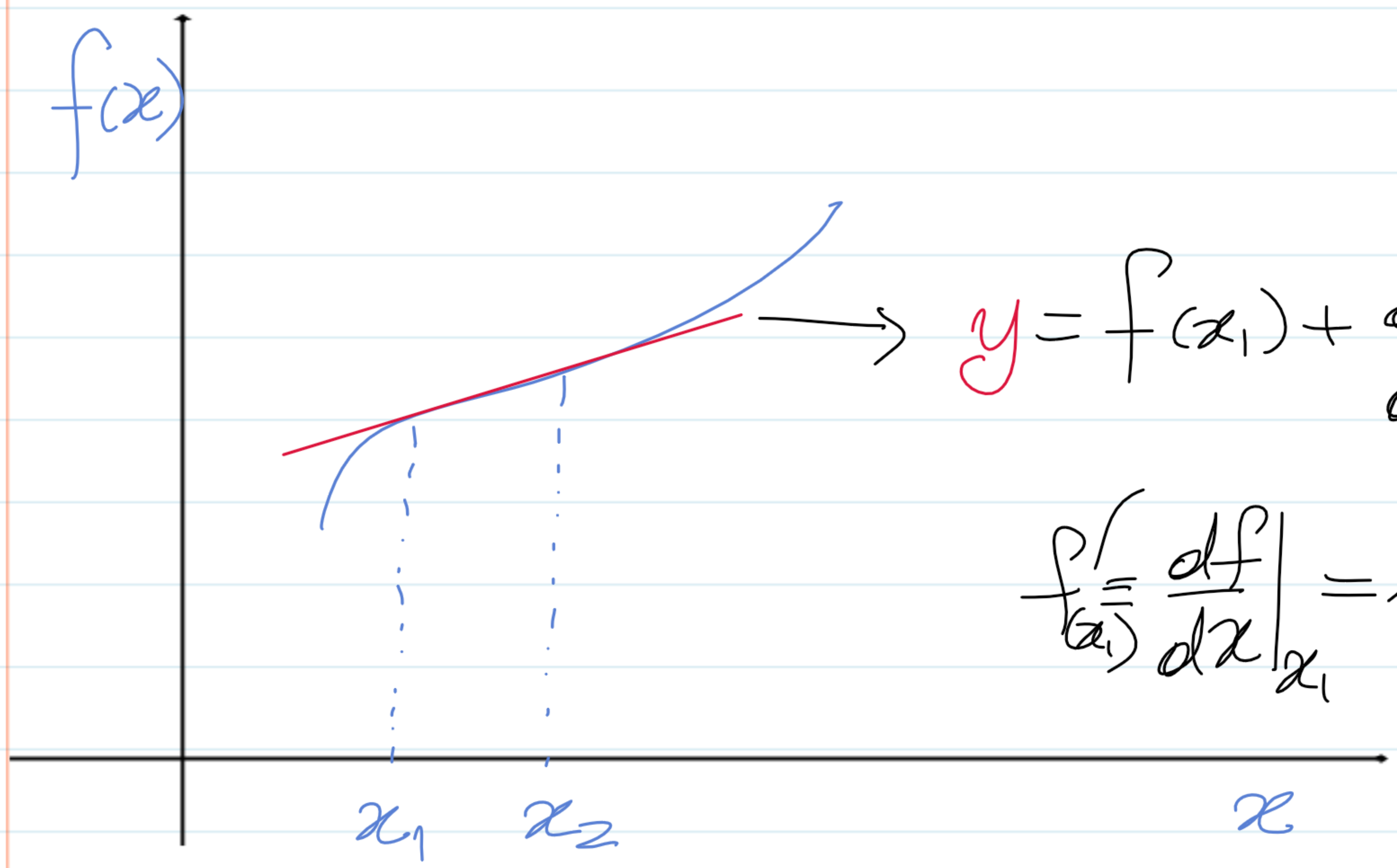
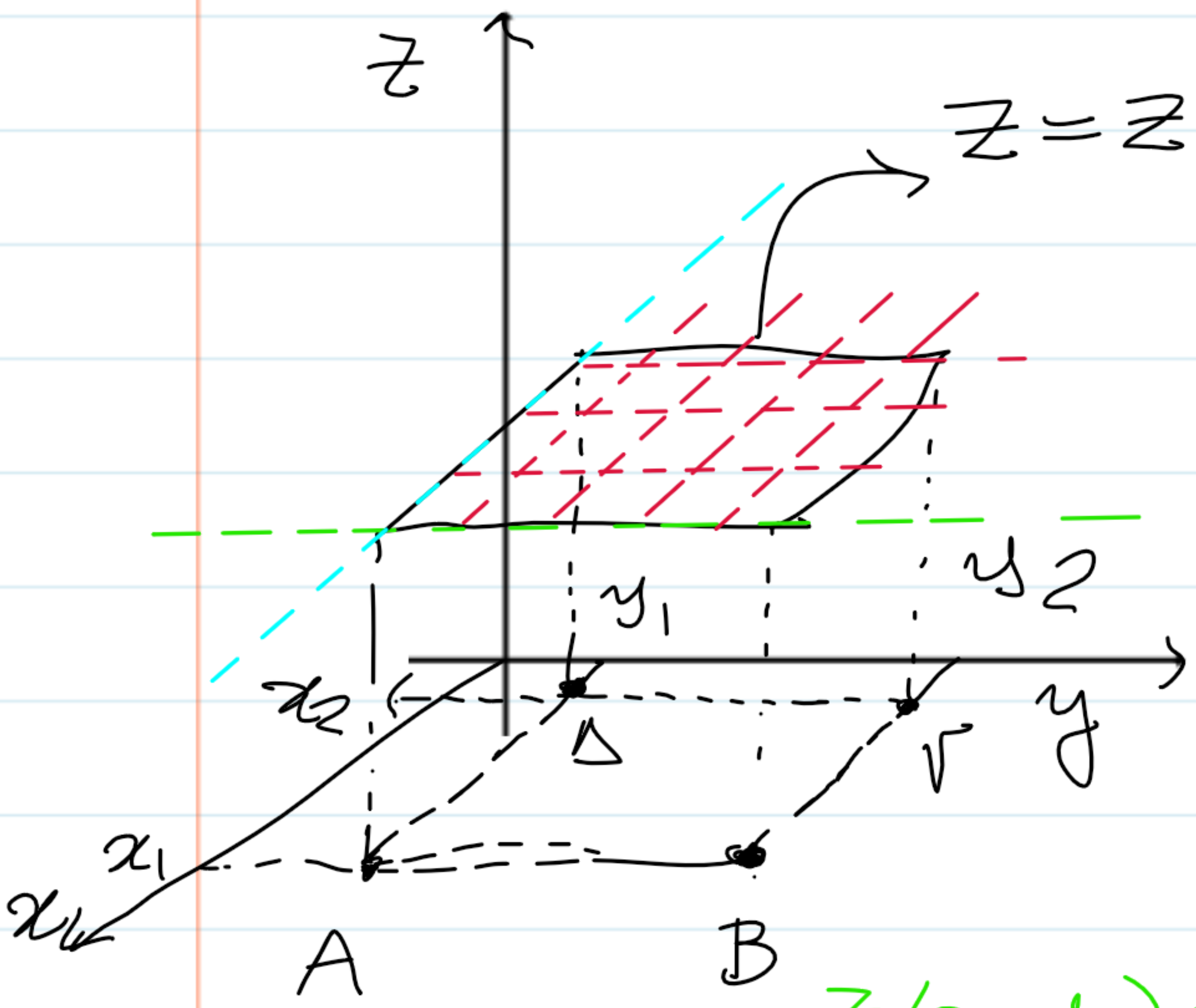
παράγωγος – μερική παράγωγος – μεικτή παράγωγος




