

Random embedding of ℓ_p^n into ℓ_r^N

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Received: 25 January 2010 / Revised: 8 July 2010 / Published online: 21 October 2010
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Abstract For any $0 < p < 2$ and any natural numbers $N > n$, we give an explicit definition of a random operator $S : \ell_p^n \rightarrow \mathbb{R}^N$ such that for every $0 < r < p < 2$ with $r \leq 1$, the operator $S_r = S : \ell_p^n \rightarrow \ell_r^N$ satisfies with overwhelming probability that $\|S_r\| \|(S_r)|_{\text{Im } S}^{-1}\| \leq C(p, r)^{n/(N-n)}$, where $C(p, r) > 0$ is a real number depending only on p and r . One of the main tools that we develop is a new type of multidimensional Esseen inequality for studying small ball probabilities.

Mathematics Subject Classification (2000) Primary 60E07 · 46B20 · 46B09; Secondary 52A21

1 Introduction

A famous result of Dvoretzky [6] says that ℓ_2^n is uniformly represented in any infinite dimensional Banach space X , which means that for any $\varepsilon > 0$ and any natural number n , $\ell_2^n \xrightarrow{1+\varepsilon} X$. For any $1 \leq p < 2$, even if it is impossible to embed ℓ_p^n uniformly in every Banach space, Maurey and Pisier [22] proved that ℓ_p^n is uniformly represented in a Banach space X if and only if X is not of stable-type p . It is possible to quantify these results when X is of finite dimension. Milman [23] proved that if E is a normed space

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of dimension N , then for any $\varepsilon \in (0, 1]$, $\ell_2^n \xrightarrow{1+\varepsilon} E$, where n depends only on ε and on a geometric parameter associated to E . If it is applied in the case of $E = \ell_1^N$, it tells that for any $\varepsilon > 0$, $\ell_2^n \xrightarrow{1+\varepsilon} \ell_1^N$, where $N = C(\varepsilon)n$ and $C(\varepsilon)$ is a function depending only on ε . Johnson and Schechtman [12] proved that for any $0 < r \leq 1$ and $0 < r < p < 2$, for any $\varepsilon \in (0, 1]$, $\ell_p^n \xrightarrow{1+\varepsilon} \ell_r^N$, where $N = C(p, r, \varepsilon)n$. Later, Pisier [25] gave a different proof and extended their result to the case of a general finite dimensional normed space E of dimension N , proving that for any $\varepsilon > 0$, $\ell_p^n \xrightarrow{1+\varepsilon} E$, where n depends only on ε and on the stable-type constant of E . All these proofs are random. Typically for Euclidean subspaces, it is possible to use matrices defined by Gaussian vectors, while for ℓ_p^n subspaces, the matrices are more complicated and defined by “approximating” p -stable vectors, for which there is no hope to get good properties of concentration around their mean. However, even if ε is taken the largest possible, say equal to 1, it is not possible to deduce the existence of an operator of rank say $\lfloor N/2 \rfloor$ satisfying the desired property. In the Euclidean case, this is a theorem of Kashin [15], who proved that for any $\eta > 0$, for any natural number n , $\ell_2^n \xrightarrow{C(\eta)} \ell_1^N$, where $N = (1 + \eta)n$ and $C(\eta)$ depends only on η . This result was generalized to the case of normed spaces with bounded volume ratio by Szarek [30], Szarek and Tomczak-Jaegermann [31]. The main subject of this paper is to prove a Kashin-type theorem for embeddings from ℓ_p^n into ℓ_r^N , where $0 < r \leq 1$, $0 < r < p < 2$, $N = (1 + \eta)n$ and $\eta > 0$. In the case $r = 1$, Naor and Zvavitch [24] proved that for any $\eta \in (0, 1)$ $\ell_p^n \xrightarrow{C(\log n, \eta)} \ell_1^{(1+\eta)n}$, where $C(\log n, \eta) = (c \log n)^{(1-\frac{1}{p})(1+\frac{1}{\eta})}$. It is important to note that they provide an explicit definition of a random operator which satisfies the desired property. Subsequently Johnson and Schechtman [13] proved that for any $1 \leq r < p < 2$, there exists an operator $T : \ell_p^n \rightarrow \ell_r^{(1+\eta)n}$, such that $\|T\| \|T_{\text{Im} T}^{-1}\| \leq C(\eta)$. However, the proof depends heavily on a result of Bourgain et al. [2] based on a theorem of Elton [7], which is valid only in ℓ_1^N . Moreover, it doesn't give any explicit construction of a random operator T . Also, the result of Naor and Zvavitch has been extended by Bernués and López-Valdes [1] who proved that $\ell_p^n \xrightarrow{C(\log n, \eta, r)} \ell_r^{(1+\eta)n}$ when $r \leq 1$.

Even if it is not the main object of that paper, there is an important related subject concerning embedding subspaces of L_p into L_r , which started with the work of [5, 4, 26, 20]. Of course, this was extended to the finite dimensional setting and we refer the reader to [3, 32, 33], where the embedding of general finite dimensional subspaces of L_p into ℓ_r^N were studied, and to the survey [14]. The notions of type and cotype are important in this theory and we refer the interested reader to the survey [21].

Before stating our main result, we need some notations. For any $s \in (0, +\infty)$, and any $d \in \mathbb{N}$, the space ℓ_s^d is the real space \mathbb{R}^d equipped with the following norm:

$$|x|_s = \left(\sum_{i=1}^d |x_i|^s \right)^{1/s}, \quad \forall x \in \mathbb{R}^d.$$

Let $0 < r < p < 2$ with $r \leq 1$ and $N \geq n$ be natural numbers. Let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of \mathbb{R}^N , Y always denotes a random vector taking the values

$\{\pm e_1, \dots, \pm e_N\}$ with probability $\frac{1}{2N}$. Let $(Y_{i,j})$ be a sequence of independent copies of Y , where $1 \leq i \leq n, j \in \mathbb{N}$. We define the following operator:

$$S : \ell_p^n \rightarrow \mathbb{R}^N \tag{1}$$

$$\alpha = (\alpha_1, \dots, \alpha_n) \mapsto \sum_{i=1}^n \alpha_i \sum_{j \geq 1} \frac{1}{j^{1/p}} Y_{i,j}.$$

Theorem 1.1 *Let $1 < p < 2$, there exist positive real numbers C_p and c_p , such that for any $\eta > 0$, and any natural numbers $n, N = (1 + \eta)n$*

$$\mathbb{P} \left\{ \forall 0 < r \leq 1 : \|S_r\| \|S_r^{-1}\| \leq C_p^{1/\eta r} \right\} \geq 1 - c \exp(-c_p n),$$

where S_r is the operator $S : \ell_p^n \rightarrow \ell_r^N$, $\|S_r^{-1}\|$ is the norm of the operator S_r^{-1} restricted to the range of S and c is an absolute positive constant.

