A NEW ISOPERIMETRIC INEQUALITY AND THE CONCENTRATION OF MEASURE PHENOMENON

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Abstract

We prove a new isoperimetric inequality for a certain product measure that improves upon some aspects of the "large deviation" consequences of the isoperimetric inequality for the Gaussian measure.

1. Introduction

We denote by γ the canonical Gaussian measure on $\mathbb{R}$, of density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ with respect to Lebesgue measure. We denote by γ^∞ the product measure on $\mathbb{R}^{\mathbb{N}}$, where each factor is endowed with γ. Throughout the paper, we set, for $\alpha = 1, 2$,

$$B_\alpha = \{x \in \mathbb{R}^{\mathbb{N}}; \quad \sum_{k \geq 1} |x_k|^\alpha \leq 1\}.$$

For two sets A, B of $\mathbb{R}^{\mathbb{N}}$, we set

$$A + B = \{x + y; \quad x \in A, \quad y \in B\}.$$

Consider a (Borel) set $A \subset \mathbb{R}^{\mathbb{N}}$, and $a \in \mathbb{R}$ such that $\gamma^\infty(A) = \gamma((-\infty, a])$. The isoperimetric inequality for the Gauss measure states that for $u \geq 0$,

$$\gamma_*^\infty(A + uB_2) \geq \gamma((-\infty, a + u]) \tag{1.1}$$

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(the inner measure is needed as $A + uB_2$ might fail to be measurable). This inequality plays a fundamental role in the theory of Gaussian processes and Gaussian measures. It was discovered independently by C. Borell [B] and B.S. Tsirelson and V.N. Sudakov [ST]. They derived this inequality from Lévy's isoperimetric inequality on the sphere, via Poincaré lemma. Later, A. Ehrhard [E1] developed a more intrinsic approach of Gaussian symmetrization that also led him to other remarkable inequalities [E2].

Very often, the Gaussian isoperimetric inequality is used under the following form

$$\gamma^\infty(A) \geq \tfrac{1}{2} \Rightarrow \gamma_*^\infty(A + uB_2) \geq \gamma((-\infty, u]) \tag{1.2}$$

where the last term is evaluated through the classical estimate $\gamma_1((-\infty, u]) \geq 1 - \frac{1}{2}e^{-u^2/2}$. For some applications (e.g. [T]) it is important to have the sharp estimate (1.2). For many others it is sufficient to know, that for some universal constant K, one has

$$\gamma^\infty(A) \geq \frac{1}{2} \Rightarrow \gamma_*^\infty(A + uB_2) \geq 1 - 2e^{-u^2/K}\,. \tag{1.3}$$

(The role of the factor 2 being to emphasize that one cares only for large values of u.)

In the terminology of V. Milman [M], (1.3) will be called the concentration of measure property (for the Gaussian measure). The main contribution of the present paper is the somewhat unexpected fact that the concentration of measure property for the Gaussian measure is the consequence of a sharper principle, itself unrelated to Gaussian measure, and that we present now. Throughout the paper, we denote by μ the probability measure on $\mathbb{R}$ of density $\frac{1}{2}e^{-|x|}$ with respect to Lebesgue measure, and by μ^∞ its product measure on $\mathbb{R}^{\mathbb{N}}$.

Theorem 1.2. *There exists a universal constant K with the following property. Consider a set $A \subset \mathbb{R}^{\mathbb{N}}$, and $a \in \mathbb{R}$ such that $\mu^\infty(A) = \mu((-\infty, a])$. Then*

$$\mu_*^\infty(A + \sqrt{u}B_2 + uB_1) \geq \mu((-\infty, a + \frac{u}{K}])\,. \tag{1.4}$$

In particular

$$\mu^\infty(A) \geq \frac{1}{2} \Rightarrow \mu_*^\infty(A + \sqrt{u}B_2 + uB_1) \geq 1 - \frac{1}{2}\exp\left(-\frac{u}{K}\right)\,. \tag{1.5}$$

A main difference between (1.4) and (1.1) is the fact that A is now enlarged by a mixture of the ℓ^1 and ℓ^2 balls, in proportions that vary with u. Consider a sequence $(t_k)_{k\geq 1}$ (having for simplicity only finitely many non-zero terms), and set $f(x) = \sum_{k\geq 1} t_k x_k$. It is possible to study the tails $\mu^\infty(\{f \geq u\})$ in an elementary way; but we will do it here using (1.5), in order to illustrate the role of the ℓ^1 and ℓ^2 balls.

Consider

$$A = \{x \in \mathbb{R}^{\mathbb{N}}; \quad \sum_{k\geq 1} t_k x_k \leq 0\}.$$

By symmetry, $\mu^\infty(A) \geq \frac{1}{2}$. If $y \in A + \sqrt{u}B_2 + uB_1$, we then have $f(y) \leq \sqrt{u}\|t\|_2 + u\|t\|_\infty$. Thus, by (1.5)

$$\mu^\infty(\{f \geq \sqrt{u}\|t\|_2 + u\|t\|_\infty\}) \leq \frac{1}{2}e^{-u/K}.$$

This implies that

$$\mu^\infty(\{f \geq u\}) \leq \frac{1}{2}e^{-u/K\|t\|_\infty}$$

for $u \geq \|t\|_2^2/\|t\|_\infty$, while, for $0 \leq u \leq \|t\|_2^2/\|t\|_\infty$ we have

$$\mu^\infty(\{f \geq u\}) \leq \frac{1}{2}e^{-u^2/K\|t\|_2^2}.$$

(For the simplicity of notations we allow the value of the constant K to vary at each occurrence.) The exponents in these bounds are exactly of the right order.

Another feature that differentiates (1.4) from (1.1) (and makes it closer to (1.3)) is the constant K on the right. It would be nice (but irrelevant for our purposes) to have a more exact form of (1.4). A natural question to ask in that direction is whether there exists, given u, a natural "smallest" set $W_u \subset \mathbb{R}^{\mathbb{N}}$ such that

$$\mu_*^\infty(A + W_u) \geq \mu((-\infty, a + u]) \tag{1.6}$$

for all $a \in \mathbb{R}$, all sets $A \subset \mathbb{R}^{\mathbb{N}}$, such that $\mu^\infty(A) = \mu((-\infty, a])$. The difficulty of that question is that the shape of W_u is likely to vary depending on u. A worthy inequality (1.6) should, in particular, allow to recover excellent tail estimates for $\mu^\infty(\{\sum_{k\geq 1} t_k x_k \geq u\})$; but in view of the variety of the estimates known for this quantity, [H] this does not appear to be a simple task.

We now explain why (1.5) improves upon (1.3). (The basic idea of this argument is due to G. Pisier [P].) Consider the increasing map ψ from $\mathbb{R}$ to $\mathbb{R}$ that transforms μ into γ. It is easy to see that

$$\forall x, y \in \mathbb{R}, \quad |\psi(x) - \psi(y)| \leq K \min(|x - y|, \quad |x - y|^{1/2}). \tag{1.7}$$

Consider the map Ψ from $\mathbb{R}^{\mathbb{N}}$ to itself given by $\Psi((x_k)_{k\geq 1}) = (\psi(x_k))_{k\geq 1}$. It transforms μ^∞ into γ^∞. Consider now a set $A \subset \mathbb{R}^{\mathbb{N}}$, with $\gamma(A) \geq 1/2$. Then $\mu(\Psi^{-1}(A)) = \gamma(A) \geq 1/2$. By (1.4) we have

$$\mu_*^\infty(\Psi^{-1}(A) + \sqrt{u}B_2 + uB_1) \geq 1 - \frac{1}{2}e^{-u/K}$$

so that

$$\gamma^\infty(\Psi(\Psi^{-1}(A)+\sqrt{u}B_2+uB_1))\geq 1-\frac{1}{2}e^{-u/K}\,.$$

Now using (1.6) it is simple to show that

$$A_u=\Psi(\Psi^{-1}(A)+\sqrt{u}B_2+uB_1)\subset A+K\sqrt{u}B_2\,;$$

thus we recover (1.3). It can, however, happen that the set A_u is much smaller than $A+K\sqrt{u}B_2$. A striking example is when A is the cube

$$A=\{x\,;\quad \forall k\leq n\,,\quad |x_k|\leq c_n\}$$

where c_n is (say) chosen such that $\gamma^\infty(A)=1/2$, so that c_n is of order $(\log n)^{1/2}$. In that case, and when $u\ll \log n$, it is simple to see that

$$A_u\subset A+K\left(\frac{\sqrt{u}}{\sqrt{\log n}}B_2+\frac{u}{\sqrt{\log n}}B_1\right)\subset A+K\left(\frac{u}{\sqrt{\log n}}\right)^{1/2}\sqrt{u}B_2\,.$$

An intriguing aspect of the improvement of (1.3) via (1.4) is that we break the rotational invariance that is fundamental to the Gaussian measure γ^∞. Thus (working in $\mathbb{R}^n$ rather than $\mathbb{R}^\infty$) if R denotes any rotation of $\mathbb{R}^n$, (with obvious notations) we have $\gamma^n((RA)_u)\geq 1-\exp(-u/K)$.

Having discovered that (1.3) is a consequence of (1.4), one must now ask whether (1.4) itself is the end of the story. We will show that in the setting of product measures, this seems to be the case. Indeed (Proposition 5.1) if for a measure θ on $\mathbb{R}$, the product measure θ^∞ on $\mathbb{R}^{\mathbb{N}}$ satisfies even a much weaker form of concentration of measure than (1.4), the function $\theta(\{|x|\geq a\})$ must decrease exponentially fast.

The main difficulty in proving Theorem 1.1 is that, in contrast with the Gaussian case, the measure μ does not have many symmetries. This limits the use of rearrangements. In section 2, we show how an induction argument reduces the proof of Theorem 1.1 to that of a certain statement in dimension 2. A special case of that statement is proved in section 3. In section 4, we use variational arguments to prove that this special case is actually the general case, thereby finishing the proof of Theorem 1.1. While the main ideas of the proof of Theorem 1.1 are rather natural, the proof requires checking a number of tedious facts.

On the other hand, the important part of Theorem 1.1 is certainly (1.5). Fortunately this is much easier to prove. In section 5, we give a simple proof of the following (that is weaker than Theorem 1.1).