$$y = f(x_1) + \left.\frac{df}{dx}\right|_{x_1}(x_2-x_1) \quad (1)$$

$$f'_{(x_1)} = \left.\frac{df}{dx}\right|_{x_1} = \lim_{x_2\to x_1}\frac{f(x_2)-f(x_1)}{(x_2-x_1)}$$

όταν $x_2 \to x_1$ τότε $(x_2-x_1) \simeq dx$ και $y \simeq f(x_2)$
επομένως η (1) γίνεται $f(x_2) = f(x_1) + f'_{(x_1)}(x_2-x_1) \Rightarrow$
$df = f(x_2) - f(x_1) = f'(x_1)dx$ ή $df = \left(\frac{df}{dx}\right)_{x_1} dx$




$z = z(x,y)$ , $A = (x_1, y_1)$, $B = (x_1, y_2)$
$\Gamma = (x_2, y_2)$, $\Delta = (x_2, y_1)$

$$z(x_2,y_1) = z(x_1,y_1) + \left.\frac{\partial z}{\partial x}\right|_{x_1,y_1}(x_2-x_1)$$

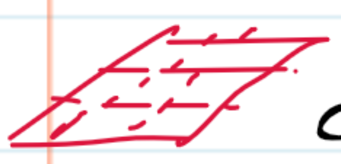
όπου $$\left.\frac{\partial z}{\partial x}\right|_{x_1,y_1} = \lim_{x_2\to x_1}\frac{\left(z(x_2,y_1) - z(x_1,y_1)\right)}{(x_2-x_1)}$$

$$z(x_1,y_2) = z(x_1,y_1) + \left.\frac{\partial z}{\partial y}\right|_{x_1,y_1}(y_2-y_1)$$

όπου $$\left.\frac{\partial z}{\partial y}\right|_{x_1,y_1} = \lim_{y_2\to y_1}\frac{\left(z(x_1,y_2) - z(x_1,y_1)\right)}{(y_2-y_1)}$$

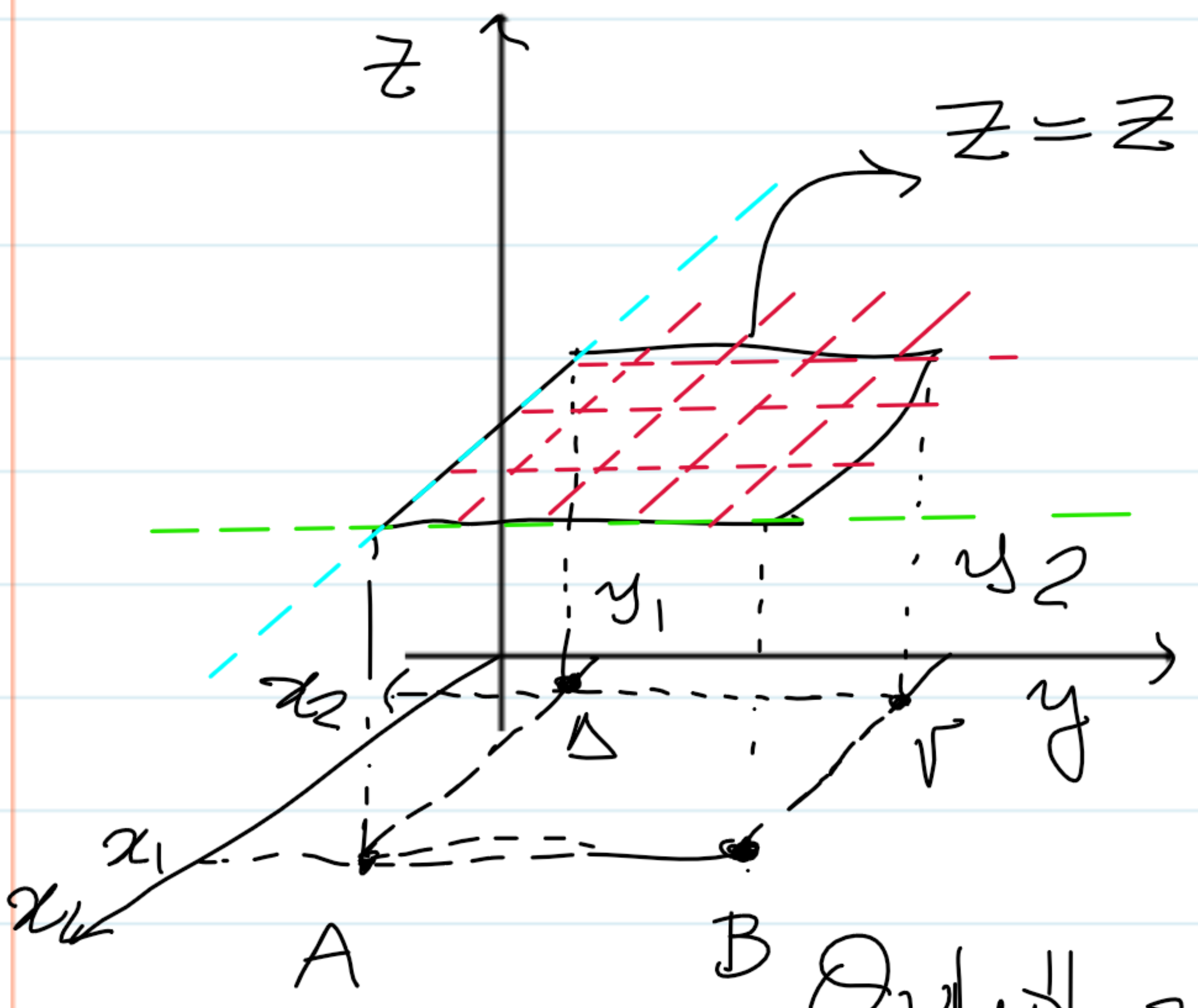
Καθώς $x_2 \to x_1$, $(x_2-x_1) \simeq dx$, ενώ όταν $y_2 \to y_1$, $(y_2-y_1) \simeq dy$
και έτσι συνολικά μπορούμε να έχουμε:

$$z(x_2,y_2) = z(x_1,y_1) + \left.\frac{\partial z}{\partial x}\right|_{x_1,y_1} dx + \left.\frac{\partial z}{\partial y}\right|_{x_1,y_1} dy \Rightarrow$$



$$dz = z(x_2,y_2) - z(x_1,y_1) = \left.\frac{\partial z}{\partial x}\right|_{x_1,y_1} dx + \left.\frac{\partial z}{\partial y}\right|_{x_1,y_1} dy = (\vec{\nabla} z)\cdot d\vec{r}$$

$$\vec{\nabla} z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \qquad d\vec{r} = (dx, dy)$$




Σε μια ομαλή συνάρτηση το αποτέλεσμα για την $z(x_2,y_2)$ θα πρέπει να είναι το ίδιο (ξεκινώντας από την $z(x_1,y_1)$) είτε ακολουθώντας τη διαδρομή $A\to\Delta\to\Gamma$ είτε την $A\to B\to\Gamma$

Θυμίζουμε $A=(x_1,y_1)$, $\Delta=(x_2,y_1)$
$B=(x_1,y_2)$, $\Gamma=(x_2,y_2)$

$A\to\Delta\to\Gamma \Rightarrow$

$$z(x_2,y_2)=z(x_1,y_1)+\left.\frac{\partial z}{\partial x}\right|_{x_1,y_1}(x_2-x_1)+\left.\frac{\partial z}{\partial y}\right|_{x_2,y_1}(y_2-y_1) \quad (1)$$

$A\to B\to\Gamma$

$$z(x_2,y_2)=z(x_1,y_1)+\left.\frac{\partial z}{\partial y}\right|_{x_1,y_1}(y_2-y_1)+\left.\frac{\partial z}{\partial x}\right|_{x_1,y_2}(x_2-x_1) \quad (2)$$

$$(1)-(2)\Rightarrow 0=\left[\left.\frac{\partial z}{\partial x}\right|_{x_1,y_1}-\left.\frac{\partial z}{\partial x}\right|_{x_1,y_2}\right](x_2-x_1)+\left[\left.\frac{\partial z}{\partial y}\right|_{x_2,y_1}-\left.\frac{\partial z}{\partial y}\right|_{x_1,y_1}\right](y_2-y_1)$$

$$\Rightarrow \left[\left.\frac{\partial z}{\partial x}\right|_{x_1,y_2}-\left.\frac{\partial z}{\partial x}\right|_{x_1,y_1}\right](x_2-x_1)=\left[\left.\frac{\partial z}{\partial y}\right|_{x_2,y_1}-\left.\frac{\partial z}{\partial y}\right|_{x_1,y_1}\right](y_2-y_1)\Rightarrow$$

$$\Rightarrow \frac{\left[\left.\frac{\partial z}{\partial x}\right|_{x_1,y_2}-\left.\frac{\partial z}{\partial x}\right|_{x_1,y_1}\right]}{(y_2-y_1)}=\frac{\left[\left.\frac{\partial z}{\partial y}\right|_{x_2,y_1}-\left.\frac{\partial z}{\partial y}\right|_{x_1,y_1}\right]}{(x_2-x_1)}$$

Καθώς $y_2\to y_1$ και $x_2\to x_1$ τα παραπάνω γίνονται μικτές παράγωγοι:

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$$