Moreover, if $0 < p \leq 1$ then for any $0 < r_2 < p$, there exist positive real numbers C_{p,r_2} and c_{p,r_2} , such that for any $\eta > 0$, and any natural numbers $n, N = (1 + \eta)n$

$$\mathbb{P} \left\{ \forall 0 < r < r_2 : \|S_r\| \|S_r^{-1}\| \leq C_{p,r_2}^{1/\eta r} \right\} \geq 1 - c \exp(-c_{p,r_2} n),$$

where c is an absolute positive constant.

This gives an explicit expression of a random operator that doesn't depend on r and satisfies the desired conclusion with overwhelming probability, for the full range of values of r . It may be surprising that these operators are already defined in [25] for the almost isometric result. Moreover, it solves completely the question of extending the theorem of [12] to a Kashin-type setting.

The proof of Theorem 1.1 is based on a splitting of the unit sphere of ℓ_p^n into subsets. In the case $p = 2$, this idea appeared in [18, 29], and was deeply developed in [27, 28] to study the smallest singular value of some random operators. After splitting the unit sphere of ℓ_p^n into two subsets, we need to study a new type of small ball estimate. For a real random variable, it is common to use an inequality due to Esseen [8]. Several multi-dimensional versions of this result are known (see [11, 34, 28, 10]). However, in our situation, none of them seem to give the inequalities we need. Theorem 3.1 is another type of multidimensional Esseen inequality that is at the heart of our proof.

The organization of the paper is as follows. In Sect. 2, we present some known tools about p -stable random vectors and about martingale inequalities that are needed later. In Sect. 3, we prove the new multidimensional Esseen-type inequality, which is particularly adapted to our context because of the main technical Lemma 4.7. Section 4 contains the proof of Theorem 1.1.

2 Preliminary results

We need several consequences of well-known results about p -stable random variables. We refer the reader to Chapter 5 of the book [16]. We recall that a real-valued symmetric

random variable θ is called p -stable for $p \in (0, 2]$ if its characteristic function is as follows: for some $\sigma \geq 0$, $\mathbb{E} \exp(it\theta) = \exp(-\sigma|t|^p)$, for any real t . When $\sigma = 1$, we say that θ is standard. Stable random variables are characterized by their fundamental “stability” property: if (θ_i) is a standard p -stable sequence, for any finite sequence (α_i) of real numbers, $\sum_i \alpha_i \theta_i$ has the same distribution as $(\sum_i |\alpha_i|^p)^{1/p} \theta_1$, in particular, for any $r < p$,

$$\left(\mathbb{E} \left| \sum_i \alpha_i \theta_i \right|^r \right)^{1/r} = s_{p,r} \left(\sum_i |\alpha_i|^p \right)^{1/p}, \tag{2}$$

where $s_{p,r} > 0$ depends only on p and r . A random vector $\Theta \in \mathbb{R}^N$ is called p -stable if for any vector $\xi \in \mathbb{R}^N$, $\langle \xi, \Theta \rangle$ is a p -stable random variable. Let (λ_i) be independent random variables with common exponential distribution $\mathbb{P}\{\lambda_i > t\} = \exp(-t)$, $t \geq 0$. Set $\Gamma_j = \sum_{i=1}^j \lambda_i$, for $j \geq 1$. We recall that Y is the random vector taking the values $\{\pm e_1, \dots, \pm e_N\}$ with probability $\frac{1}{2N}$.

Lemma 2.1 *The random vector $\tilde{\Theta} = \sum_{j \geq 1} \Gamma_j^{-1/p} Y_j$, where Y_j are independent copies of Y , is a p -stable random vector. Moreover, there is a number $s_p > 0$ depending only on p , such that $\tilde{\Theta}$ has the same distribution as $\frac{s_p}{N^{1/p}} \sum_{\ell=1}^N \theta_\ell e_\ell$, where $\theta_1, \dots, \theta_N$ are independent real-valued symmetric standard p -stable random variables.*

This result follows directly from [17]. More generally, it is known that if $(X_j)_{j \geq 1}$ is an i.i.d. sequence of symmetric random vectors in \mathbb{R}^N , then the random vector $S = \sum_{j \geq 1} \Gamma_j^{-1/p} X_j$ is p -stable, and for any $\xi \in \mathbb{R}^N$ we have

$$\mathbb{E} \exp(i \langle \xi, S \rangle) = \exp \left(-\mathbb{E} |\langle \xi, X_1 \rangle|^p / s_p^p \right).$$

For more information, we refer the reader to [19].

Let $\sigma_{p,r}$ be such that $\sigma_{p,r} s_p s_{p,r} = 1$, where $s_p, s_{p,r}$ are the constants above. Now, we define the main random operator

$$\begin{aligned} T : \ell_p^n &\rightarrow \ell_r^N \\ \alpha = (\alpha_1, \dots, \alpha_n) &\mapsto \frac{\sigma_{p,r}}{N^{1/q}} \sum_{i=1}^n \alpha_i \sum_{j \geq 1} \frac{1}{j^{1/p}} Y_{i,j}, \end{aligned} \tag{3}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and the following auxiliary operator:

$$\begin{aligned} \tilde{T} : \ell_p^n &\rightarrow \ell_r^N \\ \alpha = (\alpha_1, \dots, \alpha_n) &\mapsto \frac{\sigma_{p,r}}{N^{1/q}} \sum_{i=1}^n \alpha_i \sum_{j \geq 1} \Gamma_j^{-1/p} Y_{i,j}. \end{aligned}$$

Lemma 2.2 *Let $0 < r < p < 2$. For any $\alpha \in \ell_p^n$, we have $\mathbb{E} |\tilde{T}\alpha|_r^r = |\alpha|_p^r$.*

Proof In view of the definition of $\tilde{\Theta}$, $\tilde{T}\alpha$ has the same distribution as $\frac{\sigma_{p,r}}{N^{1/q}} \sum_{i=1}^n \alpha_i \tilde{\Theta}_i$, where $\tilde{\Theta}_i$ are independent copies of $\tilde{\Theta}$. Therefore, by property (2) and Lemma 2.1 we have

$$\begin{aligned} \mathbb{E}|\tilde{T}\alpha|_r^r &= \mathbb{E} \left| \frac{\sigma_{p,r}}{N^{1/q}} \sum_{i=1}^n \alpha_i \sum_{j \geq 1} \Gamma_j^{-1/p} Y_{i,j} \right|_r^r = \frac{\sigma_{p,r}^r}{N^{r/q}} \mathbb{E} \left| \sum_{i=1}^n \alpha_i \tilde{\Theta}_i \right|_r^r \\ &= \frac{\sigma_{p,r}^r}{N^{r/q}} \frac{s_p^r}{N^{r/p}} \mathbb{E} \left(\sum_{\ell=1}^N \left| \sum_{i=1}^n \alpha_i \theta_{i,\ell} \right|_p^r \right) = \sigma_{p,r}^r s_p^r s_{p,r}^r |\alpha|_p^r = |\alpha|_p^r, \end{aligned}$$

since $\theta_{i,\ell}$ are independent real-valued symmetric standard p -stable random variables, $\sigma_{p,r} s_p s_{p,r} = 1$ and the definition of q . □

The next two lemmas are analogous to the main lemmas in [25]. The first one uses the fact that, on the average, $\tilde{\Theta}$ behaves very much like the series $\sum_{j \geq 1} j^{-1/p} Y_j$. The second one follows from results about scalar martingale difference (see [12, Proposition 2]).

