Solutions to Exercises

Random Graphs and Complex Networks. Vol. I

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Chapter 1
Solutions to selected exercises

1.1 Solutions to the exercises of Chapter 1.

Solution to Exercise 1.2. When (1.4.7) holds with equality, then

$$1 - F_X(x) = \sum_{k=x+1}^{\infty} f_k = \sum_{k=x+1}^{\infty} k^{-\tau}.$$

Therefore, by monotonicity of $x \mapsto x^{-\tau}$,

$$1 - F_X(x) \leq \int_x^{\infty} y^{-\tau} dy = \frac{x^{1-\tau}}{\tau - 1},$$

while

$$1 - F_X(x) \geq \int_{x+1}^{\infty} y^{-\tau} dy = \frac{(x+1)^{1-\tau}}{\tau - 1}.$$

As a result, we obtain that

$$1 - F_X(x) = \frac{x^{1-\tau}}{\tau - 1}(1 + O(\frac{1}{x})).$$

For an example where (**??**) holds, but (1.4.7) fails, we can take $f_{2k+1} = 0$ for $k \geq 0$ and, for $k \geq 1$,

$$f_{2k} = \frac{1}{k^{\tau-1}} - \frac{1}{(k+1)^{\tau-1}}.$$

Then (1.4.7) fails, while

$$1 - F_X(x) = \sum_{k>x} f_k \sim \frac{1}{\lfloor x/2 \rfloor^{\tau-1}} \sim \frac{1}{x^{\tau-1}}.$$

□

Solution to Exercise 1.3. Recall that a function $x \mapsto L(x)$ is slowly varying when, for every $c > 0$,

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1.$$

For $L(x) = \log x$, we can compute

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = \lim_{x\to\infty} \frac{\log(cx)}{\log x} = \lim_{x\to\infty} \frac{\log x + \log c}{\log x} = 1.$$

For $L(x) = e^{(\log x)^\gamma}$, we compute similarly

$$\begin{aligned}\lim_{x\to\infty} \frac{L(cx)}{L(x)} &= \lim_{x\to\infty} e^{(\log(cx))^\gamma - (\log x)^\gamma}\\ &= \lim_{x\to\infty} e^{\log(x)^\gamma\left((1+\frac{\log c}{\log x})^\gamma - 1\right)}\\ &= \lim_{x\to\infty} e^{\log(x)^{\gamma-1}\gamma\log c} = 1.\end{aligned}$$

When $\gamma = 1$, however, we have that $L(x) = e^{\log x} = x$, which is regularly varying with exponent 1. $\square$

1.2 Solutions to the exercises of Chapter 2.

Solution to Exercise 2.1. Take

$$X_n = \begin{cases} Y_1 & \text{for } n \text{ even},\\ Y_2 & \text{for } n \text{ odd},\end{cases}$$

where Y_1 and Y_2 are two independent copies of a random variable which is such that $\mathbb{P}(Y_i = \mathbb{E}[Y_i]) < 1$. Then, since Y_1 and Y_2 are identical in distribution, the sequence $\{X_n\}_{n=1}^\infty$ converges in distribution. In fact, $\{X_n\}_{n=1}^\infty$ is constant in distribution.

Moreover, $X_{2n} \equiv Y_1$ and $X_{2n+1} \equiv Y_2$. Since subsequences of converging sequences are again converging, if $\{X_n\}_{n=1}^\infty$ converges in probability, the limit of $\{X_n\}_{n=1}^\infty$ should be equal to Y_1 and to Y_2. Since $\mathbb{P}(Y_1 \neq Y_2) > 0$, we obtain a contradiction. $\square$

Solution to Exercise 2.2. Note that for any $\varepsilon > 0$, we have

$$\mathbb{P}(|X_n| > \varepsilon) = \mathbb{P}(X_n = n) = \frac{1}{n} \to 0. \tag{1.2.1}$$

Therefore, $X_n \xrightarrow{\mathbb{P}} 0$, which in turn implies that $X_n \xrightarrow{d} 0$. $\square$

Solution to Exercise 2.3. The random variable X with density

$$f_X(x) = \frac{1}{\pi(1+x^2)},$$

which is a *Cauchy random variable*, does the job. $\square$

Solution to Exercise 2.4. Note that, by a Taylor expansion of the moment generating function, if $M_X(t) < \infty$ for all t, then

$$M_X(t) = \sum_{r=0}^\infty \mathbb{E}[X^r]\frac{t^r}{r!}.$$

As a result, when $M_X(t) < \infty$ for all t, we must have that

$$\lim_{r\to\infty} \mathbb{E}[X^r]\frac{t^r}{r!} = 0.$$

Thus, when $t > 1$, (2.1.8) follows. Thus, it is sufficient to show that the moment generating function $M_X(t)$ of the Poisson distribution is finite for all t. For this, we compute

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} e^{tk}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!} = \exp\{-\lambda(1-e^t)\} < \infty,$$

for all t. □

Solution to Exercise 2.5. We write out

$$\begin{aligned}\mathbb{E}[(X)_r] = \mathbb{E}[X(X-1)\cdots(X-r+1)] &= \sum_{x=0}^{\infty} x(x-1)\cdots(x-r+1)\mathbb{P}(X=x)\\ &= \sum_{x=r}^{\infty} x(x-1)\cdots(x-r+1)e^{-\lambda}\frac{\lambda^x}{x!}\\ &= \lambda^r\sum_{x=r}^{\infty} e^{-\lambda}\frac{\lambda^{x-r}}{(x-r)!} = \lambda^r. \end{aligned} \tag{1.2.2}$$

□

Solution to Exercise 2.6. Compute that

$$\mathbb{E}[X^m] = e^{-\lambda}\sum_{k=1}^{\infty} k^m\frac{\lambda^k}{k!} = \lambda e^{-\lambda}\sum_{k=1}^{\infty} k^{m-1}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{l=0}^{\infty}(l+1)^{m-1}\frac{\lambda^l}{l!} = \lambda\mathbb{E}[(X+1)^{m-1}].$$

□

Solution to Exercise 2.9. By the discussion around (2.1.16), we have that the sum $\sum_{r=k}^{n}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!}$ is alternatingly larger and smaller than $\mathbb{P}(X=k)$. Thus, it suffices to prove that, when (2.1.18) holds, then also

$$\lim_{n\to\infty}\sum_{r=k}^{n}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!} = \sum_{r=k}^{\infty}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!}. \tag{1.2.3}$$

This is equivalent to the statement that

$$\lim_{n\to\infty}\sum_{r=n}^{\infty}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!} = 0. \tag{1.2.4}$$

To prove (0.2.4), we bound

$$\Big|\sum_{r=n}^{\infty}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!}\Big| \le \sum_{r=n}^{\infty}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!} \to 0, \tag{1.2.5}$$

by (2.1.18). □

Solution to Exercise 2.10. For $r = 2$, we note that

$$\mathbb{E}[(X)_r] = \mathbb{E}[X^2] - \mathbb{E}[X], \tag{1.2.6}$$

and, for $X = \sum_{i\in\mathcal{I}} I_i$ a sum of indicators,

$$\mathbb{E}[X^2] = \sum_{i,j} \mathbb{E}[I_i I_j] = \sum_{i\neq j} \mathbb{P}(I_i = I_j = 1) + \sum_{i} \mathbb{P}(I_i = 1). \tag{1.2.7}$$

Using that $\mathbb{E}[X] = \sum_i \mathbb{P}(I_i = 1)$, we thus arrive at

$$\mathbb{E}[(X)_r] = \sum_{i\neq j} \mathbb{P}(I_i = I_j = 1), \tag{1.2.8}$$

which is (2.1.21) for $r = 2$. □

Solution to Exercise 2.11. For the Poisson distribution factorial moments are given by

$$\mathbb{E}[(X)_k] = \lambda^k$$

(recall Exercise 2.5.) We make use of Theorems 2.4 and 2.5. If X_n is binomial with parameters n and $p_n = \lambda/n$, then

$$\mathbb{E}[(X_n)_k] = \mathbb{E}[X_n(X_n-1)\cdots(X_n-k+1)] = n(n-1)\dots(n-k+1)p^k \to \lambda^k,$$

when $p = \lambda/n$ and $n \to \infty$. □

Solution to Exercise 2.12. We prove Theorem 2.7 by induction on $d \geq 1$. The induction hypothesis is that (2.1.21) holds for all measures $\mathbb{P}$ with corresponding expectations $\mathbb{E}$ and all $r_1, \dots, r_d$.

Theorem 2.7 for $d = 1$ is Theorem 2.5, which initializes the induction hypothesis. We next advance the induction hypothesis by proving (2.1.21) for $d+1$. For this, we first note that we may assume that $\mathbb{E}[(X_{d+1,n})_{r_{d+1}}] > 0$, since $(X_{d+1,n})_{r_{d+1}} \geq 0$ and when $\mathbb{E}[(X_{d+1,n})_{r_{d+1}}] = 0$, then $(X_{d+1,n})_{r_{d+1}} \equiv 0$, so that (2.1.21) follows. Then, we define the measure $\mathbb{P}_{X,d}$ by

$$\mathbb{P}_{X,d}(\mathcal{E}) = \frac{\mathbb{E}\Big[(X_{d+1,n})_{r_{d+1}} \mathbb{1}_{\mathcal{E}}\Big]}{\mathbb{E}[(X_{d+1,n})_{r_{d+1}}]}, \tag{1.2.9}$$

for all possible measurable events $\mathcal{E}$. Then,

$$\mathbb{E}[(X_{1,n})_{r_1}\cdots(X_{d,n})_{r_d}(X_{d+1,n})_{r_{d+1}}] = \mathbb{E}[(X_{d+1,n})_{r_{d+1}}]\mathbb{E}_{X,d}\Big[(X_{1,n})_{r_1}\cdots(X_{d,n})_{r_d}\Big]. \tag{1.2.10}$$

By the induction hypothesis applied to the measure $\mathbb{P}_{X,d}$, we have that

$$\mathbb{E}_{X,d}\Big[(X_{1,n})_{r_1}\cdots(X_{d,n})_{r_d}\Big] = \sum^*_{i^{(1)}_1,\dots,i^{(1)}_{r_1}\in\mathcal{I}_1} \cdots \sum^*_{i^{(d)}_1,\dots,i^{(d)}_{r_d}\in\mathcal{I}_d} \mathbb{P}_{X,d}\big(I^{(l)}_{i_s} = 1 \forall l = 1,\dots,d \& s = 1,\dots,r_l\big). \tag{1.2.11}$$

Next, we define the measure $\mathbb{P}_{\vec{i}_d}$ by

$$\mathbb{P}_{\vec{i}_d}(\mathcal{E}) = \frac{\mathbb{E}\Big[\prod_{l=1}^{d} I_{i_s}^{(l)} \mathbb{1}_{\mathcal{E}}\Big]}{\mathbb{P}\big(I_{i_s}^{(l)} = 1\ \forall l = 1, \ldots, d,\ s = 1, \ldots, r_l\big)}, \tag{1.2.12}$$

so that

$$\begin{aligned}&\mathbb{E}[(X_{d+1,n})_{r_{d+1}}]\mathbb{P}_{X,d}\big(I_{i_s}^{(l)} = 1\ \forall l = 1, \ldots, d,\ s = 1, \ldots, r_l\big)\\ &\qquad = \mathbb{E}_{\vec{i}_d}[(X_{d+1,n})_{r_{d+1}}]\mathbb{P}\big(I_{i_s}^{(l)} = 1\ \forall l = 1, \ldots, d,\ s = 1, \ldots, r_l\big).\end{aligned} \tag{1.2.13}$$

Again by Theorem 2.5,

$$\mathbb{E}_{\vec{i}_d}[(X_{d+1,n})_{r_{d+1}}] = \sum\nolimits^{*}_{i_1^{(d+1)}, \ldots, i_{r_1}^{(d+1)} \in \mathcal{I}_{d+1}} \mathbb{P}_{\vec{i}_d}(I_{i_1}^{(d+1)} = \cdots = I_{i_{r_{d+1}}}^{(d+1)} = 1). \tag{1.2.14}$$

Then, the claim for $d+1$ follows by noting that

$$\begin{aligned}&\mathbb{P}\big(I_{i_s}^{(l)} = 1\ \forall l = 1, \ldots, d,\ s = 1, \ldots, r_l\big)\mathbb{P}_{\vec{i}_d}(I_{i_1}^{(d+1)} = \cdots = I_{i_{r_{d+1}}}^{(d+1)} = 1)\\ &\qquad = \mathbb{P}\big(I_{i_s}^{(l)} = 1\ \forall l = 1, \ldots, d+1,\ s = 1, \ldots, r_l\big).\end{aligned} \tag{1.2.15}$$

□

Solution to Exercise 2.14. Observe that

$$\sum_x |p_x - q_x| = \sum_x (p_x - q_x)\mathbb{1}_{\{p_x > q_x\}} + \sum_x (q_x - p_x)\mathbb{1}_{\{q_x > p_x\}} \tag{1.2.16}$$

$$0 = 1 - 1 = \sum_x (p_x - q_x) = \sum_x (p_x - q_x)\mathbb{1}_{\{p_x > q_x\}} + \sum_x (p_x - q_x)\mathbb{1}_{\{q_x > p_x\}}. \tag{1.2.17}$$

We add the two equalities to obtain

$$\sum_x |p_x - q_x| = 2\sum_x (p_x - q_x)\mathbb{1}_{\{p_x > q_x\}}.$$

Complete the solution by observing that

$$\sum_x (p_x - \min(p_x, q_x)) = \sum_x (p_x - q_x)\mathbb{1}_{\{p_x > q_x\}}.$$

□

Solution to Exercise 2.13. The proof of (2.2.8) is the continuous equivalent of the proof of (2.2.6). Therefore, we will only prove (2.2.6).

Let Ω be the set of possible outcomes of the probability mass functions $\{p_x\}$ and $\{q_x\}$. The set Ω can be partitioned into two subsets

$$\Omega_1 = \{x \in \Omega : p_x \geq q_x\} \qquad \text{and} \qquad \Omega_2 = \{x \in \Omega : p_x < q_x\}.$$

Since $\{p_x\}$ and $\{q_x\}$ are probability distribution functions, the sum $\sum_{x\in\Omega}(p_x - q_x)$ equals zero. Therefore,

$$\begin{aligned}\sum_{x\in\Omega}|p_x - q_x| &= \sum_{x\in\Omega_1}(p_x - q_x) - \sum_{x\in\Omega_2}(p_x - q_x)\\ 0 = \sum_{x\in\Omega}(p_x - q_x) &= \sum_{x\in\Omega_1}(p_x - q_x) + \sum_{x\in\Omega_2}(p_x - q_x)\end{aligned}$$

Adding and subtracting the above equations yields

$$\sum_{x\in\Omega}|p_x - q_x| = 2\sum_{x\in\Omega_1}(p_x - q_x) = -2\sum_{x\in\Omega_2}(p_x - q_x).$$

Hence, there exists a set $A \subseteq \Omega$ such that $|F(A) - G(A)| \geq \frac{1}{2}\sum_{x\in\Omega}|p_x - q_x|$. It remains to show that $|F(A) - G(A)| \leq \frac{1}{2}\sum_{x\in\Omega}|p_x - q_x|$ for all $A \subseteq \Omega$.