Theorem 1.2. *There exists a universal constant K with the following property. Consider a set $A \subset \mathbb{R}^{\mathbb{N}}$, and for $x \in \mathbb{R}^{\mathbb{N}}$, set*

$$\theta_A(x) = \inf\left\{u \geq 0 \; ; \; x \in A + \sqrt{u}B_2 + uB_1\right\} .$$

Then

$$\int^* \exp\left(\theta_A(x)/K\right) d\mu^\infty(x) \leq 2/\mu_*^\infty(A) .$$

We observe that by Markov's inequality, this gives

$$\mu^\infty(A) \geq \tfrac{1}{2} \Rightarrow \mu_*^\infty\left(\left\{\theta_A(x) \geq Ku\right\}\right) \leq 4\exp(-u) ,$$

so that

$$\mu_*^\infty\left(A + \sqrt{u}B_2 + uB_1\right) \geq 1 - 4\exp\left(-\frac{u}{K}\right) .$$

For u large enough, this is equivalent to (1.5).

The reader is certainly advised to read the proof of Theorem 1.2 first, going back to previous sections whenever necessary.

2. Induction

We denote by $\varphi(x) = \frac{1}{2}e^{-|x|}$ the density of μ with respect to Lebesgue measure. We set

$$\Phi(x) = \int_{-\infty}^{x} \varphi(u)\, du = \mu([-\infty, x]) .$$

We set $\Phi(-\infty) = 0, \Phi(+\infty) = 1$.

The sets $\sqrt{u}B_2 + uB_1$ of Theorem 1.1 are not easy to manipulate. Our first task will be to replace them by more amenable sets.

Consider a parameter $L > 0$, to be determined later. For $x \in \mathbb{R}$, we set $\xi(x) = \frac{1}{L}\int_0^{|x|} \frac{u}{1+u}\, du$. There is no magic in this formula. The properties of ξ we really need are that $L\xi(x)$ resembles x^2 for $|x| \leq 1$ and $|x|$ for $|x| \geq 1$, and moreover (for technical reasons) that $\xi'(x)$ is strictly increasing for $x > 0$.

We now define, for $u \geq 0$,

$$V(u) = \{y \in \mathbb{R}^{\mathbb{N}} ; \quad \sum_{k \geq 1} \xi(y_k) \leq u\} \, ; \; V_n(u) = \{y \in \mathbb{R}^n \, ; \sum_{1 \leq k \leq n} \xi(y_k) \leq u\} .$$

We show that

$$V(u) \subset \sqrt{4Lu}\, B_2 + 4LuB_1 . \tag{2.0}$$

Indeed, consider $y \in V(u)$. Define $z = (z_k) \in \mathbb{R}^{\mathbb{N}}$ by $z_k = y_k$ if $|y_k| \leq 1$, $z_k = 0$ otherwise. Since $\xi(x) \geq x^2/4L$ for $|x| \leq 1$, we have $z \in \sqrt{4Lu}\, B_2$. Since $\xi(x) \geq |x|/4L$ for $|x| \geq 1$, we have $\|y - z\|_1 \leq 4Lu$. This proves (2.0).

To prove Theorem 1.1, it suffices to prove the following.

Theorem 2.1. *One can determine L with the following property. For all $A \subset \mathbb{R}^{\mathbb{N}}$,*

$$\mu^{\infty}(A) = \Phi(a) \Rightarrow \mu_*^{\infty}(A + V(u)) \geq \Phi(a + u)\,.$$

(One can then take the constant K of Theorem 1.1 equal to $4L$.) Denote now by μ^n the power of μ on $\mathbb{R}^n$. By an obvious approximation argument, it suffices to prove that we can find L such that the following holds:

(I_n) for each compact set A for $\mathbb{R}^n$, and $u \geq 0$,

$$\mu^n(A) = \Phi(a) \Rightarrow \mu^n(A + V_n(u)) \geq \Phi(a + u)\,.$$

This statement will be proved by induction on n. The first task is to prove the case $n = 1$. In that case, one has actually a much more accurate result (that is the exact analogue of (1.1)).

Proposition 2.2. *Consider a compact set $A \subset \mathbb{R}$. Then*

$$\mu(A) = \Phi(a) \Rightarrow \mu(A + [-u, u]) \geq \Phi(a + u)\,.$$

The proof uses rearrangements. We first consider the case where A is an interval $[v, w]$.

Lemma 2.3. *Consider $v \leq w$, $v + w \leq 0$. Consider $v' \leq v$ (possibly $v' = -\infty$) and w' such that $\mu([v, w]) = \mu([v', w'])$. Then $\mu([v - u, w + u]) \geq \mu([v' - u, w' + u])$.*

Proof: Define the function $y = y(x)$ by

$$\Phi(y) - \Phi(x) = \mu([x, y]) = \mu([v, w])\,.$$

Thus $y'\varphi(y) = \varphi(x)$. Set now

$$h(x) = \mu([x - u, y + u]) = \Phi(y + u) - \Phi(x - u)\,.$$

Thus

$$\begin{aligned} h'(x) &= y'\varphi(y + u) - \varphi(x - u) \\ &= \frac{1}{\varphi(y)}[\varphi(x)\varphi(y + u) - \varphi(y)\varphi(x - u)] \end{aligned}$$

has the same sign as

$$e^{-|x|-|y+u|} - e^{-|y|-|x-u|}.$$

But we have, for $u > 0$

$$\text{the function} \quad x \to |x+u| - |x| \quad \text{increases}. \tag{2.1}$$

Since y is obviously an increasing function of x, we have $x + y \leq v + w \leq 0$ for $x \leq v$, so that $y \leq -x$. Thus, by (2.1),

$$|y+u| - |y| \leq |-x+u| - |x|$$

i.e.,

$$|y+u| + |x| \leq |y| + |x-u|.$$

Thus $h'(x) \leq 0$ for $x \leq v$. This proves the result (observe that when $v' = -\infty$, we have $\lim_{x \to -\infty} y(x) = w'$). □

Lemma 2.4. *Consider $v \leq w$, and z such that $\Phi(v) + \Phi(w) = \Phi(z)$. Then $\Phi(v+u) + \Phi(w+u) \geq \Phi(z+u)$.*

Proof: This could be deducted from the previous lemma, although a direct argument is simpler. We define $y = y(x)$ by $\Phi(y) + \Phi(x) = \Phi(z)$, so that $y'\varphi(y) = -\varphi(x)$. We set $h(x) = \Phi(y+u) + \Phi(x+u)$, so that $h'(x) = y'\varphi(y+u) + \varphi(x+u)$ has the sign of

$$-\varphi(x)\varphi(y+u) + \varphi(y)\varphi(x+u) = e^{-|x+u|-|y|} - e^{-|x|-|y+u|}.$$

For $x \leq v$, we have $y \geq w$, so that $y \geq w \geq v \geq x$. By (2.1) we have $|x+u| - |x| \leq |y+u| - |y|$, i.e. $|x+u| + |y| \leq |x| + |y+u|$, so that $h' \geq 0$. The result follows by letting $x \to -\infty$, since $y \to z$. □

We now prove Proposition 2.2. It suffices to consider the case where A is a finite union of disjoint intervals. The proof proceeds by induction over the number of bounded intervals of A. When $n = 0$, either A consists of one unbounded interval, and there is nothing to prove, or else it consists of two such intervals, and the result follows from Lemma 2.4.

For the induction step from n to $n+1$, we can write $A = B \cup I$, where $I = [v, w]$ and where B is a union of intervals, at most n of which are bounded. If

$$(B + [-u, u]) \cap (I + [-u, u]) \neq \emptyset, \tag{2.2}$$

we could replace I and one of the intervals of B by one single interval containing both, without increasing $A + [-u, u]$, thus the result holds in that case. Suppose now that

$$(B + [-u, u]) \cap (I + [-u, u]) = \emptyset$$

and suppose, for definiteness, that $v \leq w$. We set

$$v' = \inf\{x \geq 0\,, \quad \exists y\,, \quad \mu([x, y]) = \mu(I)\,,\ [x - u, y + u] \cap (B + [-u, u]) = \emptyset\}$$

and we denote by w' the value of y corresponding to $x = v'$. If we replace I by $[v', w']$, we do not change $\mu(B)$, but, Lemma 2.3 shows that we decrease $\mu(B + [-u, u])$. If $v' = -\infty$, $[v', w']$ is unbounded and we have reduced to the case of n bounded intervals. If $v' > -\infty$, we have reduced to the case (2.2). □

We can and do assume $L \geq 1$. Then $\xi(x) \leq |x|$, and thus $V_1(u) \supset [-u, u]$, so that Proposition 2.1 shows that (I_1) holds.

Set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For a (Borel) function $f : \mathbb{R} \to \overline{\mathbb{R}}$, we set

$$\Phi(f) = \int_{\mathbb{R}} \Phi(f(x))\, d\mu(x) = \mu^2(\{(x, y) \in \mathbb{R}^2\,;\ y \leq f(x)\})\,.$$

In Sections 3 and 4, we will prove the following fact.

Proposition 2.5. *The parameter L can be chosen such that the following holds. Consider a non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$. Consider $u \geq 0$, and set*

$$\overline{f}(x) = \sup\{f(x) + u - \xi(x - x')\,; \quad \xi(x - x') \leq u\}\,. \tag{2.3}$$

Then

$$\Phi(f) = \Phi(a) \Rightarrow \Phi(\overline{f}) \geq \Phi(a + u)\,.$$

The rest of that section is devoted to prove that Proposition 2.5 implies that $(I_n) \Rightarrow (I_{n+1})$, thereby proving Theorem 2.1. The first task is to show that Proposition 2.5 implies a similar result when f is no longer assumed to be non-decreasing. This follows from the next result where, as well as in the rest of the paper, for a function f, we denote by $\overline{f}$ the function given by (2.3) (the value of u being fixed).

Proposition 2.6. *Consider a (Borel) function $g : \mathbb{R} \to \overline{\mathbb{R}}$. Define its non-decreasing rearrangement $f : \mathbb{R} \to \overline{\mathbb{R}}$ by*

$$f(-x) = \sup\{y\,; \quad \mu(\{g \geq y\}) \leq \Phi(x)\}\,. \tag{2.4}$$

Then f is non-decreasing, $\Phi(f)=\Phi(g)$, $\Phi(\overline{f})\le\Phi(\overline{g})$.

Proof: For notational convenience, we will show instead that the (non-increasing) function given by

$$f(x)=\sup\{y\,;\quad \mu(\{g\ge y\})\le\Phi(x)\}$$

satisfies $\Phi(f)=\Phi(g)$; $\Phi(\overline{f})\le\Phi(\overline{g})$.

To prove that $\Phi(f)=\Phi(g)$, we have to prove that the subgraphs of f and g have the same measure for μ^2. By Fubini theorem, it suffices to show that for all y, $\mu(\{f\ge y\})=\mu(\{g\ge y\})$.