Lemma 2.3 *Let $0 < r < p < 2$ be such that $r > \frac{2p}{p+2}$. There exists a positive real number $D_{p,r}$ depending only on p and r , such that for any $\alpha \in \ell_p^n$, we have*

$$\left| \mathbb{E}|T\alpha|_r^r - |\alpha|_p^r \right| \leq D_{p,r} \left(\frac{n}{N} \right)^{r/q} |\alpha|_p^r.$$

Proof If we denote by f_1, \dots, f_j the functions $f_1(t) = \dots = f_j(t) = e^{-t} 1_{t \geq 0}$ then it is well-known that $f_1 \star \dots \star f_j(t) = \frac{t^{j-1}}{(j-1)!} e^{-t} 1_{t \geq 0}$. Hence for any $x > 0$

$$\mathbb{P} \{ \Gamma_j < x \} = \int_0^x \frac{u^{j-1}}{(j-1)!} \exp(-u) du.$$

Therefore,

$$\sum_{j \geq 1} \mathbb{E} |j^{-1/p} - \Gamma_j^{-1/p}|^r = \int_0^\infty \sum_{j \geq 1} |j^{-1/p} - u^{-1/p}|^r \frac{u^{j-1}}{(j-1)!} \exp(-u) du.$$

Using Stirling’s formula, there exists a positive real number $a > 0$ such that for any $j \geq 1$, $\frac{j^j}{j!} \leq a \frac{\exp(j)}{\sqrt{j}}$. By the change of variable $u/j = t$, we get

$$\sum_{j \geq 1} \mathbb{E} |j^{-1/p} - \Gamma_j^{-1/p}|^r \leq a \int_0^\infty |1 - t^{-1/p}|^r \frac{1}{t} \sum_{j \geq 1} \left(\frac{t}{\exp(t-1)} \right)^j j^{1/2-r/p} dt.$$

Observe that $p > r > \frac{2p}{p+2} > \frac{p}{2}$ and that $-1/2 < 1/2 - r/p < 0$. The integral clearly converges near ∞ and near 0. When t is close to 1, $\frac{t}{\exp(t-1)} \underset{h \rightarrow 0}{\sim} 1 - h^2$, where $t = 1 + h$. By comparing series with integrals, it is easy to deduce that there is a real number $K_{p,r}$ depending only on r and p such that for t close to 1, we have

$$\sum_{j \geq 1} \left(\frac{t}{\exp(t-1)} \right)^j j^{1/2-r/p} \leq K_{p,r} \left(\frac{1}{\ln \left(\frac{\exp(t-1)}{t} \right)} \right)^{3/2-r/p} \underset{h \rightarrow 0}{\sim} K_{p,r} h^{2r/p-3}.$$

From this inequality, it is clear that for $r > \frac{2p}{p+2}$, the integral converges near 1. Hence, for any $0 < \frac{2p}{p+2} < r < p < 2$, there exists a real positive number $D'_{p,r}$, such that

$$\sum_{j \geq 1} \mathbb{E} |j^{-1/p} - \Gamma_j^{-1/p}|^r \leq D'_{p,r}.$$

Therefore, by Lemma 2.2 and the equation above, we get that

$$\begin{aligned} \left| \mathbb{E} |T\alpha|_r^r - |\alpha|_p^r \right| &= \left| \mathbb{E} |T\alpha|_r^r - \mathbb{E} |\tilde{T}\alpha|_r^r \right| \\ &\leq \frac{\sigma_{p,r}^r}{N^{r/q}} \sum_{i=1}^n |\alpha_i|^r \sum_{j \geq 1} \mathbb{E} |j^{-1/p} - \Gamma_j^{-1/p}|^r |Y_{i,j}|_r^r \\ &\leq \frac{\sigma_{p,r}^r}{N^{r/q}} \sum_{i=1}^n |\alpha_i|^r D'_{p,r} \leq \frac{\sigma_{p,r}^r}{N^{r/q}} D'_{p,r} |\alpha|_p^r n^{r/q} \leq D_{p,r} \left(\frac{n}{N} \right)^{r/q} |\alpha|_p^r. \end{aligned}$$

This is the announced result. □

Lemma 2.4 *Let $0 < p/2 < r < p < 2$. There exists a positive real number $b_{p,r}$ depending only on p and r , such that for any $\alpha \in \ell_p^n$ with $|\alpha|_p = 1$, and any $t > 0$, we have*

$$\mathbb{P} \left\{ \left| |T\alpha|_r^r - \mathbb{E} |T\alpha|_r^r \right| \geq t \right\} \leq 2 \exp(-b_{p,r} N t^{q/r}).$$

Proof It is exactly analogous to the inequality (2.7) from [25]. For completeness, we give a sketch of the proof following [25]. Let $(\beta_k)_{k \geq 1}$ be the non-increasing rearrangement of $(|\alpha_i| j^{-1/p})_{1 \leq i \leq n, j \geq 1}$. We first observe that $\sum_{i=1}^n \alpha_i \sum_{j \geq 1} j^{-1/p} Y_{i,j}$ has the same distribution as $\sum_{k \geq 1} \beta_k Y_k$, where Y_k are independent copies of Y . Moreover, when $s = p/r$

$$\begin{aligned} \|\beta_k^r\|_{s,\infty}^s &= \sup_{u > 0} u^s \text{card}\{(i, j) : |\alpha_i|^r j^{-r/p} > u\} \\ &\leq \sup_{u > 0} u^s \sum_{i=1}^n |\alpha_i|^p u^{-p/r} = |\alpha|_p^p = 1. \end{aligned}$$

Since $1 < s < 2$, it is well-known (see [12, Proposition 2]) that if (d_k) is a scalar martingale difference sequence, such that $|d_k| \leq \gamma_k$ a.s. and if $\|(\gamma_k)\|_{s,\infty} < \infty$, then for any $t > 0$ we have

$$\mathbb{P} \left\{ \left| \sum_{k \geq 1} d_k \right| > t \right\} \leq 2 \exp \left(- \frac{c_s t^{s'}}{\| \gamma_k \|_{s,\infty}^{s'}} \right),$$

where $1/s + 1/s' = 1$. By the definition of s , it means that $s' = q/r$. Let us denote by \mathcal{F}_j the σ -algebra generated by Y_1, \dots, Y_j , and define

$$d_k = \frac{\sigma_{p,r}^r}{N^{r/q}} \left(\mathbb{E}^{\mathcal{F}_k} \left| \sum_{j \geq 1} \beta_j Y_j \right|_r^r - \mathbb{E}^{\mathcal{F}_{k-1}} \left| \sum_{j \geq 1} \beta_j Y_j \right|_r^r \right).$$

By the triangle inequality for the ℓ_r^N -norm,

$$|d_k| \leq \frac{\sigma_{p,r}^r}{N^{r/q}} \beta_k^r$$

and by definition, $|\left|T\alpha\right|_r^r - \mathbb{E}|T\alpha|_r^r|$ has the same distribution as $|\sum_{k \geq 1} d_k|$, which ends the proof of this lemma. □

We denote by S_p^{n-1} the unit sphere of ℓ_p^n . We need the following standard lemma (see [12, Lemma 2]), involving ϑ -net with respect to $|\cdot|_p^r$ of S_p^{n-1} , where $r \leq 1$. By a ϑ -net, \mathcal{N} , we mean that for any $\alpha \in S_p^{n-1}$, we have $\inf_{y \in \mathcal{N}} |\alpha - y|_p^r \leq \vartheta$.