Let A be any subset of Ω. Just as the set Ω, the set A can be partitioned into two subsets

$$A_1 = A \cap \Omega_1 \qquad \text{and} \qquad A_2 = A \cap \Omega_2,$$

so that

$$|F(A) - G(A)| = |\sum_{x\in A_1}(p_x - q_x) + \sum_{x\in A_2}(p_x - q_x)\ | = |\alpha_A + \beta_A|.$$

Since α_A is non-negative and β_A non-positive, it holds that

$$|\alpha_A + \beta_A| \leq \max_A \big(\alpha_A, -\beta_A\big).$$

The quantity α_A satisfies

$$\alpha_A \leq \sum_{x\in\Omega_1}(p_x - q_x) = \frac{1}{2}\sum_{x\in\Omega}|p_x - q_x|,$$

while β_A satisfies

$$\beta_A \geq \sum_{x\in\Omega_2}(p_x - q_x) = -\frac{1}{2}\sum_{x\in\Omega}|p_x - q_x|.$$

Therefore,

$$|F(A) - G(A)| \leq \frac{1}{2}\sum_{x\in\Omega}|p_x - q_x| \qquad \forall A \subseteq \Omega,$$

which completes the proof. □

Solution to Exercise 2.15. By (2.2.15) and (2.2.20)

$$d_{\mathrm{TV}}(f, g) \leq \mathbb{P}(\hat{X} \neq \hat{Y}). \tag{1.2.18}$$

Therefore, the first claim follows directly from Theorem 2.10. The second claim follows by (2.2.6).

Solution to Exercise 2.18. Without any loss of generality we can take $\sigma^2 = 1$. Then for each t, and with Z a standard normal variate

$$\mathbb{P}(X \geq t) = \mathbb{P}(Z \geq t - \mu_X) \leq \mathbb{P}(Z \geq t - \mu_Y) = \mathbb{P}(Y \geq t),$$

whence $X \preceq Y$. □

Solution to Exercise 2.19. The answer is negative. Take X standard normal and $Y \sim N(0, 2)$, then $X \preceq Y$ implies

$$\mathbb{P}(Y \geq t) \geq \mathbb{P}(X \geq t) = \mathbb{P}(Y \geq t\sqrt{2}),$$

for each t. However, this is false for $t < 0$. □

Solution to Exercise 2.20. Let X be Poisson distributed with parameter λ, then

$$\mathbb{E}[e^{tX}] = \sum_{n=0}^{\infty} e^{tn} e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{\lambda(e^t - 1)}.$$

Put

$$g(t) = at - \log \mathbb{E}[e^{tX}] = at + \lambda - \lambda e^t$$

then $g'(t) = a - \lambda e^t = 0 \Leftrightarrow t = \log(a/\lambda)$. Hence, $I(a)$ in (2.4.12) is equal to $I(a) = I_\lambda(a) = a(\log(a/\lambda) - 1) + \lambda$ and with $a > \lambda$ we obtain from (2.4.9),

$$\mathbb{P}(\sum_{i=1}^{n} X_i \geq an) \leq e^{-nI_\lambda(a)}.$$

This proves (2.4.17). For $a < \lambda$, we get $g'(t) = a - \lambda e^t = 0$ for $t = \log(a/\lambda) < 0$ and we get again

$$I_\lambda(a) = a(\log a/\lambda - 1) + \lambda.$$

By (2.4.9), with $a < \lambda$, we obtain (2.4.18).

$I_\lambda(\lambda) = 0$ and $\frac{d}{da} I_\lambda(a) = \log a - \log \lambda$, so that for $a < \lambda$ the function $a \mapsto I_\lambda(a)$ decreases, whereas for $a > \lambda$ the function $a \mapsto I_\lambda(a)$ increases. Because $I_\lambda(\lambda) = 0$, this shows that for all $a \neq \lambda$, we have $I_\lambda(a) > 0$. □

Solution to Exercise 2.22. By taking expectations on both sides of (2.5.2),

$$\mathbb{E}[M_n] = \mathbb{E}[\mathbb{E}[M_{n+1} | M_1, M_2, \ldots, M_n]] = \mathbb{E}[M_{n+1}],$$

since according to the theorem of total probability:

$$\mathbb{E}[\mathbb{E}[X | Y_1, \ldots, Y_n]] = \mathbb{E}[X].$$

□

Solution to Exercise 2.23. First we show that $\mathbb{E}[|M_n|] < \infty$. Indeed, since $\mathbb{E}[|X_i|] < \infty$, $\forall i$, and since the fact that X_i is an independent sequence implies that the sequence $|X_i|$ is independent we get

$$\mathbb{E}[|M_n|] = \prod_{i=0}^{n} \mathbb{E}[|X_i|] < \infty.$$

To verify the martingale condition, we write

$$\begin{aligned}
\mathbb{E}[M_{n+1}|X_1, X_2, \ldots, X_n] &= \mathbb{E}\Big[\prod_{i=1}^{n+1} X_i \Big| X_1, X_2, \ldots, X_n\Big] \\
&= \Big(\prod_{i=1}^{n} X_i\Big) \cdot \mathbb{E}[X_{n+1}|X_1, X_2, \ldots, X_n] = M_n \mathbb{E}[X_{n+1}] = M_n \qquad \text{a.s.}
\end{aligned}$$

□

Solution to Exercise 2.24. First we show that $\mathbb{E}[|M_n|] < \infty$. Indeed, since $\mathbb{E}[|X_i|] < \infty \forall i$,

$$\mathbb{E}[|M_n|] = \mathbb{E}\Big|\sum_{i=1}^{n} X_i\Big| \leq \sum_{i=1}^{n} \mathbb{E}|X_i| < \infty.$$

To verify the martingale condition, we write

$$\begin{aligned}
\mathbb{E}[M_{n+1}|M_1, M_2, \ldots, M_n] &= \mathbb{E}[\sum_{i=1}^{n+1} X_i | X_0, X_1, \ldots, X_n] \\
&= \sum_{i=1}^{n} X_i + \mathbb{E}[X_{n+1}|X_0, X_1, \ldots, X_n] = M_n + \mathbb{E}[X_{n+1}] = M_n \qquad \text{a.s.}
\end{aligned}$$

□

Solution to Exercise 2.25. Again we first that $\mathbb{E}[|M_n|] < \infty$. Indeed, since $\mathbb{E}[|X_i|] < \infty \forall i$,

$$\mathbb{E}[|M_n|] = \mathbb{E}\Big|\mathbb{E}[Y|X_0, \ldots, X_n]\Big| \leq \mathbb{E}\Big[\mathbb{E}\big[|Y|\big|X_0, \ldots, X_n\big]\Big] = \mathbb{E}[|Y|] < \infty.$$

To verify the martingale condition, we write

$$\begin{aligned}
\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] &= \mathbb{E}\Big[\mathbb{E}[Y|X_0, \ldots, X_{n+1}]\Big|X_0, \ldots, X_n\Big] \\
&= \mathbb{E}[Y|X_0, \ldots, X_n] = M_n + \mathbb{E}[X_{n+1}] = M_n \qquad \text{a.s.}
\end{aligned}$$

□

Solution to Exercise 2.26. Since M_n is non-negative we have $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mu \leq M$, by Exercise 2.22. Hence, according to Theorem 2.22 we have convergence to some limiting random variable M_∞. □

Solution to Exercise 2.27. Since $X_i \geq 0$, we have $M_n = \prod_{i=0}^n X_i \geq 0$, hence the claim is immediate from Exercise 2.26. □

Solution to Exercise 2.28. First,

$$\mathbb{E}[|M_n|] \leq \sum_{i=1}^m \mathbb{E}[|M_n^{(i)}|] < \infty. \tag{1.2.19}$$

Secondly, since $\mathbb{E}[\max\{X, Y\}] \geq \max\{\mathbb{E}[X], \mathbb{E}[Y]\}$, we obtain

$$\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] = \mathbb{E}\Big[\max_{i=0}^m M_{n+1}^{(i)}|X_0, \ldots, X_n\Big] \geq \max_{i=0}^m \mathbb{E}[M_{n+1}^{(i)}|X_0, \ldots, X_n] \tag{1.2.20}$$

$$= \max_{i=0}^m M_n^{(i)} = M_n, \tag{1.2.21}$$

where we use that $\{M_n^{(i)}\}_{n=0}^\infty$ is a sequence of martingales with respect to $\{X_n\}_{n=0}^\infty$.

Solution to Exercise 2.29. We can write

$$M_n = \sum_{i=1}^n I_i - p, \tag{1.2.22}$$

where $\{I_i\}_{i=1}^\infty$ are i.i.d. indicator variables with $\mathbb{P}(I_i = 1) = 1 - \mathbb{P}(I_i = 0) = p$. Then, $M - n$ has the same distribution as $X - np$, while, by Exercise 2.24, the sequence $\{M_n\}_{n=0}^\infty$ is a martingale with

$$|M_n - M_{n-1}| = |I_n - p| \leq \max\{p, 1-p\} \leq 1 - p, \tag{1.2.23}$$

since $p \leq 1/2$. Thus, the claim follows from the Azuma-Hoeffding inequality (Theorem 2.25).

Solution to Exercise 2.30. Since $\mathbb{E}[X_i] = 0$, we have, by Exercise 2.24, that $M_n = \sum_{i=1}^n X_i$ is a martingale, with by hypothesis,

$$-1 \leq M_n - M_{n-1} = X_n \leq 1,$$

so that the condition of Theorem 2.25 is satisfied with $\alpha_i = \beta_i = 1$. Since $\mathbb{E}[M_n] = 0$, we have $\mu = 0$ and $\sum_{i=0}^n (\alpha_i + \beta_i)^2 = 4(n+1)$, hence from (2.5.20) we get (2.5.33).

We now compare the Azuma-Hoeffding bound (2.5.33) with the central limit approximation. With $a = x\sqrt{n+1}$, and $\sigma^2 = \mathrm{Var}(X_i)$,

$$\mathbb{P}(|M_n| \geq a) = \mathbb{P}(|M_n| \geq x\sqrt{n+1}) = \mathbb{P}(|M_n|/\sigma\sqrt{n+1} \geq x/\sigma) \to 2(1 - \Phi(x/\sigma)),$$

where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2}\, du$. A well-known approximation tells us that

$$2(1 - \Phi(t)) \sim 2\phi(t)/t = \frac{\sqrt{2}}{t\sqrt{\pi}} e^{-t^2/2},$$

so that by the central limit theorem and this approximation

$$\mathbb{P}(|M_n| \geq a) \sim \frac{\sigma\sqrt{2}}{x\sqrt{\sigma\pi}} e^{-x^2/2\sigma^2} = \frac{\sigma\sqrt{2(n+1)}}{a\sqrt{\pi}} e^{-a^2/2(n+1)\sigma^2}$$

Finally $\sigma^2 \leq 1$, so that the leading order term and with $a = x\sqrt{n+1}$, the inequality of Azuma-Hoefding is quite sharp! $\square$

1.3 Solutions to the exercises of Chapter 3.

Solution to Exercise 3.1. When $\eta = 0$, then, since η is a solution of $\eta = G_X(\eta)$, we must have that

$$p_0 = G_X(0) = 0. \tag{1.3.1}$$

$\square$

Solution to Exercise 3.2. We note that for $p = \{p_x\}_{x=0}^{\infty}$ given in (3.1.15), and writing $q = 1 - p$, we have that $\mathbb{E}[X] = 2p$, so that $\eta = 1$ when $p \leq 1/2$, and

$$G_X(s) = q + ps^2. \tag{1.3.2}$$

Since η satisfies $\eta = G(\eta)$, we obtain that

$$\eta = q + p\eta^2, \tag{1.3.3}$$

of which the solutions are

$$\eta = \frac{1 \pm \sqrt{1 - 4pq}}{2p}. \tag{1.3.4}$$

Noting further that $1 - 4pq = 1 - 4p(1-p) = 4p^2 - 4p + 1 = (2p-1)^2$, and $p > 1/2$, we arrive at

$$\eta = \frac{1 \pm (2p-1)}{2p}. \tag{1.3.5}$$

Since $\eta \in [0, 1)$ for $p > 1/2$, we must have that

$$\eta = \frac{1 - (2p-1)}{2p} = \frac{1-p}{p}. \tag{1.3.6}$$

$\square$

Solution to Exercise 3.3. We compute that

$$G_X(s) = 1 - b/p + \sum_{k=1}^{\infty} b(1-p)^{k-1} s^k = 1 - \frac{b}{p} + \frac{bs}{1-qs}, \tag{1.3.7}$$

so that

$$\mu = G_X'(1) = \frac{b}{p^2}. \tag{1.3.8}$$

As a result, $\eta = 1$ if $\mu = b/p^2 \leq 1$ follows from Theorem 3.1. Now, when $\mu = b/p^2 > 1$, then $\eta < 1$ is the solution of $G_X(\eta) = \eta$, which becomes

$$1 - \frac{b}{p} + \frac{b\eta}{1-q\eta} = \eta, \tag{1.3.9}$$

which has the solution given by (3.1.18). □

Solution to Exercise 3.4. We note that $s \mapsto G_X(s)$ in (0.3.7) has the property that for any points s, u, v

$$\frac{G_X(s) - G_X(u)}{G_X(s) - G_X(v)} = \frac{s-u}{s-v}\frac{1-qv}{1-qu}. \tag{1.3.10}$$

Taking $u = \eta, v = 1$ and using that $G_X(\eta) = \eta$ by Theorem 3.1, we obtain that, if $\eta < 1$,

$$\frac{G_X(s) - \eta}{G_X(s) - 1} = \frac{s-\eta}{s-1}\frac{p}{1-q\eta}. \tag{1.3.11}$$

By (3.1.18), we further obtain that

$$\frac{p}{1-q\eta} = \mu^{-1} = p^2/b, \tag{1.3.12}$$

so that we arrive at

$$\frac{G_X(s) - \eta}{G_X(s) - 1} = \frac{1}{\mu}\frac{s-\eta}{s-1}. \tag{1.3.13}$$

Since $G_n(s)$ is the n-fold iteration of $s \mapsto G_X(s)$, we thus arrive at

$$\frac{G_n(s) - \eta}{G_n(s) - 1} = \frac{1}{\mu^n}\frac{s-\eta}{s-1}, \tag{1.3.14}$$

of which the solution is given by the first line of (3.1.19).

When $\mu = 1$, then we have that $b = p^2$, so that

$$G_X(s) = \frac{q-(q-p)s}{1-qs}. \tag{1.3.15}$$

We now prove by induction that $G_n(s)$ is equal to the second line of (3.1.19). For $n = 1$, we have that $G_1(s) = G_X(s)$, so that the induction is initialized by (0.3.15).