Consider x such that $\Phi(x)=\mu(\{g\ge y\})$. Then $f(x)\ge y$ by definition, so that $\{f\ge y\}\supset(-\infty,x]$, and

$$\mu(\{f\ge y\})\ge\mu((-\infty,x])=\Phi(x)=\mu(\{g\ge y\})\,.$$

Consider now x such that $\Phi(x)>\mu(\{g\ge y\})$. Then $f(x)<y$, so that $\{f\ge y\}\subset[-\infty,x]$, and $\mu(\{f\ge y\})\le\Phi(x)$. Thus $\mu(\{f\ge y\})\le\mu(\{g\ge y\})$.

To prove that $\Phi(\overline{f})\le\Phi(\overline{g})$ it suffices, using Fubini theorem again, to show that for all y we have $\mu(\{\overline{f}\ge y\})\le\mu(\{\overline{g}\ge y\})$ or, equivalently, $\mu(\{\overline{f}>y\})\le\mu(\{\overline{g}>y\})$. Define b by $\Phi(b)=\mu(\{\overline{f}>y\})$. Since f, and hence $\overline{f}$, is non-increasing, we have $\overline{f}(x)>y$ for $x<b$. Consider now $x'\le x<b$. Set $s=x-x'$, and assume that $\xi(s)\le u$. Since f is non-increasing, we have

$$\Phi(x')\le\mu(\{f\ge f(x')\})=\mu(\{g\ge f(x')\})\,.$$

By Proposition 2.2, we have

$$\Phi(x)=\Phi(x'+s)\le\mu(\{g\ge f(x')\}+[-s,s])\,.$$

For $g(v)\ge f(x')$ and $|w|\le s$, we have

$$\overline{g}(v+w)\ge g(v)+u-\xi(w)\ge f(x')+u-\xi(s)\,.$$

Thus

$$\Phi(x)\le\mu(\{\overline{g}\ge f(x')+u-\xi(x-x')\})\,. \tag{2.5}$$

Since f is non-increasing, we have

$$\overline{f}(x)=\sup\{f(x')+u-\xi(x-x')\,;\quad x'\le x\,,\quad \xi(x-x')\le u\}\,.$$

Taking the supremum in (2.5) over x' gives

$$\Phi(x) \le \mu(\{\overline{g} > \overline{f}(x)\}) \le \mu(\{\overline{g} > y\})$$

and thus, taking the supremum over $x < b$ we have $\mu(\{\overline{f} > y\}) \le \mu(\{\overline{g} > y\}))$. □

We now prove the implication $(I_n) \Rightarrow (I_{n+1})$. Assuming that (I_n) holds, consider a compact subset A of $\mathbb{R}^{n+1}$. For $x \in \mathbb{R}$, we denote by A_x the set $\{z \in \mathbb{R}^n ; \quad (z,x) \in A\}$.

Step 1. For $x, x' \in \mathbb{R}$, such that $\xi(x - x') \le u$, we have

$$A_{x'} + V_n(u - \xi(x - x')) \subset (A + V_{n+1}(u))_x \,.$$

Indeed, consider $v \in V_n(u - \xi(x - x'))$. By definition,

$$\sum_{k \le n} \xi(v_k) \le u - \xi(x - x'),$$

and thus $(v, x - x') \in V_{n+1}(u)$. Consider now $z \in A_{x'}$, so that $(z, x') \in A$. We have

$$(z + v, x) = (z, x') + (v, x - x') \in A + V_{n+1}(u)$$

i.e., $z + v \in (A + V_{n+1}(u))_x$.

Step 2. Define $g(x) = \Phi^{-1}(\mu^n(A_x))$. By Fubini theorem, we have

$$\Phi(g) = \int_{\mathbb{R}} \Phi(g(x))\, d\mu(x) = \int_{\mathbb{R}} \mu^n(A_x)\, d\mu(x) = \mu^{n+1}(A) = \Phi(a)\,.$$

By Fubini theorem again, we have

$$\mu^{n+1}(A + V_{n+1}(u)) = \int_{\mathbb{R}} \mu^n((A + V_{n+1}(u))_x)\, d\mu(x)\,.$$

Step 1 shows that, whenever $\xi(x - x') \le u$, we have

$$(A + V_{n+1}(u))_x \supset A_{x'} + V_n(u - \xi(x - x'))$$

so that, by induction hypothesis (I_n), we have

$$\mu((A + V_{n+1}(u))_x) \ge \Phi(g(x') + u - \xi(x - x'))$$

and, taking the supremum over x',

$$\mu^n((A + V_{n+1}(u))_x) \ge \Phi(\overline{g}(x))\,.$$

Thus

$$\mu^{n+1}(A + V_{n+1}(u)) \ge \int_{\mathbb{R}} \Phi(\overline{g}(x)) d\mu(x) = \Phi(\overline{g})\,,$$

and the conclusion follows from Propositions 2.5, 2.6. □

3. Basic inequality

The aim of this section is to prove the following.

Proposition 3.1. *It is possible to choose the parameter L such that the following holds. Consider a non-decreasing function $f : \mathbb{R} \to \overline{\mathbb{R}}$, and set*

$$\widehat{f}(x) = \sup_{y \in \mathbb{R}} f(y) - \xi(x-y) = \sup_{y \geq x} f(y) - \xi(x-y)\,. \tag{3.1}$$

Then for $u \geq 0$, we have

$$\Phi(f) = \Phi(a) \Rightarrow \Phi(\widehat{f} + u) \geq \Phi(a+u)\,.$$

This statement is weaker than Proposition 3.5. Indeed, fixing $u \geq 0$, we have

$$\begin{aligned}\widehat{f}(x) + u &= \sup_y f(y) - \xi(x-y) + u \\ &\geq \sup\{f(y) - \xi(x-y) + u\,; \quad \xi(x-y) \leq u\} = \overline{f}(x)\,.\end{aligned}$$

Our first task is to show that Proposition 3.1 follows from the following.

Proposition 3.2. *(Basic inequality) Consider a non-decreasing function f from $\mathbb{R}$ to $\mathbb{R}$. Then*

$$\left(\int_{\mathbb{R}} e^{f(x)}\, d\mu(x)\right)\left(\int_{\mathbb{R}} e^{-\widehat{f}(x)}\, d\mu(x)\right) \leq 1\,. \tag{3.2}$$

We collect inequalities for that purpose. We note that

$$\Phi(x) = \frac{1}{2}\int_{-\infty}^{x} e^{-|t|}\, dt = \begin{cases} \frac{1}{2}e^{x} & \text{if } x \leq 0 \\ 1 - \frac{1}{2}e^{-x} & \text{if } x \geq 0\,. \end{cases}$$

Thus $\Phi(x) = \theta(e^x)$, where $\theta(t) = t/2$ for $t \leq 1$, $\theta(t) = 1 - 1/2t$ for $t \geq 1$. Since $\theta(0) = 0$ and θ' decreases, we have $\theta(t/M) \geq \theta(t)/M$ for $M \geq 1$. Thus

$$v \geq 0 \Rightarrow \Phi(x-v) \geq e^{-v}\Phi(x)\,. \tag{3.3}$$

Observe also that $\Phi(x) \leq \frac{1}{2}e^{x}$.

Lemma 3.3. *Consider a function $g : \mathbb{R} \to \overline{\mathbb{R}}$, such that $\int e^{-g}\, d\mu \leq 1$. Then $\Phi(g+v) \geq \Phi(v)$ for all $v \in \mathbb{R}$.*

Proof: Suppose first that $v \geq 0$. Since

$$1 - \Phi(x) = \Phi(-x) \leq \tfrac{1}{2}e^{-x}\,,$$

we get

$$\int_{\mathbb{R}} (1 - \Phi(g(x) + v))\, d\mu(x) \le \frac{1}{2} \int e^{-g-v}\, d\mu \le \frac{1}{2} e^{-v} = 1 - \Phi(v)$$

so that

$$\Phi(g + v) = \int \Phi(g(x) + v)\, d\mu(x) \ge \Phi(v)\,.$$

In particular, for $v = 0$, we have

$$\Phi(g) \ge \Phi(0) = \tfrac{1}{2}\,. \tag{3.4}$$

Suppose now that $v \le 0$. Then, by (3.3) and (3.4), we have

$$\Phi(g + v) = \int \Phi(g(x) + v)\, d\mu(x) \ge e^{v} \int \Phi(g(x))\, d\mu(x) = e^{v}\Phi(g) \ge \frac{1}{2} e^{v} = \Phi(v)\,. \qquad \square$$

We now deduce Proposition 3.1 from Proposition 3.2. We suppose first that $a \le 0$, so that $\Phi(a) = \frac{1}{2}e^{a}$. Since $\Phi(f(x)) \le \frac{1}{2}e^{f(x)}$, and $\Phi(a) = \Phi(f)$, we get $\int e^{f}\, d\mu \ge e^{a}$, so that $\int e^{f-a}\, d\mu \ge 1$. Obviously, we have $\widehat{f - a} = \widehat{f} - a$. The basic inequality (3.2) implies that $\int e^{-(\widehat{f}-a)} d\mu \le 1$. Lemma 3.3, used with $v = a + u$, $g = \widehat{f} - a$ concludes the proof in that case.

We suppose now that $a > 0$, and we set $h(x) = -\widehat{f}(-x) - u$. Thus

$$\widehat{h}(x) = \sup_{y} h(y) - \xi(x - y) = \sup_{y} -\widehat{f}(-y) - u - \xi(x - y)\,.$$

Since $\widehat{f}(-y) \ge f(-x) - \xi(x - y)$, it follows that

$$\widehat{h}(x) + u \le -f(-x)\,. \tag{3.5}$$

Define b by $\Phi(b) = \Phi(h)$, so that

$$\begin{aligned} 1 - \Phi(b) &= \int (1 - \Phi(-\widehat{f}(-x) - u))\, d\mu(x) = \int \Phi(\widehat{f}(-x) + u)\, d\mu(x) \\ &= \int \Phi(\widehat{f}(x) + u)\, d\mu(x) = \Phi(\widehat{f} + u)\,. \end{aligned} \tag{3.6}$$

Since $a \ge 0$, we have, since $f \le \widehat{f} + u$,

$$\tfrac{1}{2} \le \Phi(a) = \Phi(f) \le \Phi(\widehat{f} + u) = 1 - \Phi(b)$$

so that $b \le 0$. Since $b \le 0$, we already know that $\Phi(b + u) \le \Phi(\widehat{h} + u)$. By (3.5) we have, setting $\widetilde{f}(x) = -f(-x)$

$$\Phi(b + u) \le \Phi(\widehat{h} + u) \le \Phi(\widetilde{f}) = 1 - \Phi(f) = 1 - \Phi(a) = \Phi(-a)$$

and thus $a + b + u \le 0$, so that, by (3.6)

$$\Phi(a + u) \le \Phi(-b) = 1 - \Phi(b) = \Phi(\widehat{f} + u)\,. \qquad \square$$

A main step in the proof of the basic inequality is as follows.