Lemma 2.5 *Let $r \leq 1$. Then S_p^{n-1} contains a ϑ -net of cardinality at most $(1 + \frac{2}{\vartheta})^{n/r}$.*

3 Multi-dimensional Esseen type inequality

To study a small ball probability associated to a random variable, often an inequality due to Esseen [8] is being used. Several multi-dimensional versions of this result are known (see [11, 34, 28, 10]). Our aim is to present a new type of such inequalities. We say that a compact set $K \subset \mathbb{R}^N$ is star shape if the origin O belongs to the relative interior of K and if for any point $a \in K$, the segment $[O, a] \subset K$. We define the gauge associated to K by

$$\|x\|_K = \inf\{t > 0, x \in tK\}.$$

We denote by $|K|$ the volume of K in \mathbb{R}^N .

Theorem 3.1 *Let X be a random vector in \mathbb{R}^N , such that the function $\xi \mapsto \mathbb{E} \exp(i \langle \xi, X \rangle)$ belongs to $L_1(\mathbb{R}^N)$. Then for any compact star shape $K \subset \mathbb{R}^N$, for any $t > 0$*

$$\mathbb{P} \{ \|X\|_K \leq t \} \leq |K| \left(\frac{t}{2\pi} \right)^N \int_{\mathbb{R}^N} |\mathbb{E} \exp(i \langle \xi, X \rangle)| d\xi.$$

This result is useful for our purposes. We use it for $K = N^{1/r} B_r^N$ and $X = N^{1/r} T\alpha$ (with $0 < r \leq 1$). In Lemma 4.7, we estimate the L_1 norm of the characteristic function of X when α belongs to a particular subset of the unit sphere of ℓ_p^n .

Proof By a simple change of variable, it is clear that we can assume that $t = 1$. Moreover, for $\varepsilon > 0$, let ψ_ε be a C^∞ function on \mathbb{R} , such that $\psi_\varepsilon(t) = 0$ if $t \geq 1 + \varepsilon$, $\psi_\varepsilon(t) = 1$ if $t \leq 1$ and $\forall t \in \mathbb{R}, \psi_\varepsilon(t) \in [0, 1]$. Define $f(x) = \psi_\varepsilon(\|x\|_K)$ and let $(\phi_j)_{j \geq 1}$ be the approximation of the identity on \mathbb{R}^N defined for any $j \geq 1$ by

$$\forall x \in \mathbb{R}^N, \quad \phi_j(x) = j^N \exp(-j^2|x|_2^2/2)/(2\pi)^{N/2}.$$

By the definition of f and ψ_ε , $f \leq \mathbf{1}_{(1+\varepsilon)K}$ and we get $|\hat{f}| \leq (1 + \varepsilon)^N |K|$. Easy computations imply that $f \star \phi_j$ and $\widehat{f \star \phi_j}$ belong to the class of Schwarz functions on \mathbb{R}^N , so we can apply the inverse Fourier transform to deduce that for any $x \in \mathbb{R}^N$ we have

$$f \star \phi_j(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{f}(\xi) \widehat{\phi_j}(\xi) \exp(i \langle x, \xi \rangle) d\xi.$$

The function $\xi \mapsto \hat{f}(\xi) \widehat{\phi_j}(\xi) \exp(i \langle x, \xi \rangle)$ belongs to $L_1(\mathbb{R}^N)$, and we may apply Fubini's theorem to deduce that

$$\begin{aligned} \mathbb{E} f \star \phi_j(X) &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{f}(\xi) \widehat{\phi_j}(\xi) \mathbb{E} \exp(i \langle X, \xi \rangle) d\xi \\ &\leq \left(\frac{1 + \varepsilon}{2\pi} \right)^N |K| \int_{\mathbb{R}^N} |\widehat{\phi_j}(\xi)| |\mathbb{E} \exp(i \langle X, \xi \rangle)| d\xi. \end{aligned}$$

Since ψ_ε is C^∞ and bounded, we get by the dominated convergence theorem of Lebesgue

$$\lim_{y \rightarrow 0} \mathbb{E} |f(X - y) - f(X)| = 0,$$

from which it is classical to deduce that

$$\lim_{j \rightarrow \infty} \mathbb{E} |f \star \phi_j(X) - f(X)| = 0.$$

Moreover, $|\widehat{\phi}_j(\xi)|$ tends to 1 from below as $j \rightarrow \infty$ and since $\xi \mapsto \mathbb{E} \exp(i \langle X, \xi \rangle)$ belongs to $L_1(\mathbb{R}^N)$, by the dominated convergence theorem of Lebesgue, we deduce that

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} |\widehat{\phi}_j(\xi)| |\mathbb{E} \exp(i \langle X, \xi \rangle)| d\xi = \int_{\mathbb{R}^N} |\mathbb{E} \exp(i \langle X, \xi \rangle)| d\xi.$$

We conclude that for any $\varepsilon > 0$,

$$\mathbb{E} f(X) \leq \left(\frac{1 + \varepsilon}{2\pi} \right)^N |K| \int_{\mathbb{R}^N} |\mathbb{E} \exp(i \langle X, \xi \rangle)| d\xi.$$

By the definition of f and ψ we know that $f(X) \geq \mathbf{1}_K(X)$ and it is clear that

$$\mathbb{P}\{\|X\|_K \leq 1\} \leq \mathbb{E} f(X),$$

which ends the proof of this theorem, letting $\varepsilon \rightarrow 0$. □

Remark This Theorem is also a simple application of the fact that a measure whose characteristic function is integrable is absolutely continuous. Moreover, in the particular case of K being the Euclidean ball, it is possible to estimate the classical Levy concentration function of sum of independent random vectors. Indeed, it is enough to perturb the random sum X by a Gaussian vector G (with the appropriate variance). Therefore, the Levy concentration function of X has the same order as the one of $X + G$ and by properties of the characteristic function, we get integrability at infinity so that we can apply Theorem 3.1. For instance, following this argument, one may recover (and extend [9]) Theorem 3.3 from [28].