To advance the induction, we assume it for n and advance it to $n+1$. For this, we note that, since $G_n(s)$ is the n-fold iteration of $s \mapsto G_X(s)$, we have

$$G_{n+1}(s) = G_n(G_X(s)). \tag{1.3.16}$$

By the induction hypothesis, we have that $G_n(s)$ is equal to the second line of (3.1.19), so that

$$G_{n+1}(s) = \frac{nq - (nq-p)G(s)}{p + nq - nqG_X(s)} = \frac{nq(1-qs) - (nq-p)(q-(q-p)s)}{(p+nq)(1-qs) - nq(q-(q-p)s)}. \tag{1.3.17}$$

Note that, using $p = 1 - q$,

$$nq(1-qs) - (nq-p)(q-(q-p)s) = \big[nq - (nq-p)q\big] + s\big[(q-p)(nq-p) - nq^2\big] \tag{1.3.18}$$
$$= (n+1)qp - s[qp(n+1) - p^2],$$

while

$$(p+nq)(1-qs) - nq(q-(q-p)s) = \big[(p+nq) - nq^2\big] + s\big[(q-p)nq - (p+nq)q\big] \tag{1.3.19}$$
$$= [p+nqp] - s(n+1)pq = p[p+(n+1)q] - s(n+1)pq,$$

and dividing (0.3.18) by (0.3.19) advances the induction hypothesis. □

Solution to Exercise 3.5. We first note that

$$\mathbb{P}(Z_n > 0, \exists m > n \text{ such that } Z_m = 0) = \mathbb{P}(\exists m > n \text{ such that } Z_m = 0) - \mathbb{P}(Z_n = 0) = \eta - \mathbb{P}(Z_n = 0). \tag{1.3.20}$$

We next compute, using (3.1.19),

$$\mathbb{P}(Z_n = 0) = G_n(0) = \begin{cases} 1 - \mu^n \frac{1-\eta}{\mu^n - \eta} & \text{when } b \neq p^2; \\ \frac{nq}{p+nq} & \text{when } b = p^2. \end{cases} \tag{1.3.21}$$

Using that $\eta = 1$ when $b \leq p^2$ gives the first two lines of (3.1.20). When $\eta < 1$, so that $\mu > 1$, we thus obtain

$$\mathbb{P}(Z_n > 0, \ \exists m > n \text{ such that } Z_m = 0) = (1-\eta)\Big[\frac{\mu^n}{\mu^n - \eta} - 1\Big] = \frac{(1-\eta)\eta}{\mu^n - \eta}. \tag{1.3.22}$$

This proves the third line of (3.1.20). □

Solution to Exercise 3.6. By (0.3.2), we have that $G(s) = q + ps^2$. Thus, by (3.1.23), we obtain

$$G_T(s) = s\big(q + pG_T(s)^2\big), \tag{1.3.23}$$

of which the solutions are given by

$$G_T(s) = \frac{1 \pm \sqrt{1 - 4s^2pq}}{2sp}. \tag{1.3.24}$$

Since $G_T(0) = 0$, we must that that

$$G_T(s) = \frac{1 - \sqrt{1 - 4s^2pq}}{2sp}. \tag{1.3.25}$$

□

Solution to Exercise 3.7. By (0.3.7), we have $G_X(s) = 1 - \frac{b}{p} + \frac{bs}{1-qs}$. Thus, by (3.1.23), we obtain

$$G_T(s) = s\Big[1 - \frac{b}{p} + \frac{bG_T(s)}{1 - qG_T(s)}\Big]. \tag{1.3.26}$$

Multiplying by $p(1 - qG_T(s))$, and using that $p + q = 1$, leads to

$$pG_T(s)(1-qG_T(s)) = s\big[(p-b)(1-qG_T(s)) + bpG_T(s)\big] = s\big[(p-b) + (b-pq)G_T(s)\big]. \tag{1.3.27}$$

We can simplify the above to

$$pqG_T(s)^2 + (p + s(b - pq))G_T(s) + s(p - b) = 0, \tag{1.3.28}$$

of which the two solutions are given by

$$G_T(s) = \frac{-(p + sbq) \pm \sqrt{(p + s(b - pq))^2 - 4pqs(p - b)}}{2pq}. \tag{1.3.29}$$

Since $G_T(s) \geq 0$ for all $s \geq 0$, we thus arrive at

$$G_T(s) = \frac{\sqrt{(p + s(b - pq))^2 - 4pqs(p - b)} - (p + sbq)}{2pq}. \tag{1.3.30}$$

□

Solution to Exercise 3.8. Compute

$$\begin{aligned}\mathbb{E}[Z_n|Z_{n-1} = m] &= \mathbb{E}[\textstyle\sum_{i=1}^{Z_{n-1}} X_{n,i}|Z_{n-1} = m] = \mathbb{E}[\textstyle\sum_{i=1}^{m} X_{n,i}|Z_{n-1} = m]\\ &= \textstyle\sum_{i=1}^{m} \mathbb{E}[X_{n,i}] = m\mu,\end{aligned}$$

so that, by taking double expectations,

$$\mathbb{E}[Z_n] = \mathbb{E}[\mathbb{E}[Z_n|Z_{n-1}]] = \mathbb{E}[\mu Z_{n-1}] = \mu\mathbb{E}[Z_{n-1}].$$

□

Solution to Exercise 3.9. Using induction we conclude from the previous exercise that

$$\mathbb{E}[Z_n] = \mu\mathbb{E}[Z_{n-1}] = \mu^2\mathbb{E}[Z_{n-2}] = \ldots = \mu^n\mathbb{E}[Z_0] = \mu^n.$$

Hence,

$$\mathbb{E}[\mu^{-n}Z_n] = \mu^{-n}\mathbb{E}[Z_n] = 1.$$

Therefore, we have that, for all $n \geq 0$, $\mathbb{E}[|\mu^{-n}Z_n|] = \mathbb{E}[\mu^{-n}Z_n] < \infty$

By the Markov property and the calculations in the previous exercise

$$\mathbb{E}[Z_n|Z_1, \ldots, Z_{n-1}] = \mathbb{E}[Z_n|Z_{n-1}] = \mu Z_{n-1},$$

so that, with $M_n = Z_n/\mu^n$,

$$\mathbb{E}[M_n|Z_1, \ldots, Z_{n-1}] = \mathbb{E}[M_n|Z_{n-1}] = \frac{1}{\mu^n}\mu Z_{n-1} = M_{n-1},$$

almost surely. Therefore, $M_n = \mu^{-n}Z_n$ is a martingale with respect to $\{Z_n\}_{n=1}^{\infty}$. □

Solution to Exercise 3.10. For a critical BP we have $\mu = 1$, and so Z_n is a martingale. Therefore, for all n,

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_0] = 1.$$

On the other hand, if $\mathbb{P}(X = 1) < 1$, then, $\eta = 1$ by Theorem 3.1, and by monotonicity,

$$\lim_{n\to\infty} \mathbb{P}(Z_n = 0) = \mathbb{P}(\lim_{n\to\infty} Z_n = 0) = \eta = 1.$$

□

Solution to Exercise 3.11.

$$\mathbb{P}(Z_n > 0) = \mathbb{P}(Z_n \geq 1) \leq \mathbb{E}[Z_n] = \mu^n,$$

by Theorem 3.3. □

Solution to Exercise 3.12. Since $T = 1 + \sum_{n=1}^{\infty} Z_n$, we obtain by (3.2.1) that

$$\mathbb{E}[T] = 1 + \sum_{n=1}^{\infty} \mathbb{E}[Z_n] = 1 + \sum_{n=1}^{\infty} \mu^n = 1/(1-\mu). \tag{1.3.31}$$

□

Solution to Exercise 3.13. For $k = 1$, we note that, in (3.3.2), $\{T = 1\} = \{X_1 = 0\}$, so that

$$\mathbb{P}(T = 1) = p_0. \tag{1.3.32}$$

On the other hand, in (3.1.21), $T = 1$ precisely when $Z_1 = X_{1,1} = 0$, which occurs with probability p_0 as well.

For $k = 2$, since $X_i \geq 0$, we have that $\{T = 2\} = \{X_1 = 1, X_2 = 0\}$, so that

$$\mathbb{P}(T = 2) = p_0 p_1. \tag{1.3.33}$$

On the other hand, in (3.1.21), $T = 2$ precisely when $Z_1 = X_{1,1} = 1$ and $Z_2 = X_{2,1} = 0$, which occurs with probability $p_0 p_1$ as well, as required.

For $k = 3$, since $X_i \geq 0$, we have that $\{T = 3\} = \{X_1 = 2, X_2 = X_3 = 0\} \cup \{X_1 = X_2 = 1, X_3 = 0\}$, so that

$$\mathbb{P}(T = 3) = p_0^2 p_2 + p_0 p_1^2. \tag{1.3.34}$$

On the other hand, in (3.1.21),

$$\{T = 3\} = \{Z_1 = Z_2 = 1, Z_3 = 0\} \cup \{Z_1 = 2, Z_2 = 0\}, \tag{1.3.35}$$

so that $\{T = 3\} = \{X_{1,1} = X_{2,1} = 1, X_{3,1} = 0\} \cup \{X_{1,1} = 2, X_{2,1} = X_{2,2} = 0\}$, which occurs with probability $p_0^2 p_2 + p_0 p_1^2$ as well, as required. This proves the equality of $\mathbb{P}(T = k)$ for T in (3.3.2) and (3.1.21) and $k = 1, 2$ and 3. □

Solution to Exercise 3.14. We note that

$$\mathbb{P}\big(S_0 = S_{k+1} = 0, S_i > 0 \ \forall 1 \le i \le k\big) = p\mathbb{P}\big(S_1 = 1, S_i > 0 \ \forall 1 \le i \le k, S_{k+1} = 0\big), \tag{1.3.36}$$

since the first step must be upwards. By (3.3.2),

$$\mathbb{P}\big(S_1 = 1, S_i > 0 \ \forall 1 \le i \le k, S_{k+1} = 0\big) = \mathbb{P}(T = k), \tag{1.3.37}$$

which completes the proof. □

Solution to Exercise 3.15. We note that $p'_x \ge 0$ for all $x \in \mathbb{N}$. Furthermore,

$$\sum_{x=0}^{\infty} p'_x = \sum_{x=0}^{\infty} \eta^{x-1} p_x = \eta^{-1} \sum_{x=0}^{\infty} \eta^x p_x = \eta^{-1} G(\eta). \tag{1.3.38}$$

Since η satisfies $\eta = G(\eta)$, it follows also that $p' = \{p'_x\}_{x=0}^{\infty}$ sums up to 1, so that p' is a probability distribution. □

Solution to Exercise 3.16. We compute

$$G_d(s) = \sum_{x=0}^{\infty} s^x p'_x = \sum_{x=0}^{\infty} s^x \eta^{x-1} p_x = \eta^{-1} \sum_{x=0}^{\infty} (\eta s)^x p_x = \frac{1}{\eta} G_X(\eta s). \tag{1.3.39}$$

□

Solution to Exercise 3.17. We note that

$$\mathbb{E}[X'] = \sum_{x=0}^{\infty} x p'_x = \sum_{x=0}^{\infty} x \eta^{x-1} p_x = G'_X(\eta). \tag{1.3.40}$$

Now, η is the smallest solution of $\eta = G_X(\eta)$, and, when $\eta > 0$, $G_X(0) = p_0 > 0$ by Exercise 3.1. Therefore, since $s \mapsto G'_X(s)$ is increasing, we must have that $G'_X(\eta) < 1$. □

Solution to Exercise 3.18. Since $M_n = \mu^{-n} Z_n \xrightarrow{a.s.} W_\infty$ by Theorem 3.9, by Lebesques dominated convergence theorem and the fact that, for $y \ge 0$ and $s \in [0, 1]$, we have that $s^y \le 1$, it follows that

$$\mathbb{E}[s^{M_n}] \to \mathbb{E}[s^{W_\infty}]. \tag{1.3.41}$$

However,

$$\mathbb{E}[s^{M_n}] = \mathbb{E}[s^{Z_n/\mu_n}] = G_n(s^{\mu^{-n}}). \tag{1.3.42}$$

Since $G_n(s) = G_X(G_{n-1}(s))$, we thus obtain

$$\mathbb{E}[s^{M_n}] = G_X\big(G_{n-1}(s^{\mu^{-n}})\big) = G_X\Big(G_{n-1}\big((s^{\mu^{-1}})^{\mu^{-n-1}}\big)\Big) \to G_X\big(G_W(s^{1/\mu})\big), \tag{1.3.43}$$

again by (0.3.41).

Solution to Exercise 3.19. If $M_n = 0$, then $M_m = 0$ for all $m \geq n$, so that

$$\{M_\infty = 0\} = \lim_{n\to\infty} \{M_n = 0\} = \cap_{n=0}^{\infty}\{M_n = 0\}.$$

On the other hand, {extinction} $= \{\exists n : M_n = 0\}$ or {survival} $= \{\forall n, M_n > 0\}$. We hence conclude that {survival} $\subset \{M_\infty > 0\} = \cup_{n=0}^{\infty}\{M_n > 0\}$, and so

$$\mathbb{P}(M_\infty > 0|\text{survival}) = \frac{\mathbb{P}(M_\infty > 0 \cap \{\text{survival}\})}{\mathbb{P}(\text{survival})} = \frac{\mathbb{P}(M_\infty > 0)}{1-\eta} = 1,$$

because it is given that $\mathbb{P}(W_\infty > 0) = 1 - \eta$. □

Solution to Exercise 3.20. By Theorem 3.9, we have that $M_n = \mu^{-n} Z_n \xrightarrow{a.s.} W_\infty$. By Fubini's theorem, we thus obtain that

$$\mathbb{E}[W_\infty] \leq \lim_{n\to\infty} \mathbb{E}[M_n] = 1, \tag{1.3.44}$$

where the equality follows from Theorem 3.3. □

Solution to Exercise 3.28. The total offspring equals $T = 1 + \sum_{n=1}^{\infty} Z_n$, see (3.1.21). Since we search for $T \leq 3$, we must have $\sum_{n=1}^{\infty} Z_n \leq 2$ or $\sum_{n=1}^{2} Z_n \leq 2$, because $Z_k > 0$ for some $k \geq 3$ implies $Z_3 \geq 1, Z_2 \geq 1, Z_1 \geq 1$, so that $\sum_{n=1}^{\infty} Z_n \geq \sum_{n=1}^{3} Z_n \geq 3$. Then, we can write out

$$\begin{aligned}
\mathbb{P}(T=1) &= \mathbb{P}(\sum_{n=1}^{2} Z_n = 0) = \mathbb{P}(Z_1 = 0) = e^{-\lambda}, \\
\mathbb{P}(T=2) &= \mathbb{P}(\sum_{n=1}^{2} Z_n = 1) = \mathbb{P}(Z_1 = 1, Z_2 = 0) = \mathbb{P}(X_{1,1} = 1)\mathbb{P}(X_{2,1} = 0) = \lambda e^{-2\lambda} \\
\mathbb{P}(T=3) &= \mathbb{P}(\sum_{n=1}^{2} Z_n = 2) = \mathbb{P}(Z_1 = 1, Z_2 = 1, Z_3 = 0) + \mathbb{P}(Z_1 = 2, Z_2 = 0) \\
&= \mathbb{P}(X_{1,1} = 1, X_{2,1} = 1, X_{3,1} = 0) + \mathbb{P}(X_{1,1} = 2, X_{2,1} = 0, X_{2,2} = 0) \\
&= (\lambda e^{-\lambda})^2 \cdot e^{-\lambda} + e^{-\lambda}(\lambda^2/2) \cdot e^{-\lambda} \cdot e^{-\lambda} = e^{-3\lambda}\frac{3\lambda^2}{2}.
\end{aligned}$$

These answers do coincide with $\mathbb{P}(T = n) = e^{-n\lambda}\frac{(n\lambda)^{n-1}}{n!}$, for $n \leq 3$.