Proposition 3.4. *There exists a universal constant K_1 with the following property. Considering a non-decreasing function f from $\mathbb{R}$ to $\mathbb{R}$, such that $\int e^f \, d\mu = 1$. Suppose that the (right) derivative $f'(x)$ of f is $\leq 1/K_1$ for each t. Then we have*

$$\int e^{-f} \, d\mu - K_1 \int e^{-f} f'^2 \, d\mu \leq 1 \, .$$

Lemma 3.5. $\int (f - f(0))^2 \, d\mu \leq 4 \int f'^2 \, d\mu$.

Proof: Consider a function g on $\mathbb{R}$, such that $g(0) = 0$, $\|g'\|_\infty < \infty$. Integrating by parts yields

$$\begin{aligned} \int_0^\infty g^2(x) e^{-x} \, dx &= \int_0^\infty 2g(x) g'(x) e^{-x} \, dx \\ &\leq 2 \left(\int_0^\infty g(x)^2 e^{-x} \, dx \right)^{1/2} \left(\int_0^\infty g'(x)^2 e^{-x} \, dx \right)^{1/2} \end{aligned}$$

by Cauchy-Schwartz. Thus

$$\int_0^\infty g^2(x) e^{-x} \, dx \leq 4 \int_0^\infty g'(x)^2 e^{-x} \, dx \, .$$

Changing x in $-x$ we get

$$\int_{-\infty}^0 g^2(x) e^x \, dx \leq 4 \int_{-\infty}^0 g'(x)^2 e^x \, dx$$

so that

$$\int g^2 \, d\mu \leq 4 \int g'^2 \, d\mu \, . \qquad \square$$

Lemma 3.6. *If $a \geq 0$, $b \leq 0$, we have*

$$\int_a^\infty e^f \, d\mu \leq \frac{K_1}{K_1 - 1} \frac{e^{-a}}{2} e^{f(a)} \tag{3.7}$$

$$\int_{-\infty}^b e^f \, d\mu \geq \frac{K_1}{K_1 + 1} \frac{e^b}{2} e^{f(b)} \tag{3.8}$$

$$\int_{-\infty}^b e^{-f} \, d\mu \leq \frac{K_1}{K_1 - 1} \frac{e^b}{2} e^{-f(b)} \, . \tag{3.9}$$

Proof: We prove only (3.7), the others being similar. Since $\|f'\|_\infty \leq 1/K_1$, we have, for $x \geq a$

$$f(x) \leq f(a) + (x - a)/K_1$$

so that

$$\begin{aligned} \int_a^\infty e^{f(x)} \, d\mu(x) &\leq \tfrac{1}{2} \int_a^\infty e^{f(a) + (x-a)/K_1} e^{-x} \, dx \\ &= \tfrac{1}{2} e^{f(a) - a/K_1} \int_a^\infty e^{-x(1 - 1/K_1)} \, dx \\ &= \frac{K_1}{K_1 - 1} \frac{e^{-a}}{2} e^{f(a)} \, . \end{aligned} \qquad \square$$

Lemma 3.7. *$|f(0)| \le 1/2$ if K_1 is large enough.*

Proof: Using (3.8), we have

$$1 = \int e^f \, d\mu = \int_{-\infty}^{0} e^{f(x)} \, d\mu(x) + \int_0^{\infty} e^{f(x)} \, d\mu(x)$$
$$\ge \frac{K_1}{2(K_1+1)} e^{f(0)} + \frac{1}{2} e^{f(0)} = \frac{2K_1+1}{2K_1+2} e^{f(0)}$$

so that $e^{f(0)} \le \frac{2K_1+2}{2K_1+1}$. In a similar way, using (3.7), we get $e^{-f(0)} \ge \frac{2K_1-2}{2K_1-1}$. □

We now set

$$c = \inf\{x \in \mathbb{R}\,, \quad f(x) \ge -1\}\,; \quad d = \sup\{x \in \mathbb{R}\,; \quad f(x) \le 1\}\,.$$

Observe that, by the previous lemma, we have $c \le 0 \le d$.

Lemma 3.8. $K \int_c^d f'^2 \, d\mu \ge e^c + e^{-d} + \int_c^d (f(x) - f(0))^2 \, d\mu(x)$.

Proof: We apply Lemma 3.5 to the function

$$g(x) = \max(-1, \min(1, f(x))) - f(0)\,,$$

and we observe, that, since $|f(0)| \le 1/2$ by Lemma 3.7, we have $g(x)^2 \ge 1/4$ for $x < d$ or $x > c$.

Lemma 3.9. $f(0)^2 \le K \int_c^d f'^2 \, d\mu$.

Proof: For $c \le x \le d$, we have $-1 \le f(x) \le 1$, so that

$$|e^{f(x)} - e^{f(0)}| \le e^2 |f(x) - f(0)|\,.$$

Thus, using Cauchy Schwartz and Lemma 3.8

$$\left| \int_c^d (e^{f(x)} - e^{f(0)}) \, d\mu \right| \le e^2 \int_c^d |f(x) - f(0)| \, d\mu(x) \tag{3.10}$$
$$\le e^2 \left(\int_c^d |f(x) - f(0)|^2 \, d\mu(x) \right)^{1/2}$$
$$\le KT$$

where $T^2 = \int_c^d f'^2 \, d\mu$. Now by (3.7), (3.9) and Lemma 3.8 we get

$$1 - \int_c^d e^f \, d\mu = \int_{-\infty}^{c} e^f \, d\mu + \int_d^{\infty} e^f \, d\mu$$
$$\le K(e^c + e^{-d}) \le KT^2\,.$$

Since

$$e^{f(0)} - \int_c^d e^{f(0)}\, d\mu \le K(e^c + e^{-d}) \le KT^2\,,$$

we get

$$|1 - e^{f(0)}| \le K(T + T^2) \le KT$$

since $T \le 1/K_1$. This implies the result. □

Proof of Proposition 3.4. For $|t| \le 1$, we have $e^{-t} + e^t \le 2 + 4t^2$ (since the function $t^{-2}(e^{-t} + e^t - 2)$ increases). Thus, for $c \le x \le d$ we have

$$e^{-f(x)} \le 2 - e^{f(x)} + 4f^2(x)$$

so that

$$\int_c^d e^{-f}\, d\mu \le \int_c^d (2 - e^f)\, d\mu + 4\int_c^d f^2\, d\mu\,. \tag{3.11}$$

Now

$$\begin{aligned} \int e^{-f}\, d\mu &\le \int_c^d e^{-f}\, d\mu + \int_{-\infty}^c e^{-f}\, d\mu + \int_d^\infty e^{-f}\, d\mu \\ &\le \int_c^d e^{-f}\, d\mu + K(e^c + e^{-d}) \end{aligned} \tag{3.12}$$

by (3.9). Since $\int (2 - e^f)\, d\mu = 1$, we have, by (3.7)

$$\begin{aligned} \int_c^d (2 - e^f)\, d\mu &\le 1 - \int_{-\infty}^c (2 - e^f)\, d\mu - \int_d^\infty (2 - e^f)\, d\mu \\ &\le 1 + K(e^c + e^{-d})\,. \end{aligned}$$

Combining with (3.11), (3.12) yields

$$\begin{aligned} \int e^{-f}\, d\mu &\le 1 + K(e^c + e^{-d}) + 4\int_c^d f^2\, d\mu \\ &\le 1 + K(e^c + e^{-d}) + 8f(0)^2 + 8\int_c^d (f(x) - f(0))^2\, d\mu(x) \end{aligned}$$

since $f^2(x) \le 2(f(0)^2 + (f(x) - f(0))^2)$. It follows from Lemmas 3.8 and 3.9 that

$$\begin{aligned} \int e^{-f}\, d\mu &\le 1 + K\int_c^d f'^2\, d\mu \\ &\le 1 + K\int_c^d f'^2 e^{-f}\, d\mu \end{aligned}$$

since $e^{-f(x)} \ge 1/e$ for $x \le d$. □

Proof of the basic inequality. Step 1. We show that it suffices to consider the case where f is of the type $f(x) = \inf_y g(y) + \xi(x-y)$, where g is a non-decreasing step function (i.e., takes only finitely many values in $\overline{\mathbb{R}}$). Indeed, consider a non-decreasing step function $g \geq \widehat{f}$. Define $f'(x) = \inf_y g(y) + \xi(x-y)$. It is simple to see that $f' \geq f$, and $\widehat{f'} \leq g$. Since $f' \geq f$, we have $\int e^{f'}\, d\mu \geq 1$. If we know that the result holds for f', we conclude that $\int e^{-\widehat{f'}}\, d\mu \leq 1$, so that $\int e^{-g}\, d\mu \leq 1$. Since g is arbitrary, we get $\int e^{-\widehat{f}}\, d\mu \leq 1$.

Step 2. For $\sigma, \tau \in \mathbb{R}$, we define

$$R(\sigma,\tau) = \{(x,y)\,; \quad x \leq \sigma\,, \quad y \leq \tau \quad \text{or} \quad x \geq \sigma\,, \quad y \leq \tau + \xi(x-\sigma)\}\,.$$

It follows from Step 1 that we can assume that the subgraph of f is the intersection of a finite sequence of sets $R(\sigma_i, \tau_i)$, where $\sigma_i < \sigma_{i+1}$, $\tau_i < \tau_{i+1}$.

For $s \leq x$, we define

$$H(x,s) = \sup_{s \leq y \leq x} f(y) - \xi(y-s)\,.$$

That $H(x,x) = f(x)$. We now prove the following fact. Fix x, and consider the largest i such that $(x, f(x))$ is on the boundary of $R(\sigma_i, \tau_i)$. Assume that $\sigma_i < x$. Then for $\sigma_i < s < x$, we have $H(x,s) = f(x) - \xi(x-s)$.

Consider $s \leq y < x$. We have $f(y) \leq \tau_i + \xi(y - \sigma_i)$, so that

$$f(y) - \xi(y-s) \leq \tau_i + \xi(y-\sigma_i) - \xi(y-s)\,.$$

Since $\sigma_i \leq s$, and since ξ' increases on $\mathbb{R}^+$, the function $z \to \xi(z-\sigma_i) - \xi(z-s)$ increases. Thus

$$\tau_i + \xi(y-\sigma_i) - \xi(y-s) \leq \tau_i + \xi(x-\sigma_i) - \xi(x-s) = f(x) - \xi(x-s)\,.$$

Step 3. We define

$$W(x) = \int_{-\infty}^{x} e^{-H(x,s)}\, d\mu(s)\,.$$

We note that $H(x,s) \leq \widehat{f}(s)$.