4 The random embedding

Let us fix $p \in (0, 2)$, and recall that for any fixed r , such that $0 < r < p$, the operator T was defined in (4). The main theorem that we shall prove is the following:

Theorem 4.1 *Let $0 < r < p < 2$ with $r \leq 1$ be such that $r > \frac{2p}{p+2}$. For any $\eta > 0$, and any natural numbers $n, N = (1 + \eta)n$*

$$\mathbb{P} \left\{ \forall \alpha \in \ell_p^n, c(p, r, \eta) |\alpha|_p \leq |T\alpha|_r \leq C(p, r) |\alpha|_p \right\} \geq 1 - c \exp(-c_{p,r} n),$$

where $c(p, r, \eta) = c(p, r)^{1/\eta}$, $c(p, r), C(p, r), c_{p,r}$ are positive numbers depending only on p and r , and $c > 0$ is an absolute constant.

This section is organized as follows. The two first parts provide the proof of this theorem. The parameters r and p are fixed, such that they satisfy the assumption of this theorem. Observe that $\frac{2p}{p+2} > p/2$ and in this case, we may apply Lemmas 2.3 and 2.4. We first get an estimate for the upper bound. The main study of the lower

bound focuses on an estimate for small ball probabilities and is presented in the second part. The last part will be devoted to the proof of Theorem 1.1, where r is not fixed anymore.

4.1 The upper bound

Proposition 4.2 *There exist a constant $C(p, r) > 0$, such that for any natural numbers $N \geq n$*

$$\mathbb{P} \{ \|T\| \leq C(p, r) \} \geq 1 - 2 \exp(-cN/r),$$

where c is an absolute positive constant.

Proof By Lemma 2.3, let $s > 0$ be such that $D_{p,r} \leq s/2$, then we have for every $\alpha \in S_p^{n-1}$

$$|\mathbb{E}|T\alpha|_r^r - 1| \leq s/2.$$

By Lemma 2.4 with $t = s/2$ we get by the triangle inequality

$$\mathbb{P} \{ \left| |T\alpha|_r^r - 1 \right| \geq s \} \leq 2 \exp(-b_{p,r} N s^{q/r} / 2^{q/r}),$$

which implies that

$$\mathbb{P} \{ |T\alpha|_r^r \geq (1 + s) \} \leq 2 \exp(-b_{p,r} N s^{q/r} / 2^{q/r}).$$

Let \mathcal{N} be $1/2$ -net with respect to $|\cdot|_p^r$ of S_p^{n-1} . By Lemma 2.5, it can be chosen of cardinality less than $5^{n/r}$, then

$$\mathbb{P} \{ \exists y \in \mathcal{N}, |Ty|_r^r \geq (1 + s) \} \leq 2 \exp(-b_{p,r} N s^{q/r} / 2^{q/r}) 5^{n/r}.$$

For $s = 2 \max \left(D_{p,r}, \left(\frac{2 \log 5}{rb_{p,r}} \right)^{r/q} \right)$, we deduce that with probability greater than $1 - \exp(-cN/r)$, for any $y \in \mathcal{N}$ we have $|Ty|_r^r \leq (1 + s)$.

By the definition of \mathcal{N} , for any $\alpha \in S_p^{n-1}$, there exists $y \in \mathcal{N}$, such that $|\alpha - y|_p^r \leq 1/2$, therefore, by the triangle inequality

$$\|T\|^r = \sup_{\alpha \in S_p^{n-1}} |T\alpha|_r^r \leq \sup_{y \in \mathcal{N}} |Ty|_r^r + \frac{\|T\|^r}{2},$$

from which we deduce that with probability greater than $1 - 2 \exp(-cN/r)$, $\|T\| \leq (2(1 + s))^{1/r} =: C(p, r)$. □

4.2 The lower bound

We start with the following:

Theorem 4.3 *There exist positive real numbers $c(p, r), c_{p,r}$ depending only on p and r , such that for any $\eta > 0$, and any natural numbers $n, N = (1 + \eta)n$*

$$\mathbb{P} \left\{ \exists \alpha \in \ell_p^n, |T\alpha|_r \leq c(p, r, \eta)|\alpha|_p \right\} \leq c \exp(-c_{p,r}n),$$

where $c(p, r, \eta) = c(p, r)^{1/\eta}$ and $c > 0$ is an absolute constant.

For $\delta, \rho \in (0, 1)$, we define $\text{Sparse}(\delta) = \{\alpha \in \ell_p^n : |\text{supp}(\alpha)| \leq \delta n\}$, and partition the unit sphere of ℓ_p^n, S_p^{n-1} , into two sets with respect to $\text{Sparse}(\delta)$ and ρ . We define the following sets:

$$\begin{aligned} AS(\delta, \rho) &= \{\alpha \in S_p^{n-1} : \text{dist}_\rho(\alpha, \text{Sparse}(\delta)) \leq \rho^{1/r}\}, \\ NAS(\delta, \rho) &= S_p^{n-1} \setminus AS(\delta, \rho), \end{aligned}$$

where $AS(\delta, \rho)$ is the $\rho^{1/r}$ -enlargement (for the ℓ_p^n metric) of the set of sparse vectors intersected with S_p^{n-1} .

4.2.1 The case of almost sparse vectors

Let $C(p, r)$ be the number defined in Proposition 4.2.

Proposition 4.4 *There exist constants $c_{p,r} > 0$ and $\delta_{p,r}, \rho_{p,r} \in (0, 1)$, such that for any $\delta \in (0, \delta_{p,r})$ and $\rho \in (0, \rho_{p,r})$, for any natural numbers $N \geq n$*

$$\mathbb{P} \left\{ \inf_{\alpha \in AS(\delta, \rho)} |T\alpha|_r \leq \frac{1}{2^{1/r}} \ \& \ \|T\| \leq C(p, r) \right\} \leq 2 \exp(-c_{p,r}n). \tag{4}$$

Proof First we provide a small ball estimate for the sparse vectors.

$$\begin{aligned} &\mathbb{P} \left\{ \inf_{\alpha \in \text{Sparse}(\delta) \cap S_p^{n-1}} |T\alpha|_r^r \leq \frac{1}{2} \right\} \\ &= \mathbb{P} \left\{ \exists \sigma \subset \{1, \dots, n\}, |\sigma| = \delta n : \inf_{\alpha \in \mathbb{R}^\sigma \cap S_p^{n-1}} |T\alpha|_r^r \leq \frac{1}{2} \right\} \\ &\leq \binom{n}{\delta n} \mathbb{P} \left\{ \exists \alpha \in S_p^{\delta n-1}, |T\alpha|_r^r \leq \frac{1}{2} \right\}. \end{aligned}$$

By Lemma 2.3, we know that for any $\alpha \in S_p^{\delta n-1}$

$$|\mathbb{E}|T\alpha|_r^r - 1| \leq D_{p,r} \left(\frac{\delta n}{N} \right)^{r/q} \leq D_{p,r} \delta^{r/q},$$

from which we deduce that for $\delta \leq \delta'_{p,r} = 1/(4D_{p,r})^{r/q}$,

$$\forall \alpha \in S_p^{\delta n-1}, \quad |\mathbb{E}|T\alpha|_r^r - 1| \leq 1/4. \tag{5}$$