□

1.4 Solutions to the exercises of Chapter 4.

Solution to Exercise 4.3. We start by computing $\mathbb{P}(T = m)$ for $m = 1, 2, 3$. For $m = 1$, we get

$$\mathbb{P}(T = 1) = \mathbb{P}(S_1 = 0) = \mathbb{P}(X_1 = 0) = \mathbb{P}(\mathsf{Bin}(n-1, p) = 0) = (1-p)^{n-1}.$$

For $m = 2$, we get

$$\begin{aligned}
\mathbb{P}(T=2) &= \mathbb{P}(S_1>0, S_2=0) = \mathbb{P}(X_1>0, X_1+X_2=1) = \mathbb{P}(X_1=1, X_2=0)\\
&= \mathbb{P}(X_1=1)\mathbb{P}(X_2=0|X_1=1) = \mathbb{P}(\mathsf{Bin}(n-1,p)=1)\mathbb{P}(\mathsf{Bin}(n-2,p)=0)\\
&= (n-1)p(1-p)^{n-2}\cdot(1-p)^{n-2} = (n-1)p(1-p)^{2n-4}.
\end{aligned}$$

For $m = 3$, we get

$$\begin{aligned}
\mathbb{P}(T=3) &= \mathbb{P}(S_1>0, S_2>0, S_3=0) = \mathbb{P}(X_1>0, X_1+X_2>1, X_1+X_2+X_3=2)\\
&= \mathbb{P}(X_1=1, X_2=1, X_3=0) + \mathbb{P}(X_1=2, X_2=0, X_3=0)\\
&= \mathbb{P}(X_3=0|X_2=1, X_1=1)\mathbb{P}(X_2=1|X_1=1)\mathbb{P}(X_1=1)\\
&\qquad +\mathbb{P}(X_3=0|X_2=0, X_1=2)\mathbb{P}(X_2=0|X_1=2)\mathbb{P}(X_1=2)\\
&= \mathbb{P}(X_3=0|S_2=1)\mathbb{P}(X_2=1|S_1=1)\mathbb{P}(X_1=1)\\
&\quad +\mathbb{P}(X_3=0|S_2=1)\mathbb{P}(X_2=0|S_1=2)\mathbb{P}(X_1=2)\\
&= \mathbb{P}(\mathsf{Bin}(n-3,p)=0)\mathbb{P}(\mathsf{Bin}(n-2,p)=1)\mathbb{P}(\mathsf{Bin}(n-1,p)=1)\\
&\quad +\mathbb{P}(\mathsf{Bin}(n-3,p)=0)\mathbb{P}(\mathsf{Bin}(n-3,p)=0)\mathbb{P}(\mathsf{Bin}(n-1,p)=2)\\
&= (1-p)^{n-3}(n-2)p(1-p)^{n-3}(n-1)p(1-p)^{n-2}\\
&\quad +(1-p)^{n-3}(1-p)^{n-3}(n-1)(n-2)p^2(1-p)^{n-3}/2\\
&= (n-1)(n-2)p^2(1-p)^{3n-8} + (n-1)(n-2)p^2(1-p)^{3n-9}/2\\
&= (n-1)(n-2)p^2(1-p)^{3n-9}(\frac{3}{2}-p).
\end{aligned}$$

We now give the combinatoric proof. For $m = 1$,

$$\mathbb{P}(|\mathscr{C}(v)|=1) = (1-p)^{n-1},$$

because all connections from vertex 1 have to be closed. For $m = 2$,

$$\mathbb{P}(|\mathscr{C}(v)|=2) = (n-1)p(1-p)^{2n-4}$$

because you must connect one of $n-1$ vertices to vertex v and then isolate these two vertices which means that $2n-4$ connections should not be present.

For $m = 3$, the first possibility is to attach one vertex a to 1 and then a second vertex b to a, with the edge vb being closed. This gives

$$(n-1)p(1-p)^{n-2}(n-2)p(1-p)^{n-3}(1-p)^{n-3} = (n-1)(n-2)p^2(1-p)^{3n-8}.$$

The second possibility is to attach one vertex a to v and then a second vertex b to a, with the edge vb being occupied. This gives

$$\binom{n-1}{2}p(1-p)^{n-3}p(1-p)^{n-3}(1-p)^{n-3}p = \binom{n-1}{2}p^3(1-p)^{3n-9}.$$

The final possibility is that you pick two vertices attached to vertex v, and then leave both vertices without any further attachments to the other $n-3$ and being unconnected (the connected case is part of the second possibility)

$$\binom{n-1}{2}p^2(1-p)^{n-3}\cdot(1-p)^{2n-5} = \binom{n-1}{2}p^2(1-p)^{3n-8}.$$

In total, this gives

$$(n-1)(n-2)p^2(1-p)^{3n-8} + \binom{n-1}{2}p^3(1-p)^{3n-9} + \binom{n-1}{2}p^2(1-p)^{3n-9} \tag{1.4.1}$$
$$= (n-1)(n-2)p^2(1-p)^{3n-9}(1-p+\frac{p}{2}+\frac{(1-p)}{2})$$
$$= (n-1)(n-2)p^2(1-p)^{3n-9}(\frac{3}{2}-p).$$

□

Solution to Exercise 4.5. We first pick 3 different elements i, j, k from $\{1, 2, \dots, n\}$ without order. This can be done in

$$\binom{n}{3}$$

different ways. Then all three edges ij, ik, jk have to be present, which has probability p^3. The number of triangles is the sum of indicators running over all unordered triples. These indicators are dependent, but that is of no importance for the expectation, because the expectation of a sum of dependent random variables equals the sum of the expected values. Hence the expected number of occupied triangles equals:

$$\binom{n}{3}p^3.$$

□

Solution to Exercise 4.6. We pick 4 elements i, j, k, l from $\{1, 2, \dots, n\}$ This kan be done in

$$\binom{n}{4}$$

different ways. This quadruple may form an occupied square in 3 different orders, that is (i, j, k, l), (i, k, j, l) and (i, j, l, k). Hence there are

$$3 \cdot \binom{n}{4}$$

squares in which all four sides should be occupied. Hence the expected number of occupied squares equals

$$3\binom{n}{4}p^4.$$

□

Solution to Exercise 4.7. We define the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ where X_n is the number of occupied triangles in an Erdős-Rényi random graph with edge probability $p = \lambda/n$. Next we introduce the indicator function

$$I_{a,n} := \begin{cases} 0 & \text{triangle}\, a \;\text{ not connected;} \\ 1 & \text{triangle}\, i \;\text{ connected.} \end{cases}$$

Now, according to (2.1.21) we have

$$\lim_{n\to\infty} \mathbb{E}[(X_n)_r] = \lim_{n\to\infty} \sum\nolimits^*_{a_1,a_2,\dots,a_r\in\mathcal{I}} \mathbb{P}(I_{a_1,n} = 1, I_{a_2,n} = 1, \dots, I_{a_r,n} = 1). \tag{1.4.2}$$

Now, there are two types of collections of triangles, namely, sets of triangles in which all edges are distinct, or the set of triangles for which at least one edge occurs in two different triangles. In the first case, we see that the indicators $I_{a_1,n}, I_{a_2,n}, \dots, I_{a_r,n}$ are *independent*, in the second case, they are not. We first claim that the collection of $(a_1, a_2, \dots, a_r)$ for which all triangles contain different edges has size

$$(1 + o(1))\binom{n}{3}^r. \tag{1.4.3}$$

To see this, we note that the upper bound is obvious (since $\left(\binom{n}{3}\right)^r$ is the number of collections of r triangles without any restriction). For the lower bound, we note that $a_i = (k_i, l_i, m_i)$ for $k_i, l_i, m_i \in [n]$ such that $k_i < l_i < m_i$. We obtain a lower bound on the number of triangles containing different edges when we assume that all vertices k_i, l_i, m_i for $i = 1, \dots, r$ are distinct. There are precisely

$$\prod_{i=0}^{r-1} \binom{n-i}{3} \tag{1.4.4}$$

of such combinations. When r is fixed, we have that

$$\prod_{i=0}^{r-1} \binom{n-i}{3} = (1 + o(1))\binom{n}{3}^r. \tag{1.4.5}$$

Thus, the contribution to the right-hand side of (0.4.2) of collections $(a_1, a_2, \dots, a_r)$ for which all triangles contain different edges is, by independence and (0.4.3), equal to

$$(1 + o(1))\binom{n}{3}^r \Big(\frac{\lambda^3}{n^3}\Big)^r = (1 + o(1))\Big(\frac{\lambda^3}{6}\Big)^r. \tag{1.4.6}$$

We next prove that the contribution to the right-hand side of (0.4.2) of collections $(a_1, a_2, \dots, a_r)$ for which at least one edge occurs in two different triangles. We give a crude upper bound for this. We note that each edge which occurs more that once reduces the number of possible vertices involved. More precisely, when the collection of triangles $(a_1, a_2, \dots, a_r)$ contains precisely $3r - l$ edges for some $l \geq 1$, then the collection of triangles $(a_1, a_2, \ldots, a_r)$ contains at most $3r - 2l$ vertices, as can easily be seen by induction. As a result, the contribution to the right-hand side of (0.4.2) of collections $(a_1, a_2, \ldots, a_r)$ $(a_1, a_2, \ldots, a_r)$ contains precisely $3r - l$ edges is bounded by

$$n^{3r-2l}(\lambda/n)^{3r-l} = \lambda^{3r-l}n^{-l} = o(1). \tag{1.4.7}$$

Since this is negligible, we obtain that

$$\lim_{n\to\infty} \mathbb{E}[(X_n)_r] = \left(\frac{\lambda^3}{6}\right)^r. \tag{1.4.8}$$

Hence, due to Theorem 2.4 we have that the number of occupied triangles in an Erdős-Rényi random graph with edge probability $p = \lambda/n$ has an asymptotic Poisson distribution with parameter $\lambda^3/6$. □

Solution to Exercise 4.8. We have

$$\begin{aligned}\mathbb{E}[\Delta_G] &= \mathbb{E}\left[\sum_{i,j,k\in G} \mathbb{1}_{\{ij,ik,jk\,\text{occupied}\}}\right] = \sum_{i,j,k\in G} \mathbb{E}\left[\mathbb{1}_{\{ij,ik,jk\,\text{occupied}\}}\right] \\ &= n(n-1)(n-2)\left(\frac{\lambda}{n}\right)^3,\end{aligned} \tag{1.4.9}$$

and

$$\begin{aligned}\mathbb{E}[W_G] &= \mathbb{E}\left[\sum_{i,j,k\in G} I[ij, jk\,\text{occupied}]\right] = \sum_{i,j,k\in G} \mathbb{E}\left[\mathbb{1}_{\{ij,jk\,\text{occupied}\}}\right] \\ &= n(n-1)(n-2)\left(\frac{\lambda}{n}\right)^2.\end{aligned} \tag{1.4.10}$$

This yields for the clustering coefficient

$$CC_G = \lambda/n.$$

□

Solution to Exercise 4.9. We have $\mathbb{E}[W_G] = n(n-1)(n-2)p^2(1-p)$. According to the Chebychev inequality we obtain:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}[|W_G - \mathbb{E}[W]| > \epsilon] &\leq \lim_{n\to\infty} \frac{\sigma^2_{W_G}}{\epsilon^2}, \\ \lim_{n\to\infty} \mathbb{P}[|W_G - (n)(n-1)(n-2)(\frac{\lambda}{n})^2(\frac{n-\lambda}{n})| > \epsilon] &\leq \lim_{n\to\infty} \frac{\sigma^2_{W_G}}{\epsilon^2}, \\ \lim_{n\to\infty} \mathbb{P}[|W_G - n\lambda^2| > \epsilon] &\leq 0.\end{aligned}$$

Hence, $W_G/n \xrightarrow{\mathbb{P}} \lambda^2$ and, therefore, $n/W_G \xrightarrow{\mathbb{P}} 1/\lambda^2$. We have already shown in previous exercise that the number of occupied triangles has an asymptotic Poisson distribution with parameter $\frac{\lambda^3}{6}$. Δ_G is three times the number of triangles and thus $\Delta_G \xrightarrow{d} 3 \cdot \mathsf{Poi}(\frac{\lambda^3}{6})$. Slutsky's Theorem states that

$$X_n \xrightarrow{\mathbb{P}} c \text{ and } Y_n \xrightarrow{d} Y \Rightarrow X_n Y_n \xrightarrow{d} cY$$

Hence $\frac{n\Delta_G}{W_G} \xrightarrow{d} \frac{3}{\lambda^2} Y$ where $Y \sim \mathsf{Poi}(\lambda^3/6)$. □

Solution to Exercise 4.10. We have to show that for each x, the event $\{|\mathscr{C}(v)| \geq x\}$ remains true if the the number of edges increases.

Obviously by increasing the number of edges the number $|\mathscr{C}(v)|$ increases or stays the same depending on whether or not some of the added edges connect new vertices to the cluster. In both cases $\{|\mathscr{C}(v)| \geq x\}$ remains true.

□

Solution to Exercise 4.11. This is not true. Take two disjoint clusters which differ by one in size, and suppose that the larger component equals $\mathscr{C}_{\max}$, before adding the edges. Take any $v \in \mathscr{C}_{\max}$. Now add edges between the second largest component and isolated vertices. If you add two of such edges, then the new $\mathscr{C}_{\max}$ equals the union of the second largest component and the two isolated vertices. Since originally v did not belong to the second largest component and v was not isolated, because it was a member of the previous largest component, we now have $v \notin \mathscr{C}_{\max}$.

□

Solution to Exercise 4.12. As a result of (4.2.1) we have

$$\mathbb{E}_\lambda[|\mathscr{C}(v)|] = \sum_{k=1}^{\infty} \mathbb{P}(|\mathscr{C}(v)| \geq k) \leq \sum_{k=1}^{\infty} \mathbb{P}_{n,p}(T \geq k) = \mathbb{E}[T] = \frac{1}{1-\mu}, \tag{1.4.11}$$

where

$$\mu = \mathbb{E}[\text{Offspring}] = np = \lambda.$$

Hence,

$$\mathbb{E}_\lambda[|\mathscr{C}(v)|] \leq 1/(1-\lambda).$$

□

Solution to Exercise 4.14. We recall that $Z_{\geq k} = \sum_{i=1}^{n} \mathbb{1}_{\{|\mathscr{C}(i)| \geq k\}}$.

$$\begin{aligned} &|\mathscr{C}_{\max}| < k \Rightarrow |\mathscr{C}(i)| < k \forall i, \text{ which implies that } Z_{\geq k} = 0 \\ &|\mathscr{C}_{\max}| \geq k \Rightarrow |\mathscr{C}(i)| \geq k \text{ for at least } k \text{ vertices } \Rightarrow Z_{\geq k} \geq k. \end{aligned}$$

□

Solution to Exercise 4.15. Intuitively the statement is logical, for we can see M as doing n trails with succes probability p and for each trial we throw an other coin with succes probability q. The eventual amount of successes are the successes where both trails ended in succes and is thus equal to throwing n coins with succes probability pq.
There are several ways to prove this, we give two of them.

Suppose we have two binomial trials N and Y both of length n and with succes rates p, q respectively. We thus create two vectors filled with ones and zeros. For each index $i = 1, 2, \ldots, n$ we compare the vectors and in case both entries are 1, we will see this as a succes. The now counted amount of successes is of course $\mathsf{Bin}(n, pq)$ distributed.
Now we produce the first vector similarly by denoting ones and zeros for the successes and losses in trail N. For each 'one', we produce an other outcome by a $\mathsf{Be}(q)$ experiment. We count the total number of successes of these experiments and those are of course $\mathsf{Bin}(N, q)$ distributed. But now, this is the same as the experiment described above, since all Bernoulli outcomes are independent. Hence if $N \sim \mathsf{Bin}(n, p)$ and $M \sim \mathsf{Bin}(N, q)$, then $M \sim \mathsf{Bin}(n, pq)$.