Thus, for $y < x$

$$W(x) \geq \int_{-\infty}^{y} e^{-H(x,s)}\, d\mu(s) \geq \int_{-\infty}^{y} e^{-\widehat{f}(s)}\, d\mu(s)\,,$$

so that

$$\lim_{x \to \infty} W(x) \geq \int e^{-\widehat{f}}\, d\mu\,.$$

To conclude, it suffices to show that $\int W'(x)\,dx \le 1$. Since $H(x,x) = f(x)$, we have

$$W'(x) = e^{-f(x)}\varphi(x) - \int_{-\infty}^{x} \frac{\partial}{\partial x} H(x,s) e^{-H(x,s)}\, d\mu(s)\,.$$

(Thus we can differentiate under the integral sign follows from the fact that $\frac{\partial}{\partial x}H(x,s)$ is bounded by $\|\xi'\|_\infty$.) Since $H(x,s) \le f(x)$, we have

$$W'(x) \le e^{-f(x)} \left(\varphi(x) - \int_{-\infty}^{x} \frac{\partial}{\partial x} H(x,s)\, d\mu(s) \right)\,. \tag{3.13}$$

Consider the largest i such that x is on the boundary of $R(\sigma_i, \tau_i)$. Suppose first that $\sigma_i < x$. Step 2 shows that for $\sigma_i < s < x$, we have $H(x,s) = f(x) - \xi(x-s)$ so that

$$\begin{aligned} \frac{\partial}{\partial x} H(x,s) &= f'(x) - \xi'(x-s) \\ &= \xi'(x-\sigma_i) - \xi'(x-s) \end{aligned}$$

since $f(y) = \tau_i + \xi(y - \sigma_i)$ for y close enough to x.

We recall that $\xi'(t) = \frac{t}{L(t+1)}$. If $x \ge \sigma_i + 2$, we thus have, for $s \ge x - 1$

$$\frac{\partial}{\partial x} H(x,s) \ge \frac{2}{3L} - \frac{1}{2L} = \frac{1}{6L}\,.$$

If $x \le \sigma_i + 2$, for $s \ge (x+\sigma_i)/2$, setting $t = x - \sigma_i \le 2$, we have

$$\frac{\partial}{\partial x} H(x,s) \ge \frac{1}{L}\left(\frac{t}{1+t} - \frac{t/2}{1+t/2} \right) = \frac{1}{L}\frac{t}{2(t+1)(t+2)} \ge \frac{t}{24L}\,.$$

Thus, in both cases $\frac{\partial}{\partial x}H(x,s)$ is $\ge \frac{1}{KL}\min(1, x-\sigma_i)$ on an interval of length $\ge \frac{1}{2}\min(2, x-\sigma_i)$. Thus, since $\varphi(y) \ge \frac{1}{e^2}\varphi(x)$ for $|y-x| \le 2$, and since $\frac{\partial}{\partial x}H(x,s) \ge 0$, we have

$$\begin{aligned} \int_{-\infty}^{x} \frac{\partial}{\partial x} H(x,s)\, d\mu(s) &\ge \frac{1}{KL}\varphi(x)\min(1, (x-\sigma_i)^2) \\ &\ge \frac{1}{KL}\varphi(x)\frac{(x-\sigma_i)^2}{(1+|x-\sigma_i|)^2} \\ &= \frac{1}{KL}\varphi(x)(L\xi'(x-\sigma_i))^2 \\ &= \frac{L}{K}\varphi(x)\xi'(x-\sigma_i)^2 \\ &= \frac{L}{K}\varphi(x)f'(x)^2\,. \end{aligned}$$

Combining with (3.13) we get

$$W'(x) \le e^{-f(x)}\varphi(x)\left(1 - \frac{L}{K}f'(x)^2\right)\,. \tag{3.14}$$

Suppose now that $x \le \sigma_i$. In that case $f'(x) = 0$, so that (3.14) still hold. If we take $L = KK_1$, where K_1 is the constant of Proposition 3.6, we see from (3.14) that $\int_{\mathbb{R}} W'(x)dx \le 1$. This completes the proof.

4. Variational arguments

We will now show that, provided L is large enough, Proposition 2.5 follows from Proposition 3.1. The method is as follows. We fix the values of a and u. We show that there exists an f such that $\Phi(f) = \Phi(a)$, and that $\Phi(\overline{f})$ is as small as possible. We then show that $\overline{f}$ being given by (2.3), we have $\overline{f} = \widehat{f} + u$ (in which case $\Phi(\overline{f}) \geq \Phi(a+u)$ by Proposition 3.1).

The first task is to show that the infimum of $\Phi(\overline{f})$ is actually obtained. We recall that a, u are fixed, and we set

$$\mathcal{F} = \{f : \mathbb{R} \to \overline{\mathbb{R}}, \quad f \text{ non-decreasing}, \quad \Phi(f) = \Phi(a)\}.$$

Lemma 4.1. *Consider an ultrafilter $\mathcal{U}$ on $\mathcal{F}$. Define $f(x) = \lim_{g \to \mathcal{U}} g(x)$. Then $f \in \mathcal{F}$.*

Proof: It is a well known (and elementary) fact that the map $h \to \int_0^1 h(x)\,dx$ is pointwise continuous on the set $\mathcal{G}$ of non-decreasing functions from $[0,1]$ to $[0,1]$. The map $f \to \Phi(f)$ is thus pointwise continuous on the set of $\mathcal{G}'$ all non-decreasing maps from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$, as is seen by transporting $\mathcal{G}'$ to $\mathcal{G}$ by the map $f \to \Phi \circ f \circ \Phi^{-1}$. □

We define $\sigma > 0$ by $\xi(\sigma) = u$. Since $\xi(\sigma) \leq \sigma/L$, we have $\sigma \geq Lu$. We can and do assume $L \geq 2$, so that $\sigma \geq 2u$.

Lemma 4.2. *Consider an ultrafilter $\mathcal{U}$ on $\mathcal{F}$. Then if $f \in \mathcal{F}$ is given by $f(x) = \lim_{g \to \mathcal{U}} g(x)$, we have $\lim_{g \to \mathcal{U}} \Phi(\overline{g}) \geq \Phi(\overline{f})$.*

Proof: Since $\overline{g}$ is non-decreasing whenever g is, and since (as mentioned in the proof of Lemma 4.1) Φ is pointwise continuous on the set of non-decreasing functions, it suffices to show that for all x, $\lim_{g \to \mathcal{U}} \overline{g}(x) \geq \overline{f}(x)$. Given y with $|y - x| \leq \sigma$, we have

$$g(y) + u - \xi(y - x) \leq \overline{g}(x).$$

Taking the limit along $\mathcal{U}$ gives

$$f(y) + u - \xi(y - x) \leq \lim_{g \to \mathcal{U}} \overline{g}(x)$$

and thus $\overline{f}(x) \leq \lim_{g \to \mathcal{U}} \overline{g}(x)$ by definition of $\overline{f}$. □

Lemmas 4.1, 4.2 implies that we can find $f \in \mathcal{F}$ for which $\Phi(\overline{f})$ is minimal. Observe that $\Phi(\overline{f}) \leq \Phi(a+u)$, since the constant function a belongs to $\mathcal{F}$. We fix such an f, and we want to prove that $\overline{f} = \widehat{f} + u$.

Lemma 4.3. *We have* $f(2u) \in \mathbb{R}$, $f(\sigma - 2u) \in \mathbb{R}$.

Proof: We observe that $\overline{f}(x) \geq f(x+\sigma) - \xi(\sigma) + u = f(x+\sigma)$. Set $f'(x) = f(x+\sigma)$. It suffice to show that if either $f(2u) = -\infty$ or $f(\sigma - 2u) = \infty$ we would have $\Phi(f') > \Phi(a+u)$.

Consider the function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ given by

$$g(y) = \inf\{x\,; \quad f(-x) \leq y\}\,.$$

It is simple to see that $\Phi(g) = \Phi(f)$, since, by Fubini theorem, both numbers are equal to

$$\mu^2(\{(x,y)\,; \quad y \leq f(x)\})\,.$$

We set

$$g'(y) = \inf\{x\,; \quad f'(-x) \leq y\}\,.$$

Thus $g' = g + \sigma$, and $\Phi(g') = \Phi(f')$. Thus $\Phi(f') = \Phi(g+\sigma)$. Thus it suffices to show that if either $g \leq -2u$ (case $f(2u) = -\infty$) or $g \geq -\sigma + 2u$ (case $f(\sigma - 2u) = \infty$) we have $\Phi(g+\sigma) > \Phi(a+u)$.

Case $g \leq -2u$. Since $\Phi(g) = \Phi(a)$, we have $a \leq -2u$. We have

$$\begin{aligned}\Phi(g+\sigma) \geq \Phi(g+2u) &= \int \Phi(g(x)+2u)\,d\mu(x)\\ &= e^{2u}\int \Phi(g(x))\,d\mu(x)\\ &= e^{2u}\Phi(a) = \Phi(a+2u) > \Phi(a+u)\end{aligned}$$

since $\Phi(t+2u) = e^{2u}\Phi(t)$ for $t + 2u \leq 0$, and since $a \leq -2u$.

Case $g \geq -\sigma + 2u$. We define b by $\Phi(g+\sigma) = 1 - \Phi(b)$, so that $b \leq -2u$. Let $h = -(g+\sigma)$, so that $\Phi(h) = \Phi(b)$. Since $h \leq -2u$, the preceding case shows that $\Phi(h+2u) \geq \Phi(b+2u)$. But

$$\Phi(h+2u) = 1 - \Phi(g+\sigma-2u) \leq 1 - \Phi(g) = 1 - \Phi(a) = \Phi(-a)\,.$$

Thus $-a \geq b + 2u$, so that $-b \geq a + 2u$, and

$$\Phi(g+\sigma) = 1 - \Phi(b) = \Phi(-b) \geq \Phi(a+2u) > \Phi(a+u)\,. \qquad \square$$

We denote by $I =]\alpha,\beta[$ the interior of $f^{-1}(\mathbb{R})$. Thus $\alpha \leq 2u$, $\beta \geq \sigma - 2u$. We set $J = I - \sigma$. We note that

$$x < \alpha - \sigma \Rightarrow \overline{f}(x) = -\infty\,; \qquad x > \beta - \sigma \Rightarrow \overline{f}(x) = +\infty$$

$$x \in J \Rightarrow \overline{f}(x) \in \mathbb{R}\,.$$

We now start to exploit the fact that $\Phi(\overline{f})$ is as small as possible. For the convenience of notations, we write $(f+g)^-$ for $\overline{f+g}$.