Moreover, by Lemma 2.4, we know that for any $\alpha \in S_p^{\delta n-1}$

$$\mathbb{P} \left\{ |T\alpha|_r^r - \mathbb{E}|T\alpha|_r^r \geq \frac{1}{8} \right\} \leq 2 \exp(-b'_{p,r}N).$$

Let \mathcal{N} be a $1/12$ -net with respect to $|\cdot|_p^r$ of $S_p^{\delta n-1}$. By Lemma 2.5, it can be chosen such that $|\mathcal{N}| \leq 25^{\delta n/r}$, then using (5) and the union bound, we get

$$\mathbb{P} \left\{ \forall y \in \mathcal{N}, \frac{5}{8} \leq |Ty|_r^r \leq \frac{11}{8} \right\} \geq 1 - 2 \exp(-b'_{p,r}N) 25^{\delta n/r}.$$

From the classical net argument (like in the proof of Proposition 4.2), we deduce that with this probability, for any $z \in \mathbb{R}^{\delta n}$ we have

$$|Tz|_r^r \leq \frac{12}{8}|z|_p^r.$$

Moreover, for any $\alpha \in S_p^{\delta n-1}$, there exists $y \in \mathcal{N}$, such that $|\alpha - y|_p^r \leq 1/12$, which implies

$$|T\alpha|_r^r \geq |Ty|_r^r - |T(\alpha - y)|_r^r \geq 5/8 - 12/8 \cdot 1/12 = 1/2.$$

We conclude that

$$\mathbb{P} \left\{ \exists \alpha \in S_p^{\delta n-1}, |T\alpha|_r^r \leq \frac{1}{2} \right\} \leq 2 \exp(-b'_{p,r}N) 25^{\delta n/r}.$$

Since $N \geq n$, it is easy to see that there exists a number $\delta''_{p,r}$, such that for $\delta \leq \delta''_{p,r}$

$$\begin{aligned} \mathbb{P} \left\{ \inf_{\alpha \in \text{Sparse}(\delta) \cap S_p^{\delta n-1}} |T\alpha|_r^r \leq \frac{1}{2} \right\} &\leq 2 \binom{n}{\delta n} \cdot \exp(-b'_{p,r}N + C\delta n/r) \\ &\leq 2 \exp\left(\delta n \log(e/\delta) - b'_{p,r}N + C\delta n/r\right) \\ &\leq 2 \exp(-c_{p,r}n). \end{aligned} \tag{6}$$

We define $\delta_{p,r} = \min(\delta'_{p,r}, \delta''_{p,r})$.

Assume now that there exists $\alpha \in AS(\delta, \rho)$, such that $|T\alpha|_r^r \leq \frac{1}{4}$ and that $\|T\| \leq C(p, r)$, where $C(p, r)$ is defined in Proposition 4.2. Then by the definition of the

almost sparse vectors, α can be written as a sum $\alpha = y + z$, where $y \in \text{Sparse}(\delta)$ and $|z|_p^r \leq \rho$. Thus, $|y|_p^r = |\alpha - z|_p^r \geq |\alpha|_p^r - |z|_p^r \geq 1 - \rho$, and

$$|Ty|_r^r \leq |T\alpha|_r^r + \|T\|^r \cdot |z|_p^r \leq \frac{1}{4} + C(p, r)^r \rho.$$

We choose $\rho_{p,r} \in (0, 1)$ small enough, such that $\frac{1}{4} + C(p, r)^r \rho_{p,r} = \frac{1}{2}(1 - \rho_{p,r})$. Since $|y|_p^r \geq 1 - \rho$, and for any $\rho \leq \rho_{p,r}$, $\frac{1}{4} + C(p, r)^r \rho \leq \frac{1}{2}(1 - \rho)$. We have found a unit vector $u = y/|y|_p \in \text{Sparse}(\delta)$, such that $|Tu|_r^r \leq \frac{1}{2}$, so we conclude

$$\begin{aligned} &\mathbb{P} \left\{ \inf_{\alpha \in \text{NAS}(\delta, \rho)} |T\alpha|_r^r \leq \frac{1}{4} \ \& \ \|T\| \leq C(p, r) \right\} \\ &\leq \mathbb{P} \left\{ \inf_{\alpha \in \text{Sparse}(\delta) \cap S_p^{n-1}} |T\alpha|_r^r \leq \frac{1}{2} \right\}, \end{aligned}$$

which ends the proof of this lemma thanks to (6). □

4.2.2 The case of non-almost sparse vectors

The main result of this section is the following:

Proposition 4.5 *There exists a positive number $c(p, r)$ depending only on p and r , such that for any $\eta > 0$, and any natural numbers $n, N = (1 + \eta)n$,*

$$\mathbb{P} \left\{ \inf_{\alpha \in \text{NAS}(\delta, \rho)} |T\alpha|_r \leq c(p, r, \eta) \ \& \ \|T\| \leq C(p, r) \right\} \leq \exp(-cn/r),$$

where $C(p, r)$ is the constant defined in Proposition 4.2, $\delta = \delta_{p,r}$ and $\rho = \rho_{p,r}$ are chosen from Proposition 4.4 and $c > 0$ is an absolute constant. Moreover it is valid for any $c(p, r, \eta)$ such that

$$c(p, r, \eta) \leq c(p, r)^{1/\eta}.$$

We shall need the following basic properties of a non-almost sparse vector.

Lemma 4.6 *Let $\alpha \in \text{NAS}(\delta, \rho)$. Then there exists a set $I \subseteq \{1, \dots, n\}$ with $|I| \geq \frac{1}{2}\delta n \rho^{p/r}$, such that for any $k \in I$ we have*

$$\frac{\rho^{1/r}}{(2n)^{1/p}} \leq |\alpha_k| \leq \frac{1}{(\delta n)^{1/p}}.$$

Proof Consider the subsets of $\{1, \dots, n\}$ defined as

$$I_1 := \left\{ k : |\alpha_k| \leq \frac{1}{(\delta n)^{1/p}} \right\}, \quad I_2 := \left\{ k : |\alpha_k| \geq \frac{\rho^{1/r}}{(2n)^{1/p}} \right\},$$

and put $I := I_1 \cap I_2$. Since $|\alpha|_p = 1$, then $|I_1^c| \leq \delta n$ and $y := P_{I_1^c} \alpha \in \text{Sparse}(\delta)$. Recall $\alpha \in \text{NAS}(\delta, \rho)$, therefore, $|P_{I_1} \alpha|_p = |\alpha - y|_p > \rho^{1/r}$. By the definition of I_2 ,

we have $|P_{I_2^c} \alpha|_p^p \leq n \frac{\rho^{p/r}}{2^n} = \frac{\rho^{p/r}}{2}$. Hence,

$$|P_I \alpha|_p^p \geq |P_{I_1} \alpha|_p^p - |P_{I_2^c} \alpha|_p^p \geq \frac{\rho^{p/r}}{2}.$$

Moreover, $|P_I \alpha|_p^p \leq |P_I \alpha|_\infty^p \cdot |I| \leq \frac{1}{\delta n} |I|$, which proves that $|I| \geq \frac{1}{2} \delta n \rho^{p/r}$. \square

First, we need to evaluate the small ball probability associated to a non-almost sparse vector. To do this, we use Theorem 3.1 and we need the following main technical lemma.