We will also give an analytical proof, which is somewhat more enhanced. We wish to show that $\mathbb{P}(M = m) = \binom{n}{m}(pq)^m(1-pq)^{n-m}$. Off course we have

$$\begin{aligned}
\mathbb{P}(M = m) &= \sum_{i=m}^{n} \mathbb{P}(N = i) \cdot \binom{i}{m} \cdot q^m \cdot (1-q)^i - m, \\
&= \sum_{i=m}^{n} \binom{n}{i} \cdot (p)^i \cdot (1-p)^{n-i} \cdot \binom{i}{m} \cdot q^m \cdot (1-q)^i - m.
\end{aligned}$$

Rearranging terms yields

$$\mathbb{P}(M = m) = \frac{(1-p)^n q^m}{(1-q)^m} \sum_{i=m}^{n} \binom{n}{i}\binom{i}{m} \frac{p^i}{(1-p)^i}(1-q)^i.$$

Further analysis yields

$$\begin{aligned}
\mathbb{P}(M=m) &= (1-p)^n\Big(\frac{q}{1-q}\Big)^m \sum_{i=m}^{n} \frac{n!}{i!(n-i)!}\,\frac{i!}{m!(i-m)!}\,\Big(\frac{p(1-q)}{1-p}\Big)^i \\
&= (1-p)^n\Big(\frac{q}{1-q}\Big)^m \frac{n!}{m!}\sum_{i=m}^{n} \frac{1}{(n-i)!(i-m)!}\Big(\frac{p(1-q)}{1-p}\Big)^i \\
&= (1-p)^n(\frac{q}{1-q})^m \frac{n!m!}{\sum}^{n-m}{}_{k=0} \frac{1}{(n-k-m)!(m+k-m)!}\Big(\frac{p(1-q)}{1-p}\Big)^{k+m} \\
&= (1-p)^n(\frac{q}{1-q})^m \frac{n!}{m!(n-m)!}\sum_{k=0}^{n-m} \frac{(n-m)!}{(n-k-m)!k!}\Big(\frac{p(1-q)}{1-p}\Big)^{k+m} \\
&= \binom{n}{m}\sum_{k=0}^{n-m}\binom{n-m}{k}p^{k+m}(1-p)^{n-m-k}q^m(1-q)^{k+m-m} \\
&= \binom{n}{m}p^m q^m \sum_{k=0}^{n-m}\binom{n-m}{k}p^{k}(1-p)^{n-m-k}(1-q)^{k}
\end{aligned}$$

It is now sufficient to show that $\sum_{k=0}^{n-m}\binom{n-m}{k}p^k(1-p)^{n-m-k}(1-q)^k = (1-pq)^{n-m}$.

$$\begin{aligned}
\sum_{k=0}^{n-m}\binom{n-m}{k}p^k(1-p)^{n-m-k}(1-q)^k &= (1-p)^{n-m}\sum_{k=0}^{n-m}\binom{n-m}{k}\Big(\frac{p-pq}{1-p}\Big)^k \\
&= (1-p)^{n-m}\Big(1+\frac{p-pq}{1-p}\Big)^{n-m} \\
&= (1-p)^{n-m}\Big(\frac{1-p+p-pq}{1-p}\Big)^n - m \\
&= (1-pq)^{n-m}.
\end{aligned}$$

Now we can use this result to proof that $N_t \sim \mathsf{Bin}(n,(1-p)^t)$ by using induction. The initial value $N_0 = n-1$ is given, hence

$$\begin{aligned}
N_0 &= n-1; \\
N_1 &= \mathsf{Bin}(n-1, 1-p); \\
N_2 &= \mathsf{Bin}(N_1, 1-p) = \mathsf{Bin}(n-1,(1-p)^2); \\
&\vdots \\
N_t &= \mathsf{Bin}(n-1,(1-p)^t).
\end{aligned}$$

□

Solution to Exercise 4.17. The extinction probability η satisfies

$$\eta_\lambda = G_X(\eta_\lambda) = \mathbb{E}[\eta_\lambda^X] = e^{-\lambda+\lambda\eta_\lambda}$$

Hence,

$$\zeta_\lambda = 1-\eta_\lambda = 1-e^{-\lambda+\lambda\eta} = 1-e^{-\lambda\zeta_\lambda}.$$

This equation has only two solutions, one of which is $\zeta_\lambda = 0$, the other must be the survival probability. □

Solution to Exercise 4.18. We compute that

$$\begin{aligned}\chi(\lambda) &= \mathbb{E}_\lambda[|\mathscr{C}(1)|] = \mathbb{E}_\lambda\left[\sum_{j=1}^{n} \mathbb{1}_{\{j\in\mathscr{C}(1)\}}\right] = 1 + \sum_{j=2}^{n} \mathbb{E}_\lambda[\mathbb{1}_{\{j\in\mathscr{C}(1)\}}] \\ &= 1 + \sum_{j=2}^{n} \mathbb{E}_\lambda[\mathbb{1}_{\{1\leftrightarrow j\}}] = 1 + \sum_{j=2}^{n} \mathbb{P}_\lambda(1\leftrightarrow j) = 1+(n-1)\mathbb{P}_\lambda(1\leftrightarrow 2). \qquad (1.4.12)\end{aligned}$$

□

Solution to Exercise 4.19. In this exercise we denote by $|\mathscr{C}_{(1)}| \geq |\mathscr{C}_{(2)}| \geq \ldots$, the components ordered by their size. Relation (4.4.1) reads that for $\nu \in (\frac{1}{2}, 1)$:

$$\mathbb{P}\Big(\big||\mathscr{C}_{\max}| - n\zeta_\lambda\big| \geq n^\nu\Big) = O(n^{-\delta}).$$

Observe that

$$\begin{aligned}\mathbb{P}_\lambda(1\leftrightarrow 2) &= \mathbb{P}_\lambda(\exists \mathscr{C}(k): 1\in\mathscr{C}(k), 2\in\mathscr{C}(k)) \\ &= \sum_{l\geq 1} \mathbb{P}_\lambda(1,2\in\mathscr{C}_{(l)}) = \mathbb{P}_\lambda(1,2\in\mathscr{C}_{(1)}) + \sum_{l\geq 2}\mathbb{P}_\lambda(1,2\in\mathscr{C}_{(l)}) \\ &= \frac{(n\zeta_\lambda \pm n^\nu)^2}{n^2} + O(n^{-\delta}) + \sum_{l\geq 2}\mathbb{P}_\lambda(1,2\in\mathscr{C}_{(l)}).\end{aligned}$$

For $l\geq 2$, we have $|\mathscr{C}_{(l)}| \leq K\log n$ with high probability, hence

$$\mathbb{P}_\lambda(1,2\in\mathscr{C}_{(l)}) \leq \frac{K^2\log^2 n}{n^2} + O(n^{-2}),$$

so that

$$\sum_{l\geq 2}\mathbb{P}_\lambda(1,2\in\mathscr{C}_{(l)}) \leq \frac{K^2\log^2 n}{n} + O(n^{-1}) \to 0.$$

Together, this shows that

$$\mathbb{P}_\lambda(1\leftrightarrow 2) = \zeta_\lambda^2 + O(n^{-\delta}),$$

for some $\delta > 0$. □

Solution to Exercise 4.20. Combining Exercise 4.18 and Exercise 4.19, yields

$$\chi(\lambda) = 1+(n-1)\zeta_\lambda^2(1+o(1)) = n\zeta_\lambda^2(1+o(1)).$$

□

Solution to Exercise 4.16. We have that the cluster of i has size l. Furthermore, we have $\mathbb{P}_\lambda\big(i \longleftrightarrow j \big| |\mathscr{C}(i)| = l\big) + \mathbb{P}_\lambda\big(i \not\longleftrightarrow j \big| |\mathscr{C}(i)| = l\big) = 1$ Of course $i, j \in [n]$ and $j \neq i$. So, having i fixed, gives us $n-1$ choices for j in $\mathrm{ER}_n(p)$ and $l-1$ choices for j in $\mathscr{C}(i)$. Hence,

$$\mathbb{P}_\lambda\big(i \longleftrightarrow j \big| |\mathscr{C}(i)| = l\big) = \frac{l-1}{n-1},$$

and thus

$$\mathbb{P}_\lambda\big(i \not\longleftrightarrow j \big| |\mathscr{C}(i)| = l\big) = 1 - \frac{l-1}{n-1}.$$

□

Solution to Exercise 4.21. According to the duality principle we have that the random graph obtained by removing the largest component of a supercritical Erdős-Rényi random graph is again an Erdős-Rényi random graph of size $m \sim n\eta_\lambda = \frac{\mu_\lambda n}{\lambda}$ where $\mu_\lambda < 1 < \lambda$ are conjugates as in (3.6.6) and the remaining graph is thus in the subcritical regime. Hence, studying the second largest component in a supercritical graph is close to studying the largest component in the remaining graph.
Now, as a result of Theorems 4.4 and 4.5 we have that for some $\epsilon > 0$

$$\lim_{n\to\infty} \Big(\mathbb{P}\big(\frac{|\mathscr{C}_{\max}|}{\log m} > I_{\mu_\lambda}^{-1} + \epsilon\big) + \mathbb{P}\big(\frac{|\mathscr{C}_{\max}|}{\log m} < I_{\mu_\lambda}^{-1} - \epsilon\big)\Big) = 0.$$

Hence, $\frac{|\mathscr{C}_{\max}|}{\log m} \xrightarrow{\mathbb{P}} I_{\mu_\lambda}^{-1}$. But since we have that $n - m = \zeta_\lambda n(1 + o(1))$ and thus $m = n(1-\zeta_\lambda)$, we have that $\frac{\log m}{\log n} \to 1$ as $n \to \infty$. Hence $\frac{|\mathscr{C}_{\max}|}{\log n} \xrightarrow{\mathbb{P}} I_{\mu_\lambda}^{-1}$. □

Solution to Exercise 4.22. Denote

$$Z_n = \frac{X_n - a_n p_n}{\sqrt{a_n p_n(1-p_n)}}, \tag{1.4.13}$$

so that we need to prove that Z_n converges is distribution to a standard normal random variable Z. For this, it suffices to prove that the moment generating function $M_{Z_n}(t) = \mathbb{E}[e^{tZ_n}]$ of Z_n converges to that of Z.

Since the variance of X_n goes to infinity, the same holds for a_n. Now we write X_n as to be a sum of a_n Bernoulli variables $X_n = \sum_{i=1}^{a_n} Y_i$, where $\{Y_i\}_{1\leq i\leq a_n}$ are independent random variables with $Y_i \sim \mathsf{Be}(p_n)$. Thus, we note that the moment generating function of X_n equals

$$M_{X_n}(t) = \mathbb{E}[e^{tX_n}] = \mathbb{E}[e^{tY_1}]^{a_n}. \tag{1.4.14}$$

We further prove, using a simple Taylor expansion,

$$\log \mathbb{E}[e^{tY_1}] = \log\big(p_n e^t + (1-p_n)\big) = p_n t + \frac{t^2}{2} p_n(1-p_n) + O(|t|^3 p_n). \tag{1.4.15}$$

Thus, with $t_n = t/\sqrt{a_n p_n(1-p_n)}$, we have that

$$M_{Z_n}(t) = M_{X_n}(t_t) e^{a_n p_n t_n} = e^{a_n \log \mathbb{E}[e^{tY_1}]} = e^{\frac{t_n^2}{2} p_n(1-p_n) + O(|t_n|^3 a_n p_n)} = e^{t^2/2 + o(1)}. \tag{1.4.16}$$

We conclude that $\lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2}$, which is the moment generating function of a standard normal distribution. Theorem 2.3(b) implies that $Z_n \xrightarrow{d} Z$, as required. Hence, the CLT follows and (4.5.15) implies (4.5.16). □

Solution to Exercise 4.25. We have that $n\lambda/2$ edges are added in a total system of $n(n-1)/2$ edges. This intuitively yields for p in the classical notation for the ER graphs to be $p = \frac{n\lambda/2}{n(n-1)/2}$ and $\lambda' = n \cdot p$, so that one would expect subcritical behavior $|\mathscr{C}_{\max}|/\log n \xrightarrow{\mathbb{P}} I_\lambda^{-1}$. We now provide the details of this argument.

We make use of the crucial relation (4.6.1), and further note that when we increase M, then we make the event $|\mathscr{C}_{\max}| \geq k$ more likely. This is a related version of monotonicity as in Section 4.1.1. In particular, from (4.6.1), it follows that for any increasing event E, and with $p = \lambda/n$,

$$\begin{aligned}\mathbb{P}_\lambda(E) &= \sum_{m=1}^{n(n-1)/2} \mathbb{P}_m(E)\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, p) = m\big) \\ &\geq \sum_{m=M}^{\infty} \mathbb{P}_m(E)\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, p) = m\big) \\ &\geq \mathbb{P}_M(E)\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, p) \geq M\big).\end{aligned} \tag{1.4.17}$$

In particular, when p is chosen such that $\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, p) \geq M\big) = 1 - o(1)$, then $\mathbb{P}_M(E) = o(1)$ follows when $\mathbb{P}_\lambda(E) = o(1)$.

Take $a > I_\lambda^{-1}$ and let $k_n = a \log n$. Then we shall first show that $\mathbb{P}_{n,M}(|\mathscr{C}_{\max}| \geq k_n) = o(1)$. For this, we use the above monotonicity to note that, for every λ',

$$\mathbb{P}_{n,M}(|\mathscr{C}_{\max}| \geq k_n) \leq \mathbb{P}_{\lambda'}(|\mathscr{C}_{\max}| \geq k_n)/\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, \lambda'/n) \geq M\big). \tag{1.4.18}$$

For any $\lambda' > \lambda$, we have $\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, \lambda'/n) \geq M\big) = 1 + o(1)$. Now, since $\lambda \mapsto I_\lambda^{-1}$ is continuous, we can take $\lambda' > \lambda$ such that $I_{\lambda'}^{-1} < a$, we further obtain by Theorem 4.4 that $\mathbb{P}_{\lambda'}(|\mathscr{C}_{\max}| \geq k_n) = o(1)$, so that $\mathbb{P}_{n,M}(|\mathscr{C}_{\max}| \geq k_n) = o(1)$ follows.

Next, take $a < I_\lambda^{-1}$, take $k_n = a \log n$, and we next wish to prove that $\mathbb{P}_{n,M}(|\mathscr{C}_{\max}| \leq k_n) = o(1)$. For this, we make use of a related bound as in (0.4.17), namely, for a *decreasing event* F, we obtain

$$\begin{aligned}\mathbb{P}_\lambda(F) &= \sum_{m=1}^{n(n-1)/2} \mathbb{P}_m(F)\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, p) = m\big) \\ &\geq \sum_{m=1}^{M} \mathbb{P}_m(F)\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, p) = M\big) \\ &\geq \mathbb{P}_M(F)\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, p) \leq M\big).\end{aligned} \tag{1.4.19}$$

Now, we take $p = \lambda'/n$ where $\lambda' < \lambda$, so that $\mathbb{P}\big(\mathsf{Bin}(n(n-1)/2, p) \leq M\big) = 1 - o(1)$. Then, we pick $\lambda' < \lambda$ such that $I_{\lambda'}^{-1} > a$ and use Theorem 4.5. We conclude that, with high probability, $|\mathscr{C}_{\max}|/\log n \leq I_\lambda^{-1} + \varepsilon)$ for any $\varepsilon > 0$, and, again with high probability, $|\mathscr{C}_{\max}|/\log n \geq I_\lambda^{-1} - \varepsilon)$ for any $\varepsilon > 0$. This yields directly that $|\mathscr{C}_{\max}|/\log n \xrightarrow{\mathbb{P}} I_\lambda^{-1}$. □

1.5 Solutions to the exercises of Chapter 5.

Solution to Exercise 5.1. Fix some $r > 0$, then

$$\chi(1) \geq \sum_{k=1}^{rn^{2/3}} \mathbb{P}(|\mathscr{C}(1)| \geq k) = \sum_{k=1}^{rn^{2/3}} P_{\geq k}(1). \tag{1.5.1}$$

By Proposition 5.2, we have the bounds

$$P_{\geq k}(1) \geq \frac{c_1}{\sqrt{k}}.$$

Substituting this bounds into (0.5.1) yields

$$\chi(1) \geq \sum_{k=1}^{rn^{2/3}} \frac{c_1}{\sqrt{k}} \geq c_1' r n^{1/3},$$

where $c_1' > 0$ and $r > 0$. □

Solution to Exercise 5.2. By Theorem 3.16, we have that

$$\frac{1}{\lambda}\, e^{-I_\lambda t} \mathbb{P}_1^*(T^* = t) = \frac{1}{\lambda}\, e^{-(\lambda-1-\log\lambda)t} \frac{t^{t-1}}{t!} e^{-t}.$$

Rearranging the terms in this equation we get

$$\frac{1}{\lambda}\, e^{-I_\lambda t} \mathbb{P}_1^*(T^* = t) = \frac{1}{\lambda}\, \left(e^{\log\lambda}\right)^t \frac{t^{t-1}}{t!}\, e^{-\lambda t} = \frac{(\lambda t)^{t-1}}{t!}\, e^{-\lambda t}.$$

□

Solution to Exercise ??. Take some $l \in \mathbb{N}$ such that $l < n$, then $\chi_{n-l}(\lambda \frac{n-l}{n})$ is the expected component size in the graph $\mathrm{ER}(n-l, p)$. We have to prove that the expected component size in the graph $\mathrm{ER}(n-l, p)$ is smaller than the expected component size in the graph $\mathrm{ER}(n-l+1, p)$ for all $0 < p \leq 1$. Consider the graph $\mathrm{ER}(n-l+1, p)$. This graph can be created from $\mathrm{ER}(n-l, p)$ by adding the vertex $n-l+1$ and independently connecting this vertex to each of the vertices $1, 2, \ldots, n-l$.