Lemma 4.4. *Consider a bounded continuous function v from $\mathbb{R}$ to $\mathbb{R}$, that is zero outside I. Suppose that $\int v\varphi(f)\,d\mu \geq 0$ (resp. > 0). Then*

$$\limsup_{s\to 0_+} s^{-1}(\Phi((f+sv)^-) - \Phi(\overline{f})) \geq 0 \qquad (resp. > 0).$$

Proof: Since $\varphi \leq 1/2$, Φ is lipschitz. Since v is bounded, by dominated convergence we have

$$\int v\varphi(f)\,d\mu = \lim_{s\to 0} s^{-1}(\Phi(f+sv) - \Phi(f)).$$

Set

$$\delta = \int v\varphi(f)\,d\mu\,; \quad \gamma = \limsup_{s\to 0_+} s^{-1}(\Phi((f+sv)^-) - \Phi(\overline{f}).$$

and fix $\delta' < \delta$, $\gamma' > \gamma$. Thus for all s small enough we have

$$\Phi(f+sv) \geq \Phi(f) + \delta' s \tag{4.1}$$

$$\Phi((f+sv)^-) \leq \Phi(\overline{f}) + \gamma' s\,. \tag{4.2}$$

Consider the non-decreasing rearrangement g_s of $f+sv$, defined in Proposition 2.6. From that proposition follows that $\Phi(g_s) = \Phi(f+sv)$, $\Phi(\overline{g}_s) \leq \Phi((f+sv)^-)$. Thus, from (4.2), for s small enough, we have

$$\Phi(\overline{g}_s) \leq \Phi(\overline{f}) + s\gamma'\,. \tag{4.3}$$

Consider the number $t(s)$ such that $\Phi(g_s + t(s)) = \Phi(a)$, so that $g_s + t(s) \in \mathcal{F}$. Since $\Phi(\overline{f})$ is the minimum of $\Phi(\overline{g})$ for $g \in \mathcal{F}$, we have

$$\Phi(\overline{f}) \leq \Phi((g_s + t(s))^-) = \Phi(\overline{g}_s + t(s))\,.$$

Combining with (4.3), we see that

$$\Phi(\overline{g}_s + t(s)) - \Phi(\overline{g}_s) \geq -s\gamma'\,. \tag{4.4}$$

It is clear that $g_s + t(s)$ is the non-increasing rearrangement of $f + sv + t(s)$. Thus

$$\Phi(f) = \Phi(a) = \Phi(g_s + t(s)) = \Phi(f + sv + t(s))\,,$$

and by (4.1), for s small enough, we have

$$\Phi(f+sv) - \Phi(f + sv + t(s)) \geq \delta' s\,. \tag{4.5}$$

Since $\Phi(x) - \Phi(y) \leq x - y$ for $x > y$, we have, for all functions h

$$\forall t \geq 0\,, \qquad \Phi(h+t) - \Phi(h) \leq t\,. \tag{4.6}$$

We suppose now that $\delta' > 0$. Since Φ is increasing, (4.5) show that $t(s) \leq 0$, and using (4.6) for $t = -t(s)$, $h = f + sv + t(s)$, we get $-t(s) \geq \delta' s$, i.e. $t(s) \leq -\delta' s$. By (4.4) we now have

$$\begin{aligned} \gamma' &\geq \int s^{-1}(\Phi(\overline{g}_s(x)) - \Phi(\overline{g}_s(x) + t(s))\, d\mu(x) \\ &\geq \int s^{-1}(\Phi(\overline{g}_s(x)) - \Phi(\overline{g}_s(x) - \delta' s))\, d\mu(x)\,. \end{aligned}$$

For $A > 0$, $x, y \in [-A, A]$, $x < y$, we have

$$\Phi(y) - \Phi(x) \geq (x - y)\varphi(A)\,.$$

It clearly follows that for some constant B that depends on f, u only, we have $\gamma' \geq B\delta'$. Thus $\delta > 0$ implies $\gamma > 0$.

Suppose now that $\gamma' < 0$. By (4.4) we see now that $t(s) \geq 0$, and by (4.6) that $t(s) \geq (-\gamma')s$. By (4.5) we have

$$s^{-1}(\Phi(f + sv + (-\gamma')s) - \Phi(f + sv)) \leq -\delta'\,.$$

Letting $s \to 0$, we get by Fatou's lemma that

$$(-\gamma') \int \varphi(f)\, d\mu \leq -\delta'$$

so that $\gamma < 0$ implies $\delta < 0$, and thus $\delta \geq 0$ implies $\gamma \geq 0$. □

For $x \in J$, we define

$$A_x = \bigcap_{\varepsilon > 0} \text{ closure } \{y; \quad x \leq y \leq x + \sigma\,, \quad \overline{f}(x) \leq f(y) - \xi(y - x) + u + \varepsilon\}\,.$$

While the idea of the following result is well known, we provide a proof since we cannot find an exact reference.

Lemma 4.5. $\lim\limits_{s \to 0_+} s^{-1}((f + sv)^-(x) - \overline{f}(x)) = \sup\limits_{z \in A_x} v(z)$.

Proof: a) Consider $z \in A_x$, and $\varepsilon > 0$. Thus we can find y such that $x \leq y \leq x + \sigma$ and that

$$v(y) \geq v(z) - \varepsilon\,; \qquad \overline{f}(x) \leq f(y) - \xi(y - x) + u + \varepsilon\,.$$

Since

$$(f+sv)^{-}(x) \geq f(y)+sv(y)-\xi(y-x)+u\,,$$

we get

$$(f+sv)^{-}(x)-\overline{f}(x) \geq sv(y)-\varepsilon \geq sv(z)-\varepsilon(1+s)\,.$$

This holds for all $\varepsilon > 0$. Thus

$$(f+sv)^{-}(x)-\overline{f}(x) \geq sv(z)$$

and thus

$$\liminf_{s\to 0_+} s^{-1}((f+sv)^{-}(x)-\overline{f}(x)) \geq v(z)\,.$$

Since z is arbitrary in A_ε, we have

$$\liminf_{s\to 0_+} s^{-1}((f+sv)^{-}(x)-\overline{f}(x)) \geq \sup_{z\in A_x} v(z)\,.$$

b) Consider now $B > \sup_{z\in A_x} v(z)$. Since v is continuous, the set $\{v < B\}$ is a neighborhood of A_x. Thus, by definition of A_x we can find $\varepsilon > 0$ such that

$$f(y)-\xi(y-x)+u+\varepsilon \geq \overline{f}(x) \Rightarrow v(y) < B\,.$$

Distinguishing whether $v(y) < B$ or not, we have

$$\begin{aligned}(f+sv)^{-}(x) &= \sup\{f(y)-\xi(y-x)+u+sv(y)\,; \quad x \leq y \leq x+\sigma\}\\ &\leq \max\{\overline{f}(x)+sB\,, \quad \overline{f}(x)-\varepsilon+s\|v\|_\infty\}\end{aligned}$$

so that

$$s^{-1}((f+sv)^{-}(x)-\overline{f}(x)) \leq \max\left(B, \|v\|_\infty - \frac{\varepsilon}{s}\right)$$

and hence

$$\limsup_{s\to 0_+} s^{-1}((f+sv)^{-}(x)-\overline{f}(x)) \leq B$$

for all $B > \sup_{z\in A_x} v(z)$. □

Lemma 4.6. *Consider $x_1, x_2 \in J$, $x_1 < x_2$, $z_1 \in A_{x_1}$, $z_2 \in A_{x_2}$. Then $z_1 \leq z_2$.*

Proof: We first observe that for $z \in A_x$, there is a sequence (z_n) converging to z such that

$$\overline{f}(x) = \lim_{n\to\infty} f(z_n)-\xi(z_n-x)+u\,.$$

Since f is non-decreasing, setting $f^+(z) = \lim_{y \to z, y > z} f(y)$, we have

$$\overline{f}(x) \leq f^+(z) - \xi(z - x) + u\,. \tag{4.7}$$

We now argue by contradiction, and assume that $z_2 < z_1$. Thus $x_1 < x_2 \leq z_2 < z_1 \leq x_1 + \sigma$. For $x_1 \leq y \leq x_1 + \sigma$, we have

$$f(y) - \xi(y - x_1) + u \leq \overline{f}(x_1)$$

so that, letting $y \to z_2$, $y > z_2$ we get

$$f^+(z_2) - \xi(z_2 - x_1) + u \leq \overline{f}(x_1) \leq f^+(z_1) - \xi(z_1 - x_1) + u \tag{4.8}$$

where the second inequality follows from (4.7). A similar argument, using now the fact that $z_1 \leq x_1 + \sigma < x_2 + \sigma$ gives

$$f^+(z_1) - \xi(z_1 - x_2) + u \leq \overline{f}(x_2) \leq f^+(z_2) - \xi(z_2 - x_2) + u\,.$$

Adding with (4.8) gives

$$\xi(z_1 - x_1) + \xi(z_2 - x_2) \leq \xi(z_2 - x_1) + \xi(z_1 - x_2)\,.$$

so that

$$\xi(z_1 - x_2) - \xi(z_1 - x_1) \geq \xi(z_2 - x_2) - \xi(z_2 - x_1)\,.$$

But since ξ' increases strictly on $\mathbb{R}^+$, and since $x_1 < x_2$, the function $z \to \xi(z - x_2) - \xi(z - x_1)$ decreases strictly for $z > x_1$. This contradicts the fact that $z_1 > z_2$. □

Proposition 4.7. *For all $x \in J$, A_x has exactly one point.*

Proof: By definition, A_x is not empty. Suppose that for some $x \in J$, A_x contains two points. Then we can find $z_1 < z_2$ such that A_x contains a point $< z_1$ and a point $> z_2$. By Lemma 4.6, for $y \neq x$, we have $A_y \cap [z_1, z_2] = \emptyset$. Consider now a positive continuous function v that is supported by $[z_1, z_2]$. It follows from Lemma 4.5 that for $y \neq x$, we have

$$\lim_{s \to 0_+} s^{-1}\big((f + sv)^-(y) - \overline{f}(y)\big) = 0\,.$$

By dominated convergence, we have

$$\lim_{s \to 0_+} s^{-1}(\Phi((f + sv)^-) - \Phi(\overline{f})) = 0\,.$$

But since $\int v\varphi(f)\,d\mu > 0$, this contradicts Lemma 4.3. □

We define $A(x)$ by $A_x = \{A(x)\}$. We note that, by (4.7) we have

$$\overline{f}(x) \leq f^+(A(x)) - \xi(A(x) - x) + u\,. \tag{4.9}$$

Moreover, if f is continuous at $A(x)$, we have

$$f(x) = f(A(x)) - \xi(A(x) - x) + u\,. \tag{4.9'}$$

For $y \in I$, we set

$$w(y) = \sup\{x \in J\,; \quad A(x) \leq y\}\,.$$

Proposition 4.8. *For some constant C we have*

$$\forall y \in I\,, \quad \int_{\alpha-\sigma}^{w(y)} \varphi(\overline{f}(x))\,d\mu(x) = C\int_{\sigma}^{y} \varphi(f(x))\,d\mu(x)\,. \tag{4.10}$$

In particular w is one to one and continuous.