Lemma 4.7 *For any vector $\alpha \in NAS(\delta, \rho)$, the function*

$$\xi \mapsto \mathbb{E} \exp(i N^{1/r} \langle \xi, T\alpha \rangle),$$

belongs to $L_1(\mathbb{R}^N)$. Moreover,

$$\int_{\mathbb{R}^N} |\mathbb{E} \exp(i N^{1/r} \langle \xi, T\alpha \rangle)| d\xi \leq C(p, r, \delta, \rho)^N.$$

Proof Let X be the random vector $N^{1/r} T\alpha$. By the definition of $T\alpha$ and of the random vector Y , and by independence of the sequence $(Y_{k,j})$ we have

$$\begin{aligned} |\mathbb{E} \exp(i \langle \xi, X \rangle)| &= \prod_{k=1}^n \prod_{j \geq 1} \left| \mathbb{E} \exp \left(i N^{1/p} \sigma_{p,r} \frac{\alpha_k}{j^{1/p}} \langle \xi, Y_{k,j} \rangle \right) \right| \\ &= \prod_{i=1}^n \prod_{j \geq 1} \left| \frac{1}{N} \sum_{\ell=1}^N \cos \left(\sigma_{p,r} N^{1/p} \frac{\alpha_i}{j^{1/p}} \xi_\ell \right) \right|. \end{aligned}$$

Since $|\cos \theta| = |\cos(\theta + m\pi)|$ for any integer $m \in \mathbb{Z}$, we know that

$$|\cos \theta| \leq 1 - c_1 \min_{m \in \mathbb{Z}} |\theta - \pi m|^2,$$

where c_1 is a positive number. It is also clear that for any real number x , $1 - x \leq \exp(-x)$, therefore, we have

$$\begin{aligned} |\mathbb{E} \exp(i \langle \xi, X \rangle)| &\leq \prod_{i=1}^n \prod_{j \geq 1} \left(1 - \frac{c_1}{N} \sum_{\ell=1}^N \min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} \xi_\ell - \pi m \right|^2 \right) \\ &\leq \prod_{i=1}^n \prod_{j \geq 1} \exp \left(-\frac{c_1}{N} \sum_{\ell=1}^N \min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} \xi_\ell - \pi m \right|^2 \right) \\ &\leq \prod_{\ell=1}^N \prod_{i=1}^n \prod_{j \geq 1} \exp \left(-\frac{c_1}{N} \min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} \xi_\ell - \pi m \right|^2 \right). \end{aligned}$$

We shall now prove that the function

$$H : z \mapsto \prod_{i=1}^n \prod_{j \geq 1} \exp \left(-\frac{c_1}{N} \min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} z - \pi m \right|^2 \right),$$

belongs to $L_1(\mathbb{R})$. Moreover, if R is defined by

$$R = \left(\sigma_{p,r} \rho^{1/r} \left(\frac{1 + \eta}{2} \right)^{1/p} \right)^{-1},$$

then the function H satisfies for any $|z| \leq R$,

$$H(z) \leq \exp(-b_{p,r} \delta^{2/p} \rho^{4/r} z^2) \tag{7}$$

and for any $|z| > R$,

$$H(z) \leq \exp(-b_{p,r} \delta \rho^{2p/r} z^p), \tag{8}$$

where $b_{p,r}$ is a positive real numbers depending only on p and r . From now on, we emphasize that the value of $b_{p,r}$ may change from line to line, but it depends only on p and r . Once it will be proved, then we easily conclude the proof of this lemma, since

$$\int_{\mathbb{R}^N} |\mathbb{E} \exp(i \langle \xi, X \rangle)| d\xi \leq \left(\int_{\mathbb{R}} H(z) dz \right)^N.$$

Let $I \subseteq \{1, \dots, n\}$ be the subset given by Lemma 4.6, then

$$H(z) \leq \prod_{i \in I} \prod_{j \geq 1} \exp \left(-\frac{c_1}{N} \min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} z - \pi m \right|^2 \right).$$

By Lemma 4.6 for any $i \in I$, we know $\frac{\rho^{1/r}}{(2n)^{1/p}} \leq |\alpha_i| \leq \frac{1}{(\delta n)^{1/p}}$.

Let us first assume that $|z| \leq R$, then for any $i \in I$ we have

$$\left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} z \right| \leq \frac{2^{1/p}}{\delta^{1/p} \rho^{1/r} j^{1/p}},$$

and we deduce that for $j \geq j_0 = \lceil 2^p / \delta \rho^{p/r} \rceil \geq 2$

$$\min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} z - \pi m \right|^2 = \frac{\sigma_{p,r}^2 N^{2/p} \alpha_i^2}{j^{2/p}} z^2.$$

We get that

$$\begin{aligned}
 H(z) &\leq \prod_{i \in I} \prod_{j \geq j_0} \exp\left(-\frac{c_1}{N} \min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} z - \pi m \right|^2\right) \\
 &\leq \prod_{i \in I} \prod_{j \geq j_0} \exp\left(-\frac{c_1}{N} \frac{\sigma_{p,r}^2 N^{2/p} \alpha_i^2}{j^{2/p}} z^2\right).
 \end{aligned}$$

By the definition of I in Lemma 4.6

$$\sum_{i \in I} \alpha_i^2 \geq |I| \cdot \frac{\rho^{2/r}}{(2n)^{2/p}} \geq \frac{1}{2} \delta n \rho^{p/r} \cdot \frac{\rho^{2/r}}{(2n)^{2/p}} = \frac{1}{2^{1+2/p}} \delta \rho^{2/r+p/r} n^{1-2/p},$$

and by the definition of j_0

$$\sum_{j \geq j_0} j^{-2/p} \geq b_{p,r} (j_0 - 1)^{1-2/p} \geq b_{p,r} (j_0/2)^{1-2/p} \geq \frac{b_{p,r}}{\delta^{1-2/p} \rho^{p/r-2/r}}.$$

We conclude that

$$H(z) \leq \exp\left(-b_{p,r} (1 + \eta)^{2/p-1} \delta^{2/p} \rho^{4/r} z^2\right) \leq \exp\left(-b_{p,r} \delta^{2/p} \rho^{4/r} z^2\right),$$

which ends the proof of (7).