Let $\mathscr{C}'(1)$ denote the component of $\mathrm{ER}(n-l, p)$ which contains vertex 1 and $\mathscr{C}(1)$ represents the component of $\mathrm{ER}(n-l+1, p)$ which contains vertex 1. By the construction of $\mathrm{ER}(n-l+1, p)$, it follows that

$$\mathbb{P}(|\mathscr{C}(1)| = k) = \begin{cases} (1-p)^{n-l+1} & \text{if } k = 1, \\ \mathbb{P}(|\mathscr{C}'(1)| = k)(1-p)^k + \mathbb{P}(|\mathscr{C}'(1)| = k-1)(1-(1-p)^{k-1}) & \text{if } 2 \leq k \leq n, \\ \mathbb{P}(|\mathscr{C}'(1)| = n)(1-(1-p)^n) & \text{if } k = n+1. \end{cases}$$

Hence, the expected size of $\mathscr{C}(1)$ is

$$\begin{aligned}\mathbb{E}[|\mathscr{C}(1)|] &= \sum_{k=1}^{n+1} \mathbb{P}(|\mathscr{C}(1)| = k)k \\ &= (1-p)^{n-l+1} + \sum_{k=2}^{n} \left[\mathbb{P}(|\mathscr{C}'(1)| = k)(1-p)^k + \mathbb{P}(|\mathscr{C}'(1)| = k-1)(1-(1-p)^{k-1})\right] k \\ &\quad + \mathbb{P}(|\mathscr{C}'(1)| = n)(1-(1-p)^n)(n+1).\end{aligned}$$

Rewriting this expression for the expected size of $\mathscr{C}(1)$ yields

$$\begin{aligned}\mathbb{E}[|\mathscr{C}(1)|] &= (1-p)^{n-l+1} + \mathbb{P}(|\mathscr{C}'(1)| = 1)2p + \sum_{k=2}^{n-1} \mathbb{P}(|\mathscr{C}'(1)| = k)k \\ &\quad + \sum_{k=2}^{n-1} \mathbb{P}(\mathscr{C}'(1) = k)(1-(1-p)^{k-1}) + \mathbb{P}(|\mathscr{C}'(1)| = n)(n + (1-(1-p)^n)) \\ &\geq (1+p)\mathbb{P}(|\mathscr{C}'(1)| = 1) + \sum_{k=2}^{n-1} k\mathbb{P}(\mathscr{C}'(1) = k) \geq \mathbb{E}[|\mathscr{C}(1)'|].\end{aligned}$$

□

Solution to Exercise ??. By (**??**), we have that

$$\frac{\partial}{\partial\lambda}\chi_n(\lambda) = (n-1)\frac{\partial}{\partial\lambda}\tau_n(\lambda).$$

For the derivative of $\tau_n(\lambda)$ we use (**??**) to obtain

$$\frac{\partial}{\partial\lambda}\chi_n(\lambda) \leq \sum_{l=1}^{n} l\mathbb{P}_\lambda(|\mathscr{C}(1)| = l)\chi_{n-l}(\lambda\frac{n-l}{n}).$$

The function $l \mapsto \chi_{n-l}(\lambda\frac{n-l}{n})$ is decreasing (see Exercise **??**), hence

$$\frac{\partial}{\partial\lambda}\chi_n(\lambda) \leq \chi_n(\lambda)\sum_{l=1}^{n} l\mathbb{P}_\lambda(|\mathscr{C}(1)| = l) = \chi_n(\lambda)^2,$$

or

$$\frac{\frac{\partial}{\partial\lambda}\chi_n(\lambda)}{\chi_n(\lambda)^2} \leq 1. \tag{1.5.2}$$

The second part of the exercise relies on integration. Integrate both the left-hand and the right-hand side of (0.5.2) between λ and 1.

$$\frac{1}{\chi_n(\lambda)} - \frac{1}{\chi_n(1)} \leq 1 - \lambda$$

Bring a term to the other side to obtain

$$\frac{1}{\chi_n(\lambda)} \leq \frac{1}{\chi_n(1)} + 1 - \lambda,$$

which is equivalent to

$$\chi_n(\lambda) \geq \frac{1}{\chi_n(1)^{-1} + (1-\lambda)}.$$

□

Solution to Exercise 5.3. Using (5.3.8) and (5.3.10) we see that

$$\begin{aligned}
\mathbb{E}_\lambda[Y^2] &= n\mathbb{P}_\lambda(|\mathscr{C}(1)| = 1) + n(n-1)\left(\frac{\lambda}{n(1-\frac{\lambda}{n})} + 1\right)\mathbb{P}_\lambda(|\mathscr{C}(1)| = 1)^2 \\
&= n\left(1 - \frac{\lambda}{n}\right)^{n-1} + n(n-1)\left(1 - \frac{\lambda}{n}\right)^{2n-3} \\
&= n\left(1 - \frac{\lambda}{n}\right)^{n-1}\left(1 + (n-1)\left(1 - \frac{\lambda}{n}\right)^{n-2}\right).
\end{aligned}$$

Consider the first power, taking the logarithm yields

$$\log n + (n-1)\log(1 - \frac{\lambda}{n}) = \log n + (n-1)\log(1 - \frac{\log n + t}{n}).$$

Taylor expanding the logarithm gives

$$\log n + (n-1)\log(1 - \frac{\log n + t}{n}) = \log n - (n-1)\Big[\frac{\log n + t}{n} + O\Big((\frac{\log n + t}{n})^2\Big)\Big].$$

The latter expression can be simplified to

$$\begin{aligned}
\log n - (n-1)\Big[\frac{\log n + t}{n} + O\Big((\frac{\log n + t}{n})^2\Big)\Big] &= \log n - \frac{n-1}{n}\log n - \frac{n-1}{n}t + O\Big(\frac{(\log n + t)^2}{n}\Big) \\
&= -t + \frac{\log n}{n} + \frac{t}{n} + O\Big(\frac{(\log n + t)^2}{n}\Big),
\end{aligned}$$

and, as n tends to infinity,

$$-t + \frac{\log n}{n} + \frac{t}{n} + O\Big(\frac{(\log n + t)^2}{n}\Big) \to -t.$$

Hence,

$$\lim_{n\to\infty} n\left(1 - \frac{\lambda}{n}\right)^{n-1} = e^{-t}.$$

A similar argument gives that as $n \to \infty$

$$\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^{n-2} = e^{-t}.$$

Therefore, we conclude

$$\lim_{n\to\infty}\mathbb{E}_\lambda[Y^2] = e^{-t}(1 - e^{-t}),$$

which is the second moment of a Poisson random variable with mean e^{-t}.

□

1.6 Solutions to the exercises of Chapter 6.

Solution to Exercise 6.1. By the definition of p_{ij} (6.1.1), the numerator of p_{ij} is $(n\lambda)^2(n-\lambda)^{-2}$. The denominator of p_{ij} is

$$\sum_{i=1}^{n} \frac{n\lambda}{n-\lambda} + \left(\frac{n\lambda}{n-\lambda}\right)^2 = \frac{n^2\lambda}{n-\lambda} + \left(\frac{n\lambda}{n-\lambda}\right)^2 = \frac{n^2\lambda(n-\lambda)+(n\lambda)^2}{(n-\lambda)^2} = \frac{n^3\lambda}{(n-\lambda)^2}.$$

Dividing the numerator of p_{ij} by its denominator gives

$$p_{ij} = \frac{(n\lambda)^2}{n^3\lambda} = \frac{\lambda}{n}.$$

□

Solution to Exercise 6.2. Consider the distribution function $F_n(x) = \mathbb{P}(w_U \leq x)$ of a uniformly chosen vertex U and let $x \geq 0$. The law of total probability gives that

$$\begin{aligned} \mathbb{P}(w_U \leq x) &= \sum_{i=1}^{n} \mathbb{P}(w_U \leq x | U = i)\mathbb{P}(U = i) \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{w_i \leq x\}}, \qquad x \geq 0, \end{aligned} \tag{1.6.1}$$

as desired. □

Solution to Exercise 6.4. By (6.1.17), $F_n(x) = \frac{1}{n}(\lfloor nF(x)\rfloor + 1) \wedge 1$. To prove pointwise convergence of this function to $F(x)$, we shall first examine its behavior when $F(x)$ gets close to 1. Consider the case where $\frac{1}{n}(\lfloor nF(x)\rfloor + 1) > 1$, or equivalently, $\lfloor nF(x)\rfloor > n-1$, which is in turn equivalent to $F(x) > \frac{n-1}{n}$. Now fixing x gives us two possibilities: either $F(x) = 1$ or there is an n such that $F(x) \leq \frac{n-1}{n}$. In the first case, we have that

$$\begin{aligned} \left|\left[\frac{1}{n}(\lfloor nF(x)\rfloor + 1) \wedge 1\right] - F(x)\right| &= \left|\left[\frac{1}{n}(\lfloor n\rfloor + 1) \wedge 1\right] - 1\right| \\ &= |1-1| = 0. \end{aligned} \tag{1.6.2}$$

In the second case, we have that for large enough n

$$\begin{aligned} \left|\left[\frac{1}{n}(\lfloor nF(x)\rfloor + 1) \wedge 1\right] - F(x)\right| &= \left|\frac{1}{n}(\lfloor nF(x)\rfloor + 1) - \frac{nF(x)}{n}\right| \\ &= \left|\frac{\lfloor nF(x)\rfloor - nF(x) + 1}{n}\right| \leq \left|\frac{1}{n}\right| \to 0, \end{aligned} \tag{1.6.3}$$

which proves the pointwise convergence of F_n to F, as desired. □

Solution to Exercise 6.5. We note that $x \mapsto F(x)$ is non-decreasing, since it is a distribution function. This implies that $x \mapsto 1 - F(x)$ is non-increasing, so that $u \mapsto [1-F]^{-1}(u)$ is non-increasing.

To see (6.1.19), we let U be a uniform random variable, and note that

$$\frac{1}{n}\sum_{i=1}^{n} h(w_i) = \mathbb{E}\Big[h\Big([1-F]^{-1}(\lceil Un\rceil/n)\Big)\Big]. \tag{1.6.4}$$

Now, $\lceil Un\rceil/n \geq U$ a.s., and since $u \mapsto [1-F]^{-1}(u)$ is non-increasing, we obtain that $[1-F]^{-1}(\lceil Un\rceil/n) \leq [1-F]^{-1}(U)$ a.s. Further, again since $x \mapsto h(x)$ is non-decreasing,

$$h\Big([1-F]^{-1}(\lceil Un\rceil/n)\Big) \leq h\big([1-F]^{-1}(U)\big). \tag{1.6.5}$$

Thus,

$$\frac{1}{n}\sum_{i=1}^{n} h(w_i) \leq \mathbb{E}\Big[h\Big([1-F]^{-1}(U)\Big)\Big] = \mathbb{E}[h(W)], \tag{1.6.6}$$

since $[1-F]^{-1}(U)$ has distribution function F when U is uniform on $(0,1)$ (recall the remark below (6.1.16)). □

Solution to Exercise 6.6. Using the non-decreasing function $h(x) = x^{\alpha}$ in Exercise 6.5, we have that for a uniform random variable U

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} w_i^{\alpha} &= \int_0^1 [1-F]^{-1}\left(\frac{\lceil un\rceil}{n}\right)\frac{1}{n}du \\ &= \mathbb{E}\Big[\Big([1-F]^{-1}(\lceil Un\rceil/n)\Big)^{\alpha}\Big].\end{aligned} \tag{1.6.7}$$

We also know that $\lceil Un\rceil/n \geq U$ a.s., and since $u \mapsto [1-F]^{-1}(u)$ is non-increasing by Exercise 6.5 and $x \mapsto x^{\alpha}$ is non-decreasing, we obtain that

$$\frac{1}{n}\left([1-F]^{-1}(\lceil Un\rceil/n)\right)^{\alpha} \leq \frac{1}{n}\left([1-F]^{-1}(U)\right)^{\alpha}. \tag{1.6.8}$$

The right hand side function is integrable with value $\mathbb{E}[W^{\alpha}]$, by assumption. Therefore, by the dominated convergence theorem (Theorem A.17), we have that the integral of the left hand side converges to the integral of its pointwise limit. Since $\lceil Un\rceil/n$ converges in distribution to U, we get that $[1-F]^{-1}(\lceil Un\rceil/n) \to [1-F]^{-1}(U)$, as desired. □

Solution to Exercise 6.7. By (6.1.14),

$$w_i = [1-F]^{-1}(i/n). \tag{1.6.9}$$

Now apply the function $[1-F]$ to both sides to get

$$[1-F](w_i) = i/n, \tag{1.6.10}$$

which, by the assumption, can be bounded from above by

$$i/n = [1-F](w_i) \leq c w_i^{-(\tau-1)}. \tag{1.6.11}$$

This inequality can be rewritten to

$$i^{-\frac{1}{\tau-1}}(cn)^{\frac{1}{\tau-1}} \geq w_i, \tag{1.6.12}$$

where the left hand side is a descending function in i for $\tau > 1$. This implies

$$w_i \leq w_1 \leq c^{\frac{1}{\tau-1}} n^{\frac{1}{\tau-1}}, \quad \forall i \in [n], \tag{1.6.13}$$

giving the $c' = c^{\frac{1}{\tau-1}}$ as desired. □

Solution to Exercise 6.9. A mixed Poisson variable X has the property that $\mathbb{P}(X=0) = \mathbb{E}[e^{-W}]$ is strictly positive, unless W is infinite whp. Therefore, the random variable Y with $\mathbb{P}(Y=1) = \frac{1}{2}$ and $\mathbb{P}(Y=2) = \frac{1}{2}$ cannot be represented by a mixed Poisson variable. □

Solution to Exercise 6.10. By definition, the characteristic function of X is

$$\mathbb{E}[e^{itX}] = \sum_{n=0}^{\infty} \mathrm{e}^{itn}\mathbb{P}(X=n) = \sum_{n=0}^{\infty} \mathrm{e}^{itn}\left(\int_0^\infty f_W(w)\frac{\mathrm{e}^{-w}w^n}{n!}dw\right),$$

where $f_W(w)$ is the density function of W evaluated in w. Since all terms are non-negative we can interchange summation and integration. Rearranging the terms gives

$$\begin{aligned}\mathbb{E}[e^{itX}] &= \int_0^\infty f_W(w)\mathrm{e}^{-w}\left(\sum_{n=0}^{\infty}\frac{(\mathrm{e}^{it}w)^n}{n!}\right)dw = \int_0^\infty f_W(w)\mathrm{e}^{-w}\exp(e^{it}w)dw \\ &= \int_0^\infty f_W(w)\exp((\mathrm{e}^{it}-1)w)dw.\end{aligned}$$

The latter expression is the moment generating function of W evaluated in $e^{it}-1$. □

Solution to Exercise 6.11. By the tower rule, we have that $\mathbb{E}[\mathbb{E}[X|W]] = \mathbb{E}[X]$. Computing the expected value on the left hand side gives