Proof: It follows from Lemma 4.5 and dominated convergence that, for a bounded continuous function v supported by J, we have

$$\lim_{s\to 0_+} s^{-1}(\Phi((f+sv)^-) - \Phi(\overline{f})) = \int_J v(A(x))\varphi(\overline{f}(x))\,d\mu(x)\,.$$

By Lemma 4.3, we have

$$\int_I v(x)\varphi(f(x))\,d\mu(x) = 0 \Rightarrow \int_J v(A(x))\varphi(\overline{f}(x))\,d\mu(x) = 0\,.$$

Thus, there exists a constant C such that for all continuous bounded functions v with support in I, we have

$$\int_J v(A(x))\varphi(\overline{f}(x))\,d\mu(x) = C\int_I v(x)\varphi(f(x))\,d\mu(x)\,.$$

If we approximate the indicator function of $]\alpha, y]$ by v we get

$$\int_B \varphi(\overline{f}(x))\,d\mu(x) = C\int_\alpha^y \varphi(f(x))\,d\mu(x)$$

where $B = \{x \geq \alpha - \sigma\,, \quad A(x) \leq y\}$.

We observe that $A(w(x)) = x$ for $x \in I$ and $w(A(x)) = x$ for $x \in J$.

We now define

$$Z = \{y \in I\,; \quad w(y) = y - \sigma\}\,.$$

Since w is continuous, Z is closed in I. Our objective is to prove that $Z = \emptyset$ (at which point the proof will be almost finished).

Lemma 4.9. *Suppose that f is not continuous at $y \in I$. Then $y \in Z$.*

Proof: Consider $z < y$. By (4.9) we have

$$\begin{aligned}\overline{f}(w(z)) &\le f^+(A(w(z))) - \xi(A(w(z)) - w(z)) + u \\ &\le f^+(z) - \xi(z - w(z)) + u \\ &< f^+(y) - \xi(y - w(z)) + u\end{aligned}$$

as soon as $(y-z)\|\xi'\|_\infty \le \frac{1}{L}(y-z) < f^+(y) - f^-(y)$. Thus, by definition of $\overline{f}$, we must have $w(z)+\sigma \le y$. Since w is continuous, letting $z \to y$ we have $w(y)+\sigma \le y$, so that $w(y) = y-\sigma$. □

Lemma 4.10. *For $x \in I$, we have $f(x+\sigma) \le \overline{f}(x) \le f(x+\sigma) + u$. Thus*

$$e^{-u}\varphi(f(x+\sigma)) \le \varphi(\overline{f}(x)) \le e^u \varphi(f(x+\sigma)) . \tag{4.11}$$

Proof: We have

$$\overline{f}(x) \ge f(x+\sigma) - \xi(\sigma) + u = f(x+\sigma)$$

and also

$$\overline{f}(x) \le \sup_{|y-x|\le\sigma} f(y) - \xi(y-x) + u \le f(x+\sigma) + u . \qquad \square$$

Lemma 4.11. *Consider $y_1 < y_2$, $y_1, y_2 \in Z \cup \{\alpha, \beta\}$. Then if we set $-|-\infty-\sigma|+|-\infty| = -\sigma$; $-|\infty-\sigma|+|\infty| = \sigma$, we have*

$$e^{-u-|y_1-\sigma|+|y_1|} \le C \le e^{u-|y_2-\sigma|+|y_2|} .$$

Proof: By (4.10) we have

$$\int_{y_1-\sigma}^{y_2-\sigma} \varphi(\overline{f}(x))\,d\mu(x) = C\int_{y_1}^{y_2} \varphi(f(x))\,d\mu(x) .$$

Thus, by (4.11),

$$\begin{aligned}C\int_{y_1}^{y_2} \varphi(f(x))\,d\mu(x) &\le e^u \int_{y_1-\sigma}^{y_2-\sigma} \varphi(f(x+\sigma))\varphi(x)\,dx \\ &\le e^u \int_{y_1}^{y_2} \varphi(f(x))\varphi(x-\sigma)\,dx \\ &\le e^u \sup_{y_1\le x\le y_2} \frac{\varphi(x-\sigma)}{\varphi(x)} \int_{y_1}^{y_2} \varphi(f(x))\,d\mu(x) .\end{aligned}$$

Thus

$$\begin{aligned}C &\le e^u \sup\{e^{-|x-\sigma|+|x|};\quad y_1 \le x \le y_2\} \\ &\le e^{u-|y_2-\sigma|+|y_2|}\end{aligned}$$

since the function $x \to -|x-\sigma|+|x|$ is non-decreasing. The other inequality is proved in as similar way. □

Corollary 4.12. *For $y_1, y_2 \in Z$, we have*

$$-|y_1-\sigma|+|y_1| \leq 2u + (-|y_2-\sigma|+|y_2|)\,.$$

(Note that we no longer assume $y_1 \leq y_2$. The case of interest is actually $y_1 > y_2$.)

Proof: We use Lemma 4.11 for the pair α, y_2, and then for the pair y_1, β. □

Lemma 4.13. *If $y \in I$, $y \notin Z$, we have*

$$\limsup_{z \to y+} \frac{f(z)-f(y)}{z-y} \leq \xi'(\sigma)\,.$$

Proof: By Lemma 4.9, f is continuous at y. Thus for $|z-w(y)|<\sigma$, we have

$$f(z)-\xi(z-w(y))+u \leq \overline{f}(w(y)) = f(y)-\xi(y-w(y))+u\,.$$

Using this for $y < z < w(y)+\sigma$, we get

$$\frac{f(z)-f(y)}{z-y} \leq \frac{\xi(z-w(y))-\xi(y-w(y))}{z-y}\,.$$

This implies the result since $\xi'(y-w(y)) \leq \xi'(\sigma)$ as ξ' increases on $\mathbb{R}^+$. □

Lemma 4.14. *$\sigma\xi'(\sigma) \leq 2u$.*

Proof:

$$\begin{aligned} u = \xi(\sigma) &= \frac{1}{L}\int_0^\sigma \frac{x}{1+x}\,dx \geq \frac{1}{L(1+\sigma)}\int_0^\sigma x\,dx \\ &= \frac{1}{2L}\frac{\sigma^2}{1+\sigma} = \frac{1}{2}\sigma\xi'(\sigma)\,. \end{aligned}$$ □

Lemma 4.15. *Suppose $[x, A(x)] \cap Z = \emptyset$. Then*

$$f(x)+u \leq \overline{f}(x) \leq f(x)+3u\,.$$

Proof: Since $\xi(0)=0$, we have $\overline{f}(x) \geq f(x)+u$. Since $\overline{f}$ is continuous at $A(x)$ by Lemma 4.9, we have by (4.9′) that

$$\begin{aligned} \overline{f}(x) &= f(A(x))-\xi(A(x)-x)+u \\ &\leq f(A(x))+u\,. \end{aligned}$$

By Lemmas 4.13, 4.14, we have

$$f(A(x)) \leq f(x)+(A(x)-x)\xi'(\sigma) \leq f(x)+\sigma\xi'(\sigma) \leq f(x)+2u\,.$$ □

Lemma 4.16. a) *Suppose that for some $\alpha' > \alpha$, we have $(\alpha, \alpha') \cap Z = \emptyset$. Then $\alpha = -\infty$ and $C \le e^{3u}$.*

b) *Suppose that for some $\beta' < \beta$, we have $(\beta', \beta) \cap Z = \emptyset$. Then $\beta = \infty$ and $C \ge e^{-3u}$.*

Proof: We prove only a) since b) is similar. The fact that $\alpha = -\infty$ follows from Lemmas 4.9 and 4.13. By Proposition 4.8 we have

$$C \int_{-\infty}^{\alpha'} \varphi(f(x))\, d\mu(x) = \int_{-\infty}^{w(\alpha')} \varphi(\overline{f}(x))\, d\mu(x)\,.$$

Now for $x < w(\alpha')$, we have $A(x) < \alpha'$, so that $|\overline{f}(x) - f(x)| \le 3u$ by Lemma 4.15 and $\varphi(\overline{f}(x)) \le e^{3u} \varphi(f(x))$. Thus

$$\begin{aligned} C \int_{-\infty}^{\alpha'} \varphi(f(x))\, d\mu(x) &\le e^{3u} \int_{-\infty}^{w(\alpha')} \varphi(f(x))\, d\mu(x) \\ &\le e^{3u} \int_{-\infty}^{\alpha'} \varphi(f(x))\, d\mu(x)\,. \end{aligned}$$

□

Proposition 4.17. *Suppose that $L \ge 30$. Then $Z = \emptyset$.*

Proof: **Step 1** We have $\xi(x) < |x|/L$, so that $\sigma > Lu \ge 30u$. Since $\alpha \le 2u$, $\beta \ge \sigma - 2u$, the value of the function $-|x - \sigma| + |x|$ at β (resp. α) is at least $\sigma - 4u$ (resp. at most $-\sigma + 4u$). Thus Corollary 4.12 shows that α, β cannot be both cluster points of Z, for otherwise, we would have

$$\sigma - 4u \le 2u + (-\sigma + 4u)\,.$$

For definiteness, we assume that $\alpha \notin Z$. By Lemma 4.16 a), we have $\alpha = -\infty$, $C \le e^{3u}$. By Lemma 4.11, we have $-|y - \sigma| + |y| \le 4u$ for $y \in Z$, so that $y \le \sigma/2 + 2u$. Since $\beta \ge \sigma - u$, and $\sigma > 6u$, β cannot be a cluster point of Z. By Proposition 4.15 b), we have $\beta = \infty$, $C \ge e^{-3u}$. Thus, by Lemma 4.11, we have $-4u \le -|y - \sigma| + |y|$ for $y \in Z$, so that $z \ge \sigma/2 - 2u$. Thus $Z \subset \left[\frac{\sigma}{2} - 2u\,, \quad \frac{\sigma}{2} + 2u\right]$.