Now, we assume that $|z| > R$, then for any $i \in I$ we have $\sigma_{p,r} N^{1/p} |\alpha_i| |z| \geq 1$. Let j_0 be given by

$$j_0 = \lceil \sigma_{p,r}^p N |\alpha_i|^p |z|^p \rceil + 1,$$

then it is obvious that $j_0 \geq 2$ and $j_0 - 1 \geq j_0/2$. Moreover, for any $j \geq j_0$

$$\min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} z - \pi m \right|^2 = \frac{\sigma_{p,r}^2 N^{2/p} \alpha_i^2}{j^{2/p}} z^2.$$

We get that

$$\begin{aligned}
 H(z) &\leq \prod_{i \in I} \prod_{j \geq j_0} \exp\left(-\frac{c_1}{N} \min_{m \in \mathbb{Z}} \left| \frac{\sigma_{p,r} N^{1/p} \alpha_i}{j^{1/p}} z - \pi m \right|^2\right) \\
 &\leq \prod_{i \in I} \prod_{j \geq j_0} \exp\left(-\frac{c_1}{N} \frac{\sigma_{p,r}^2 N^{2/p} \alpha_i^2}{j^{2/p}} z^2\right).
 \end{aligned}$$

By the definition of j_0

$$\sum_{j \geq j_0} j^{-2/p} \geq b_{p,r}(j_0 - 1)^{1-2/p} \geq b_{p,r}(j_0/2)^{1-2/p} \geq b_{p,r} (\sigma_{p,r}^p N |\alpha_i|^p |z|^p)^{1-2/p},$$

and by the definition of I in Lemma 4.6,

$$\sum_{i \in I} \alpha_i^p \geq |I| \frac{\rho^p}{2n} \geq \frac{1}{4} \delta \rho^{2p/r},$$

hence, we deduce that

$$H(z) \leq \exp\left(-b_{p,r} \delta \rho^{2p/r} |z|^p\right),$$

which proves (8), and ends the proof of this lemma. □

Proof of Proposition 4.5 The numbers δ and ρ are now chosen and depend only on p and r . We will not write these numbers anymore. By Lemma 4.7, we can apply Theorem 3.1 with $K = N^{1/r} B_p^N$ and $X = N^{1/r} T\alpha$ for a fixed α , which belongs to the non-almost sparse vectors. Observe that $\|X\|_K = |T\alpha|_r$ and it is known that there is a positive real number A (independent of r and N), such that $|K| \leq A^N$ therefore, we get that for any $u > 0$

$$\mathbb{P}\{|T\alpha|_r \leq u\} \leq (c_{p,r} u)^N,$$

where $c_{p,r}$ depends only on p and r (it comes from Lemma 4.7). Let \mathcal{N}' be a $\vartheta/2$ -net with respect to $|\cdot|_p^r$ for the unit sphere S_p^{n-1} . It is easy to construct from \mathcal{N}' a ϑ -net, \mathcal{N} , with respect to $|\cdot|_p^r$ for the set of the $NAS(\delta, \rho)$ vectors such that $|\mathcal{N}| \leq |\mathcal{N}'|$.

Therefore by Lemma 2.5, we have $|\mathcal{N}| \leq \left(\frac{5}{\vartheta}\right)^{n/r}$. We choose $\vartheta = \frac{1}{2} \left(\frac{u}{C(p,r)}\right)^r$, where $C(p, r)$ is defined in Proposition 4.2. Since $N = (1 + \eta)n$, we get by the union bound that

$$\begin{aligned} \mathbb{P}\{\exists y \in \mathcal{N}, |Ty|_r^r \leq u^r\} &\leq (c_{p,r} u)^{(1+\eta)n} \left(\frac{5}{\vartheta}\right)^{n/r} \\ &= (c_{p,r} u)^{(1+\eta)n} \left(\frac{10C(p,r)^r}{u^r}\right)^{n/r}. \end{aligned}$$

For $u = \left(c_{p,r}^{1+\eta} \cdot 20^{1/r} \cdot C(p,r)\right)^{-1/\eta}$ we get that

$$\mathbb{P}\{\exists y \in \mathcal{N}, |Ty|_r^r \leq u^r\} \leq \exp(-cn/r).$$

If there exists $\alpha \in NAS(\delta, \rho)$ such that $|T\alpha|_r^r \leq u^r/2$ and if $\|T\| \leq C(p, r)$, then by definition of \mathcal{N} , there is a vector $y \in \mathcal{N}$, such that $|\alpha - y|_p^r \leq \vartheta$, and by the triangle

inequality, we have

$$|Ty|_r^r \leq |T\alpha|_r^r + |T(\alpha - y)|_r^r \leq u^r/2 + \vartheta \cdot C(p, r)^r = u^r.$$

We conclude that for $c(p, r, \eta) = c(p, r)^{1/\eta}$,

$$\mathbb{P} \{ \exists \alpha \in NAS(\delta, \rho), |T\alpha|_r \leq c(p, r, \eta) \ \& \ \|T\| \leq C(p, r) \} \leq \exp(-cn/r),$$

thanks to the last inequality. □

Proof of Theorem 4.3 It is enough to remark that $S_p^{n-1} = AS(\delta, \rho) \cup NAS(\delta, \rho)$, and to combine the results of Propositions 4.2, 4.4 and 4.5. □

Proof of Theorem 4.1 It is just a combination of Theorem 4.3 with the Proposition 4.2. □

4.3 Conclusion

Proof of Theorem 1.1 In the case $1 < p < 2$, let r_1 and r_2 be such that $0 < \frac{2p}{p+2} < r_1 < 1$ and $r_2 = 1$. By Theorem 4.1, we deduce that with overwhelming probability, the operator S is simultaneously a nice embedding from ℓ_p^n to both $\ell_{r_1}^N$ and $\ell_{r_2}^N$. Indeed, the operator S is just a multiplicity of the operator T ,

$$\frac{\sigma_{p,r}}{N^{1/q}} S\alpha = \frac{\sigma_{p,r}}{N^{1/q}} \sum_{i=1}^n \alpha_i \sum_{j \geq 1} \frac{1}{j^{1/p}} Y_{i,j} = T\alpha,$$

which means that with probability greater than $1 - 2c \exp(-c_p n)$, for any $\alpha \in \ell_p^n$, we have

$$\begin{aligned} c(p)^{1/\eta} N^{1/r_1-1/p} |\alpha|_p &\leq |S\alpha|_{r_1} \leq C(p) N^{1/r_1-1/p} |\alpha|_p, \\ c(p)^{1/\eta} N^{1/r_2-1/p} |\alpha|_p &\leq |S\alpha|_{r_2} \leq C(p) N^{1/r_2-1/p} |\alpha|_p, \end{aligned}$$

where the numbers $c_p, c(p)$ and $C(p)$ are the worse constants that come from Theorem 4.1 for the fixed choice of r_1 and r_2 . Now, by a standard extrapolation argument, it is easy to deduce that

$$c'(p)^{1/\eta r} N^{1/r-1/p} |\alpha|_p \leq |S\alpha|_r \leq C(p) N^{1/r-1/p} |\alpha|_p,$$

where $c'(p)$ is another positive real number depending only on p . The result follows by interpolation for $r_1 < r < 1$.

For the last part, we repeat exactly the same argument in the case $0 < p \leq 1$, but we have to fix $r_2 < p$, then we obtain the desired conclusion for every $0 < r \leq r_2$ with constants depending only on p and r_2 . □

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