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X|W]] &= \sum_w \mathbb{E}[X|W=w]\mathbb{P}(W=w) \\ &= \sum_w \mathbb{P}(W=w)\sum_k k e^{-w}\frac{w^k}{k!} \\ &= \sum_w w\cdot\mathbb{P}(W=w)\cdot e^{-w}\sum_k \frac{w^{(k-1)}}{(k-1)!} \\ &= \sum_w w\cdot\mathbb{P}(W=w) = \mathbb{E}[W],\end{aligned} \tag{1.6.14}$$

so $\mathbb{E}[X] = \mathbb{E}[W]$. For the second moment of X, we consider $\mathbb{E}[\mathbb{E}[X(X-1)|W]] = \mathbb{E}[X(X-1)]$. Computing the expected value on the left hand side gives

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X(X-1)|W]] &= \sum_w \mathbb{E}[X(X-1)|W=w]\mathbb{P}(W=w)\\ &= \sum_w \mathbb{P}(W=w)\sum_k k(k-1)e^{-w}\frac{w^k}{k!}\\ &= \sum_w w^2\cdot\mathbb{P}(W=w)\cdot e^{-w}\sum_k \frac{w^{(k-2)}}{(k-2)!}\\ &= \sum_w w^2\cdot\mathbb{P}(W=w) = \mathbb{E}[W^2]. \end{aligned} \tag{1.6.15}$$

Now, we have that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[W^2] + \mathbb{E}[W] - \mathbb{E}[W]^2$, which is the sum of the variance and expected value of W. □

Solution to Exercise 6.13. Suppose there exists a $\varepsilon > 0$ such that $\varepsilon \le w_i \le \varepsilon^{-1}$ for every i. Now take the coupling D_i' as in (**??**). Now, by (**??**), we obtain that

$$\begin{aligned}\mathbb{P}\Big((D_1,\dots,D_m) \neq (\hat{D}_1,\dots,\hat{D}_m)\Big) &\le 2\sum_{i,j=1}^m p_{ij}\\ &= 2\sum_{i,j=1}^m \frac{w_iw_j}{l_n + w_iw_j}. \end{aligned} \tag{1.6.16}$$

Now $l_n = \sum_{i=1}^n w_i \ge n\varepsilon$ and $\varepsilon^2 \le w_iw_j \le \varepsilon^{-2}$. Therefore,

$$2\sum_{i,j=1}^m \frac{w_iw_j}{l_n + w_iw_j} \le 2m^2\frac{\varepsilon^{-2}}{n\varepsilon + \varepsilon^2} = o(1), \tag{1.6.17}$$

since $m = o(\sqrt{n})$. □

Solution to Exercise 6.14. We have to prove

$$\max_k |\mathbb{E}[P_k^{(n)}] - p_k| \le \frac{\varepsilon}{2}. \tag{1.6.18}$$

We have

$$\max_k |\mathbb{E}[P_k^{(n)}] - p_k| \le \frac{\varepsilon}{2} \Leftrightarrow \forall_k |\mathbb{E}[p_k^{(n)}] - p_k| \le \frac{\varepsilon}{2}. \tag{1.6.19}$$

Furthermore the following limit is given

$$\lim_{n\to\infty} \mathbb{E}[P_k^{(n)}] = \lim_{n\to\infty} \mathbb{P}(D_1 = k) = p_k. \tag{1.6.20}$$

Hence we can write

$$\forall_{\varepsilon>0}\forall_k\exists_{M_k}\forall_{n>M_k}|\mathbb{E}[P_k^{(n)}] - p_k| \le \frac{\varepsilon}{2} \tag{1.6.21}$$

Taking $M := \max_k M_k$ we obtain

$$\forall_{\varepsilon>0}\exists_M\forall_k\forall_{n>M}|\mathbb{E}[P_k^{(n)}]-p_k|\le\tfrac{\varepsilon}{2}$$
$$\Leftrightarrow$$
$$\forall_{\varepsilon>0}\exists_M\forall_{n>M}\max_k|\mathbb{E}[P_k^{(n)}]-p_k|\le\tfrac{\varepsilon}{2}.$$

□

Solution to Exercise 6.15. Using the hint, we get

$$\begin{aligned}\mathbb{P}(\max_{i=1}^{n} W_i\ge\varepsilon n) &\le \sum_{i=1}^{n}\mathbb{P}(W_i\ge\varepsilon n)\\ &= n\mathbb{P}(W_1\ge\varepsilon n).\end{aligned}\tag{1.6.22}$$

This probability can be rewritten, and applying the Markov inequality now gives

$$n\mathbb{P}(W_1\ge\varepsilon n)=n\mathbb{P}(\mathbb{1}_{\{W_1\ge\varepsilon n\}}W_1\ge\varepsilon n)\le\mathbb{P}(W_1\ge\varepsilon n)\mathbb{E}[W_1]\to 0.\tag{1.6.23}$$

Therefore, $\max_{i=1}^{n} W_i$ is $o(n)$ whp, and

$$\frac{1}{n^2}\sum_{i=1}^{n}W_i^2\le\frac{1}{n}\max_{i=1}^{n}W_i^2\to 0,\tag{1.6.24}$$

as desired. □

Solution to Exercise 6.17. Using partial integration we obtain for the mean of W_1

$$\begin{aligned}\mathbb{E}[W_1] &= \int_0^\infty xf(x)dx=[xF(x)-x]_{x=0}^{\infty}+\int_0^\infty[1-F(x)]dx\\ &= \Big(\lim_{R\to\infty}RF(R)-R\Big)-0+\int_0^\infty[1-F(x)]dx\\ &= \int_0^\infty 1-F(x)dx\end{aligned}$$

Hence,

$$\mathbb{E}[W_1]=\infty\Leftrightarrow\int_0^\infty[1-F(x)]dx=\infty.\tag{1.6.25}$$

□

Solution to Exercise 6.20. It suffices to prove that $\prod_{1\le i<j\le n}(u_iu_j)^{x_{ij}}=\prod_{i=1}^{n}u_i^{d_i(x)}$, where $d_i(x)=\sum_{j=1}^{n}x_{ij}$.
The proof will be given by a simple counting argument. Consider the powers of u_k in the left hand side, for some $k=1,\dots,n$. For $k<j\le n$, the left hand side contains the terms $u_k^{x_{kj}}$, whereas for $1\le i<k$, it contains the terms $u_k^{x_{ik}}$. When combined, and using the fact that $x_{ij}=x_{ji}$ for all i,j, we see that the powers of u_k in the left hand side can be written as $\sum_{j\ne k}x_{kj}$. But since, $x_{ii}=0$ for all i, this equals $\sum_{j=1}^{n}x_{ij}=d_i(x)$, as required. □

Solution to Exercise 6.21. We pick $t_k = t$ and $t_i = 1$ for all $i \neq k$. Then,

$$\mathbb{E}[t^{D_k}] = \prod_{1\leq i\leq n: i\neq k} \frac{l_n + w_i w_k t}{l_n + w_i w_k}$$
$$= e^{w_k(t-1)\sum_{1\leq i\leq n: i\neq k} \frac{w_i}{l_n} + R_n}, \tag{1.6.26}$$

where

$$R_n = \sum_{1\leq i\leq n: i\neq k} \log\Big(1 + \frac{w_i w_k t}{l_n}\Big) - \log\Big(1 + \frac{w_i w_k}{l_n}\Big) - w_k(t-1) \sum_{1\leq i\leq n: i\neq k} \frac{w_i}{l_n}$$
$$= \sum_{1\leq i\leq n: i\neq k} \log(l_n + w_i w_k t) - \log(l_n + w_i w_k) - w_k(t-1) \sum_{1\leq i\leq n: i\neq k} \frac{w_i}{l_n}. \tag{1.6.27}$$

A Taylor expansion of $x \mapsto \log(a + x)$ yields that

$$\log(a + x) = \log(a) + \frac{x}{a} + O(\frac{x^2}{a^2}). \tag{1.6.28}$$

Therefore, applying the above with $a = l_n$ and $x = w_i w_k$, yields that, for t bounded,

$$R_n = O(w_k^2 \sum_{i=1}^n \frac{w_i^2}{\ell_n^2}) = o(1), \tag{1.6.29}$$

by Exercise 6.3, so that

$$\mathbb{E}[t^{D_k}] = e^{w_k(t-1)\sum_{1\leq i\leq n: i\neq k} \frac{w_i}{\ell_n}} (1 + o(1))$$
$$= e^{w_k(t-1)}(1 + o(1)), \tag{1.6.30}$$

since w_k is fixed. Since the generating function of the degree converges, the degree of vertex k converges in distribution to a random variable with generating function $e^{w_k(t-1)}$ (recall Theorem 2.3(c)). The probability generating function of a Poisson random variable with mean λ is given by $e^{\lambda(t-1)}$, which completes the proof of Theorem 6.7(a).

For Theorem 6.7(b), we use similar ideas, now taking $t_i = t_i$ for $i \leq m$ and $t_i = 0$ for $i > m$. Then,

$$\mathbb{E}[\prod_{i=1}^m t_i^{D_i}] = \prod_{1\leq i\leq m, i<j\leq n} \frac{l_n + w_i w_j t_i}{l_n + w_i w_j} = \prod_{i=1}^m e^{w_i(t_i-1)}(1 + o(1)), \tag{1.6.31}$$

so that the claim follows. □

Solution to Exercise 6.22. The degree of vertex k converges in distribution to a random variable with generating function $e^{w_k(t-1)}$. We take $w_i = \frac{\lambda}{1-\lambda/n}$ which yields for the generating function $e^{\frac{\lambda(t-1)}{1-\lambda/n}}$. This gives us for the degree a $\mathsf{Poi}(\frac{\lambda}{1-\lambda/n})$ random variable, which for large n is close to a $\mathsf{Poi}(\lambda)$ random variable.

Solution to Exercise 6.23. The Erdős-Rényi Random Graph is obtained by taking $W_i \equiv \frac{\lambda}{1-\frac{\lambda}{n}}$. Since $p_{ij} = \lambda/n \to 0$, Theorem 6.7(b) states that the degrees are asymptotically independent.

□

Solution to Exercise 6.24. Let X be a mixed Poisson random variable with mixing distribution $\gamma W^{\tau-1}$. The generating function of X now becomes

$$\begin{aligned} G_X(t) = \mathbb{E}[t^X] &= \sum_{k=0}^{\infty} t^k \mathbb{P}(X = k) \\ &= \sum_{k=0}^{\infty} t^k \mathbb{E}[e^{-\gamma W^{\tau-1}} \frac{(\gamma W^{\tau-1})^k}{k!}] \\ &= \mathbb{E}\left[e^{-\gamma W^{\tau-1}} \sum_{k=0}^{\infty} \frac{(\gamma W^{\tau-1} t)^k}{k!}\right] \\ &= \mathbb{E}[e^{(t-1)\gamma W^{\tau-1}}] \end{aligned} \tag{1.6.32}$$

□

Solution to Exercise 6.25. By using partial integration we obtain

$$\begin{aligned} \mathbb{E}[h(X)] &= \int_0^{\infty} h(x) f(x) dx \\ &= [h(x)(F(x)) - 1]_{x=0}^{\infty} - \int_0^{\infty} h'(x)[F(x) - 1] dx \\ &= \Big(\lim_{R\to\infty} h(R)(1 - F(R))\Big) - h(0)(1 - F(0)) + \int_0^{\infty} h'(x)[1 - F(x)] dx \\ &= \int_0^{\infty} h'(x)[1 - F(x)] dx. \end{aligned}$$

□

Solution to Exercise 6.27. By definition, $p^{(n)}$ and $q^{(n)}$ are asymptotically equivalent if for every sequence (x_n) of events

$$\lim_{n\to\infty} p^{(n)}_{x_n} - q^{(n)}_{x_n} = 0. \tag{1.6.33}$$

By taking the sequence of events $x_n \equiv x \in \mathcal{X}$ for all n, this means that asymptotical equivalence implies that also

$$\lim_{n\to\infty} \max_{x\in\mathcal{X}} |p^{(n)}_x - q^{(n)}_x| = \lim_{n\to\infty} d_{\mathrm{TV}}(p^{(n)}, q^{(n)}) = 0. \tag{1.6.34}$$

Conversely, if the total variation distance converges to zero, which means that the maximum over all $x \in \mathcal{X}$ of the difference $p^{(n)}_x - q^{(n)}_x$ converges in absolute value to zero. Since this maximum is taken over all $x \in \mathcal{X}$, it will certainly hold for all $x \in (x_n) \subseteq \mathcal{X}$ as well. Therefore, it follows that for any sequence of events, $p^{(n)}_{x_n} - q^{(n)}_{x_n}$ must converge to zero as well, which implies asymptotical equivalence. /ensol

Solution to Exercise 6.28. We recall that

$$d_{\mathrm{TV}}(M,M') = \sup_{A\subset \mathbb{Z}} |\mathbb{P}(M\in A) - \mathbb{P}(M'\in A)|. \tag{1.6.35}$$

Now, for binomial random variables with the same m and with success probabilities p and q respectively, we have that

$$\frac{\mathbb{P}(M=k)}{\mathbb{P}(M'=k)} = \big(\frac{p}{q}\big)^k\big(\frac{1-p}{1-q}\big)^{m-k} = \big(\frac{1-p}{1-q}\big)^m\big(\frac{p(1-q)}{q(1-p)}\big)^k, \tag{1.6.36}$$

which is monotonically increasing or decreasing for $p\neq q$. As a result, we have that the supremum in (0.6.35) is attained for a set $A=\{0,\ldots,j\}$ for some $j\in\mathbb{N}$, i.e.,

$$d_{\mathrm{TV}}(M,M') = \sup_{j\in\mathbb{N}} |\mathbb{P}(M\leq j) - \mathbb{P}(M'\leq j)|. \tag{1.6.37}$$

Now assume that $\lim_{N\to\infty} m(p-q)/\sqrt{mp} = \alpha\in(-\infty,\infty)$. Then, by Exercise 4.22, $(M-mp)/\sqrt{mp} \xrightarrow{d} Z\sim\mathcal{N}(0,1)$ and $(M'-mp)/\sqrt{mp} \xrightarrow{d} Z' sim \mathcal{N}(\alpha,1)$, where $\mathcal{N}(\mu,\sigma^2)$ denotes a normal random variable with mean μ and variance σ^2. Therefore, we arrive at

$$\begin{aligned} d_{\mathrm{TV}}(M,M') &= \sup_{j\in\mathbb{N}} |\mathbb{P}(M\leq j) - \mathbb{P}(M'\leq j)| = \sup_{x\in\mathbb{R}} |\mathbb{P}(Z\leq x) - \mathbb{P}(Z'\leq x)| + o(1)\\ &\to \Phi(\alpha/2) - \Phi(-\alpha/2), \end{aligned} \tag{1.6.38}$$

where $x\mapsto \Phi(x)$ is the distribution function of a standard normal random variable. Thus, $d_{\mathrm{TV}}(M,M') = o(1)$ precisely when $\alpha=0$, which implies that $m(p-q)/\sqrt{mp} = o(1)$.

Solution to Exercise 6.29. We write

$$\begin{aligned} d_{\mathrm{TV}}(p,q) &= \frac{1}{2}\sum_x |p_x - q_x| = \frac{1}{2}\sum_x (\sqrt{p_x}+\sqrt{q_x})|\sqrt{p_x}-\sqrt{q_x}|\\ &= \frac{1}{2}\sum_x \sqrt{p_x}|\sqrt{p_x}-\sqrt{q_x}| + \frac{1}{2}\sum_x \sqrt{q_x}|\sqrt{p_x}-\sqrt{q_x}|. \end{aligned} \tag{1.6.39}$$

By the Cauchy-Schwarz inequality, we obtain that

$$\sum_x \sqrt{p_x}|\sqrt{p_x}-\sqrt{q_x}| \leq \sqrt{\sum_x p_x}\sqrt{\sum_x(\sqrt{p_x}-\sqrt{q_x})^2} \leq 2^{-1/2} d_{\mathrm{H}}(p,q). \tag{1.6.40}$$

The same bound applies to the second sum on the right-hand side of (0.6.39), which proves the upper bound in (6.6.11).