Step 2 Suppose now that $Z \ne \emptyset$. Consider the smallest point $z \in Z$. We have

$$\int_{-\infty}^{z-\sigma} \varphi(\overline{f}(x))\, d\mu(x) = C \int_{-\infty}^{z} \varphi(f(x))\, d\mu(x)\,.$$

By Lemma 4.14, we have $\varphi(\overline{f}(x)) \le e^{3u} \varphi(f(x))$ for $x \le z - \sigma$; since $z \ge 0$, we have

$$C \int_{-\infty}^{0} \varphi(f(x))\, d\mu(x) \le e^{3u} \int_{-\infty}^{z-\sigma} \varphi(f(x))\, d\mu(x)\,.$$

By Lemma 4.13, for $x \leq 0$, we have $|f(x) - f(0)| \leq |x|\xi'(\sigma)$, so that

$$\varphi(f(x)) \geq e^{x\xi'(\sigma)}\varphi(f(0)).$$

For $x \leq z - \sigma$, we have $|f(x) - f(z-\sigma)| \leq |z - \sigma - x|\xi'(\sigma)$, again by Lemma 4.13, so that

$$\varphi(f(x)) \leq e^{(z-\sigma-x)\xi'(\sigma)}\varphi(f(z-\sigma)).$$

The proof of Lemma 3.6 then shows that (since $z - \sigma \leq 0$)

$$\int_{-\infty}^{0} \varphi(f(x))\, d\mu(x) \geq \frac{1}{2}(1 + \xi'(\sigma))^{-1}\varphi(f(0))$$

$$\int_{-\infty}^{z-\sigma} \varphi(f(x))\, d\mu(x) \leq \frac{1}{2}(1 - \xi'(\sigma))^{-1}e^{-|z-\sigma|}\varphi(f(z-\sigma)).$$

Thus

$$C \leq e^{3u}(1 + \xi'(\sigma))(1 - \xi'(\sigma))^{-1}e^{-|z-\sigma|}\frac{\varphi(f(z-\sigma))}{\varphi(f(0))}.$$

Now, since $|z - \sigma| \leq \sigma$

$$|f(z-\sigma) - f(0)| \leq \sigma\xi'(\sigma) \leq 2u$$

so that $\varphi(f(z-\sigma)) \leq e^{2u}\varphi(f(0))$. Since $|z - \sigma| \geq \sigma/2 - 2u$, we have

$$C \leq e^{7u-\sigma/2}\frac{1 + \xi'(\sigma)}{1 - \xi'(\sigma)}.$$

We note that $\frac{1+x}{1-x} \leq 1 + 3x$ for $x \leq 1/3$. Since $\xi'(\sigma) \leq 1/L$, and since $\xi'(\sigma) \leq \sigma/L$, we get since $\sigma \geq Lu$

$$\begin{aligned} C &\leq e^{7u-\sigma/2+3\xi'(\sigma)} \leq e^{7u-\sigma\left(\frac{1}{2}-\frac{3}{L}\right)} \\ &\leq e^{7u-Lu\left(\frac{1}{2}-\frac{3}{L}\right)}. \end{aligned}$$

Since $L \geq 30$, this contradicts the fact that $C \geq e^{-3u}$. □

We can now finish the proof of Theorem 1.1.

Proposition 4.18. *Suppose that $Z = \emptyset$. Then $\overline{f} = \widehat{f} + u$.*

Proof: We have shown that when $Z = \emptyset$, $I = \mathbb{R}$. By Lemma 4.13, for $y \geq x + \sigma$, we have, since ξ' is increasing

$$f(y) - \xi(y - x) \leq f(x + \sigma) - \xi(\sigma)$$

so that

$$f(x + \sigma) - \xi(\sigma) + u \geq f(y) - \xi(y - x) + u.$$

Thus

$$\widehat{f}(x) + u = \sup_{y} f(y) - \xi(y - x) + u \leq \sup_{y \leq x+\sigma} f(y) - \xi(y - x) + u = \overline{f}(x). \quad □$$

5. Proof of Theorem 1.2

For a compact set A of $\mathbb{R}^n$, we set

$$h_A(x) = \inf\left\{u \geq 0 \ ; \ x \in A + V_n(u)\right\} .$$

To prove Theorem 1.2, it suffices, by (2.0) and an obvious approximation argument, to show that for all n we have

$$\int_{\mathbb{R}^n} \exp h_A(x) d\mu^n(x) \leq 2/\mu^n(A) , \qquad (H_n)$$

provided the parameter L has been chosen large enough.

The proof of (H_n) will be by induction over n. We first consider the case $n = 1$. Define a by $\mu(A) = e^{-a}$. Since $\mu([-a,a]) = 1-e^{-a}$, there exists $b \in A$ with $|b| \leq a$. Thus $|x-b| \leq |x|+a$ and since $\xi(u) \leq u/L$, we have $h_A(x) \leq (|x| + a)/L$. Thus, if we assume $L \geq 2$, we have

$$\begin{aligned}\int_{\mathbb{R}} \exp h_A(x) d\mu(x) &\leq \tfrac{1}{2} \int_{\mathbb{R}} e^{(|x|+a)/L} e^{-|x|} dx \\ &= \frac{1}{1-1/L} e^{a/L} \leq 2e^a = 2/\mu(A) .\end{aligned}$$

This proves (H_1). Let us point out that a more cautious computation using e.g. Proposition 2.2 allows the removal of the factor 2 in the right hand side of (H_1) (and, as the proof will show, also in the right hand side of Theorem 1.2).

Consider now a compact subset A of $\mathbb{R}^{n+1}$. For $x \in \mathbb{R}$, we denote by A_x the set $\{z \in \mathbb{R}^n \ ; \ (z,x) \in A\}$. Consider $x' \in \mathbb{R}$, $z \in \mathbb{R}^n$, and u such that $z \in A_{x'} + V_n(u)$. Then, by definition of $V_{n+1}(u)$, we have, for all $x \in \mathbb{R}$ that

$$(z,x) \in A + V_{n+1}\big(u + \xi(x-x')\big) .$$

Thus we have

$$h_A\big((z,x)\big) \leq \xi(x-x') + h_{A_{x'}}(z)$$

and hence

$$\exp h_A\big((z,x)\big) \leq \exp \xi(x-x') \exp h_{A_{x'}}(z) .$$

By the induction hypothesis, we have

$$\int_{\mathbb{R}^n} \exp h_A\big((z,x)\big) d\mu^n(z) \leq 2 \frac{e^{\xi(x-x')}}{\mu^n(A_{x'})} .$$

Thus

$$\int_{\mathbb{R}^n} \exp h_A\big((z,x)\big) d\mu^n(z) \leq 2 \inf_{x' \in \mathbb{R}} \frac{e^{\xi(x-x')}}{\mu^n(A_{x'})} .$$

By Fubini theorem, we have

$$\int_{\mathbb{R}^{n+1}} e^{h_A(y)} d\mu^{n+1}(y) \leq 2 \int_{\mathbb{R}} \inf_{x' \in \mathbb{R}} \frac{e^{\xi(x-x')}}{\mu^n(A_{x'})} d\mu(x) . \tag{5.1}$$

Consider the function $g(x)$ given by $\exp g(x) = \mu^n(A_x)$. By Fubini theorem, we have $\int \exp g(x) d\mu(x) = \mu^{n+1}(A)$. On the other hand, the right hand side of (5.1) is

$$2 \int_{\mathbb{R}} e^{-\widehat{g}(x)} d\mu(x)$$

where $\widehat{g}(x) = \sup_{x' \in \mathbb{R}} \big(g(x') - \xi(x - x')\big)$. Thus to prove that $(H_n) \Rightarrow (H_{n+1})$, it suffices to show that

$$\int_{\mathbb{R}} e^{-\widehat{g}(x)} d\mu(x) \int_{\mathbb{R}} e^{g(x)} d\mu(x) \leq 1 .$$

The basic inequality (3.2) shows that this it true when g is non-decreasing. To reduce the general case to that case, consider the non-decreasing rearrangement f of g given by (2.4). Since $\mu(\{f \geq y\}) = \mu(\{g \geq y\})$ for all $y \in \mathbb{R}$, we have $\int_{\mathbb{R}} e^{g(x)} d\mu(x) = \int_{\mathbb{R}} e^{f(x)} d\mu(x)$. It suffices to show that for all t,

$$\mu(\{\widehat{f}(x) > y\}) \leq \mu(\{\widehat{g}(x) > y\}) .$$

The argument to prove this is identical to that of Proposition 2.6. Theorem 1.2 is proved.

Finally, we prove that even a weak form of the concentration of measure property can hold only for powers of measures with exponential tails.

Proposition 5.1. *Consider a probability θ on $\mathbb{R}$, and its power θ^∞ on $\mathbb{R}^{\mathbb{N}}$. Assume that there exists $u_0 \in \mathbb{R}$ with the following property. For each Borel set $A \subset \mathbb{R}^{\mathbb{N}}$*

$$\theta^\infty(A) \geq 1/2 \Rightarrow \theta^\infty(A + u_0 B_\infty) \geq \frac{3}{4} ,$$

where $B_\infty = \{x \in \mathbb{R}^{\mathbb{N}}; \forall k \geq 1, |x_k| \leq 1\}$. Then $\theta(\{|x| \geq u\}) \leq K \exp(-u/K)$.

Comment. The choice of the number $\frac{3}{4}$ is not magical. Any number $> \frac{1}{2}$ would serve the same purpose.

Proof: Consider $u > 0$. Let n be the smallest integer such that $\theta([-u, u])^n \geq 1/2$. Set

$$A = \{x \in \mathbb{R}^{\mathbb{N}} \ ; \ \forall k \leq n \ , \ |x_k| \leq u\}$$

so that $\theta^\infty(A) \geq 1/2$. Then

$$A + u_0 B_\infty = \{x \in \mathbb{R}^{\mathbb{N}}; \quad \forall k \leq n, \quad |x_k| \leq u + u_0\} .$$

Thus

$$\theta^{\infty}(A+u_0B_{\infty}) = \theta\big([-u-u_0, u+u_0]\big)^n \geq \tfrac{3}{4}\ .$$

Thus, we have

$$\theta\big(\{|x|>u\}\big) \leq 1-\big(\tfrac{1}{2}\big)^{1/n} \Rightarrow \theta\big(\{|x|>u+u_0\}\big) \leq 1-\big(\tfrac{3}{4}\big)^{1/n}\ .$$

Since $1-\big(\frac{1}{2}\big)^{1/n}$ is of order $\frac{1}{n}\log 2$, while $1-\big(\frac{3}{4}\big)^{1/n}$ is of order $\frac{1}{n}\log\frac{4}{3}$, it follows that for u large enough we have

$$\theta\big(\{|x|\geq u+u_0\}\big) \leq \gamma\theta\big(\{|x|\geq u\}\big)\ ,$$

where $\gamma < 1$. The result follows easily. □

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