For the lower bound, we bound

$$d_{\mathrm{H}}(p,q)^2 = \frac{1}{2}\sum_x(\sqrt{p_x}-\sqrt{q_x})^2 \leq \frac{1}{2}\sum_x(\sqrt{p_x}+\sqrt{q_x})|\sqrt{p_x}-\sqrt{q_x}| = d_{\mathrm{TV}}(p,q). \tag{1.6.41}$$

Solution to Exercise 6.30. By exercise 6.27, we have that $p^{(n)} = \{p_x^{(n)}\}_{x\in\mathcal{X}}$ and $q^{(n)} = \{q_x^{(n)}\}_{x\in\mathcal{X}}$ are asymptotically equivalent if and only if their total variation distance converges to zero. By exercise 6.29, we know that (6.6.11) holds, and therefore also

$$2^{-1/2} d_{\mathrm{TV}}(p^{(n)}, q^{(n)}) \le d_{\mathrm{H}}(p^{(n)}, q^{(n)}) \le \sqrt{d_{\mathrm{TV}}(p^{(n)}, q^{(n)})}. \tag{1.6.42}$$

Both the left and right hand side of those inequalities converge to zero if $d_{\mathrm{TV}}(p^{(n)}, q^{(n)}) \to 0$, which implies by the sandwich theorem that $d_{\mathrm{H}}(p^{(n)}, q^{(n)}) \to 0$. Conversely, if $d_{\mathrm{H}}(p^{(n)}, q^{(n)}) \to 0$, by (6.6.11) we have that $d_{\mathrm{TV}}(p^{(n)}, q^{(n)}) \to 0$. □

Solution to Exercise 6.31. We bound

$$\begin{aligned}\rho(p,q) &= \big(\sqrt{p}-\sqrt{q}\big)^2 + \big(\sqrt{1-p}-\sqrt{1-q}\big)^2 \\ &= (p-q)^2\big((\sqrt{p}+\sqrt{q})^{-2} + (\sqrt{1-p}+\sqrt{1-q})^{-2}\big).\end{aligned} \tag{1.6.43}$$

Solution to Exercise 6.32. We wish to show that $\mathbb{P}(Y=k) = e^{-\lambda p}\frac{(\lambda p)^k}{k!}$. We will use that in the case of X fixed, Y is simply a $\mathsf{Bin}(X,p)$ random variable. We have

$$\begin{aligned}\mathbb{P}(Y=k) &= \mathbb{P}\Big(\sum_{i=0}^{X} I_i = k\Big) = \sum_{x=k}^{\infty} \mathbb{P}(X=x)\cdot \mathbb{P}\Big(\sum_{i=0}^{x} I_i = k\Big) \\ &= \sum_{x=k}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!}\cdot \binom{x}{k} p^k(1-p)^{x-k} = e^{-\lambda}\sum_{x=k}^{\infty}\frac{\lambda^x}{x!}\cdot\frac{x!}{(x-k)!k!}p^k(1-p)^{x-k} \\ &= e^{-\lambda}\frac{(\lambda p)^k}{k!}\sum_{x=k}^{\infty}\frac{\lambda^{x-k}(1-p)^{x-k}}{(x-k)!} = e^{-\lambda}\frac{(\lambda p)^k}{k!}\sum_{x=0}^{\infty}\frac{(\lambda-\lambda p)^x}{x!} \\ &= e^{-\lambda}e^{\lambda-\lambda p}\frac{(\lambda p)^k}{k!} = e^{-\lambda p}\frac{(\lambda p)^k}{k!}\end{aligned}$$

If we define Y to be the number of edges between i and j at time t and X the same at time $t-1$. Furthermore we define I_k to be the decision of keeping edge k or not. It is given that $X \sim \mathsf{Poi}(\frac{W_iW_j}{L_{t-1}})$ and $I_k \sim \mathsf{Be}(1-\frac{W_t}{L_t})$. According to what is shown above we now obtain for Y to be a Poisson random variable with parameter

$$\frac{W_iW_j}{L_{t-1}}\cdot\Big(1-\frac{W_t}{L_t}\Big) = W_iW_j\frac{1}{L_{t-1}}\frac{L_t-W_t}{L_t} = W_iW_j\frac{1}{L_{t-1}}\frac{L_{t-1}}{L_T} = \frac{W_iW_j}{L_t} \tag{1.6.44}$$

□

Solution to Exercise 6.33. A graph is simple when it has no self loops or double edges between vertices. Therefore, the Norros-Reittu random graph is simple at time n if for all i $X_{ii}=0$, and for all $i \neq j$ $X_{ij}=0$ or $X_{ij}=1$. By Exercise 6.32, we know that the number of edges X_{ij} between i and j at time n are Poisson with parameter

$\frac{w_i w_j}{\ell_n}$. The probability then becomes

$$\begin{aligned}
\mathbb{P}(\mathrm{NR}_n(\boldsymbol{w}) \text{ simple}) &= \mathbb{P}(0 \leq X_{ij} \leq 1, \forall i \neq j)\mathbb{P}(X_{ii} = 0, \forall i)\\
&= \prod_{1\leq i<j\leq n} (\mathbb{P}(X_{ij} = 0) + \mathbb{P}(X_{ij} = 1)) \prod_{k=1}^{n} \mathbb{P}(X_{kk} = 0)\\
&= \prod_{1\leq i<j\leq n} \mathrm{e}^{-\frac{w_i w_j}{\ell_n}} (1 + \frac{w_i w_j}{\ell_n}) \prod_{k=1}^{n} \mathrm{e}^{-\frac{w_k^2}{\ell_n}}\\
&= \mathrm{e}^{-\sum_{1\leq i\leq j\leq n} \frac{w_i w_j}{\ell_n}} \prod_{1\leq i<j\leq n} (1 + \frac{w_i w_j}{\ell_n}). \qquad (1.6.45)
\end{aligned}$$

□

Solution to Exercise 6.34. Let $X_{ij} \sim \mathsf{Poi}(\frac{w_i w_j}{\ell_n})$ be the number of edges between vertex i and j at time n. The degree of vertex k at time n becomes $\sum_{j=1}^{n} X_{kj}$, and because X_{kj} is Poisson with mean $\frac{w_k w_j}{L_n}$, the sum will be Poisson with mean $\sum_{j=1}^{n} \frac{w_k w_j}{\ell_n} = w_k \frac{\sum_{j=1}^{n} w_j}{\ell_n} = W_k$. Therefore, since the w_i are i.i.d, the degree at time n has a mixed Poisson distribution with mixing distribution F_w

Solution to Exercise 6.35. Couple $X_n = X(G_n)$ and $X'_n = X(G'_n)$ by coupling the edge occupation statuses X_{ij} of G_n and X'_{ij} of G'_n such that (6.7.12) holds. Let $(\hat{X}_n, \hat{X}'_n)$ be this coupling and let E_n and E'_n be the sets of edges of the coupled versions of G_n and G'_n, respectively. Then, since X is increasing

$$\mathbb{P}(\hat{X}_n \leq \hat{X}'_n) \geq \mathbb{P}(E_n \subseteq E'_n) = \mathbb{P}(X_{ij} \leq X'_{ij} \forall i, j \in [n]) = 1, \qquad (1.6.46)$$

which proves the stochastic domination by Lemma 2.12.

1.7 Solutions to the exercises of Chapter 7.

Solution to Exercise 7.1. Consider for instance the graph of size $n = 4$ with degrees $\{d_1, \ldots, d_4\} = \{3, 3, 1, 1\}$ or the graph of size $n = 5$ with degrees $\{d_1, \ldots, d_5\} = \{4, 4, 3, 2, 1\}$. □

Solution to Exercise 7.2. For $2m$ vertices we use m pairing steps, each time pairing two vertices with each other. For step $i + 1$, we have already paired $2i$ vertices. The next vertex can thus be paired with $2m - 2i - 1$ other possible vertices. This gives for all pairing steps the total amount of possibilities to be

$$(2m - 1)(2m - 3) \cdots (2m - (2m - 2) - 1) = (2m - 1)!!. \qquad (1.7.1)$$

□

Solution to Exercise 7.8. We can write

$$\mathbb{P}\Big(L_n \text{ is odd}\Big) \quad = \quad \mathbb{P}\Big((-1)^{L_n} = -1\Big) = \frac{1}{2}\Big(1 - \mathbb{E}[(-1)^{L_n}]\Big). \tag{1.7.2}$$

To compute $\mathbb{E}[(-1)^{L_n}]$, we use the characteristic function $\phi_{D_1}(t) = \mathbb{E}[e^{itD_1}]$ as follows:

$$\phi_{D_1}(\pi) \quad = \quad \mathbb{E}[(-1)^{D_1}] \tag{1.7.3}$$

Since $(-1)^{L_n} = (-1)^{\sum D_i}$ where $\{D_i\}_{i=1}^n$ are i.i.d. random variables, we have for the characteristic function of L_n, $\phi_{L_n}(\pi) = (\phi_{D_1}(\pi))^n$. Furthermore, we have

$$\phi_{D_1}(\pi) = -\mathbb{P}(D_1 \text{ is odd}) + \mathbb{P}(D_1 \text{ is even}). \tag{1.7.4}$$

Now we assume $\mathbb{P}(D_1 \text{ is odd}) \notin \{0, 1\}$. This gives us

$$-1 < \mathbb{P}(D_1 \text{ is even}) - \mathbb{P}(D_1 \text{ is odd}) < 1, \tag{1.7.5}$$

so that $|\phi_{D_1}(\pi)| < 1$, which by (0.7.2) leads directly to the statement that $\mathbb{P}(L_n \text{ is odd})$ is exponentially close to $\frac{1}{2}$.

$\square$

Solution to Exercise 7.10. We compute

$$\sum_{k=1}^{\infty} k p_k^{(n)} \quad = \quad \sum_{k=1}^{\infty} k\Big(\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{\tilde{d}_i = k\}}\Big) = \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{\infty} k\mathbb{1}_{\{\tilde{d}_i = k\}} = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{\ell_n}{n}$$

$\square$

Solution to Exercise 7.11. The probability that there are at least three edges between i and j is bounded above by

$$\frac{d_i(d_i-1)(d_i-2)d_j(d_j-1)(d_j-2)}{(\ell_n-1)(\ell_n-3)(\ell_n-5)}. \tag{1.7.6}$$

Thus, by Boole's inequality, the probability that there exist vertices $i \neq j$ such that there are at least *three* edges between i and j is bounded above by

$$\sum_{i,j=1}^{n} \frac{d_i(d_i-1)(d_i-2)d_j(d_j-1)(d_j-2)}{(\ell_n-1)(\ell_n-3)(\ell_n-5)} = o(1), \tag{1.7.7}$$

since $d_i = o(\sqrt{n})$ when Condition 7.7(a)-(c) holds (this follows by applying Exercise 6.3 to the weights $\boldsymbol{w} = \boldsymbol{d}$) as well as $\ell_n \geq n$. We conclude that the probability that there are $i, j \in [n]$ such that there are at least three edges between i and j is $o(1)$ as $n \to \infty$. As a result, (S_n, M_n) converges in distribution to (S, M) precisely when $(S_n, \widetilde{M}_n)$ converges in distribution to (S, M).

Solution to Exercise 7.12. We start by evaluating (7.4.18) from the right- to the left-hand side.

$$\begin{aligned}\mu\mathbb{E}[(X+1)^{r-1}] &= \mu\sum_{k=1}^{\infty}(k+1)^{r-1}\frac{\mathrm{e}^{-\mu}\mu^k}{k!} = \sum_{k=1}^{\infty}(k+1)^r\frac{\mathrm{e}^{-\mu}\mu^{k+1}}{(k+1)!};\\ &= \sum_{n=1}^{\infty}n^r\frac{\mathrm{e}^{-\mu}\mu^n}{n!} = \sum_{x=0}^{\infty}x^r\frac{\mathrm{e}^{-\mu}\mu^x}{x!} = \mathbb{E}[X^r].\end{aligned}$$

Now we can use the independency of the two random variables and the result above for the evaluation of (7.4.19).

$$\mathbb{E}[X^rY^s] = \mathbb{E}[X^r]\mathbb{E}[Y^s] = \mathbb{E}[X^r]\mu_{Y}\mathbb{E}[(Y+1)^{s-1}] = \mu_{Y}\mathbb{E}[X^r(Y+1)^{s-1}].$$

□

Solution to Exercise 7.13. We use a two-dimensional extension of Theorem 2.3(e), stating that when the mixed moments $\mathbb{E}[X_n^rY_n^s]$ converge to the moments $\mathbb{E}[X^rY^s]$ for each $r,s=0,1,2,\ldots$, and the moments of X and Y satisfy (2.1.8), then (X_n,Y_n) converges in distribution to (X,Y). See also Theorem 2.6 for the equivalent statement for the factorial moments instead of the normal moments, from which the above claim actually follows. Therefore, we are left to prove the asymptotics of the mixed moments of (S_n,M_n).

To prove that $\mathbb{E}[S_n^rM_n^s]$ converge to the moments $\mathbb{E}[S^rM^s]$, we again make use of induction, now in both r and s.

Proposition 7.12 follows when we prove that

$$\lim_{n\to\infty}\mathbb{E}[S_n^r] = \mathbb{E}[S^r] = \mu_{S}\mathbb{E}[(S+1)^{r-1}], \tag{1.7.8}$$

and

$$\lim_{n\to\infty}\mathbb{E}[S_n^rM_n^s] = \mathbb{E}[S^rM^s] = \mu_{M}\mathbb{E}[S^r(M+1)^{s-1}], \tag{1.7.9}$$

where the second equalities in (0.7.8) and (0.7.9) follow from (7.4.18) and (7.4.19).

To prove (0.7.8), we use the shape of S_n in (7.3.20), which we restate here as

$$S_n = \sum_{i=1}^{n}\sum_{1\le a<b\le d_i}I_{ab,i}. \tag{1.7.10}$$

Then, we prove by induction on r that

$$\lim_{n\to\infty}\mathbb{E}[S_n^r] = \mathbb{E}[S^r]. \tag{1.7.11}$$

The induction hypothesis is that (0.7.11) is true for all $r'\le r-1$, for $\mathrm{CM}_n(\boldsymbol{d})$ when $n\to\infty$ and for all $(d_i)_{i\in[n]}$ satisfying Condition 7.7(a)-(c) We prove (0.7.11) by induction on r. For $r=0$, the statement is trivial, which initializes the induction hypothesis.

To advance the induction hypothesis, we write out

$$\mathbb{E}[S_n^r] = \sum_{i=1}^n \sum_{1\leq a<b\leq d_i} \mathbb{E}[I_{ab,i} S_n^{r-1}]$$
$$= \sum_{i=1}^n \sum_{1\leq a<b\leq d_i} \mathbb{P}(I_{ab,i}=1)\mathbb{E}[S_n^{r-1} | I_{ab,i}=1]. \tag{1.7.12}$$

When $I_{ab,i}=1$, then the *remaining* stubs need to be paired in a uniform manner. The number of self-loops in the total graph in this pairing has the same distribution as

$$1+S_n', \tag{1.7.13}$$

where S_n' is the number of self-loops in the configuration model where with degrees $(d_i')_{i\in[n]}$, where $d_i'=d_i-2$, and $d_j'=d_j$ for all $j\neq i$. The added 1 in (0.7.13) originates from $I_{ab,i}$. By construction, the degrees $(d_i')_{i\in[n]}$ still satisfy Condition 7.7(a)-(c). By the induction hypothesis, for all $k\leq r-1$

$$\lim_{n\to\infty} \mathbb{E}[(S_n')^k] = \mathbb{E}[S^k]. \tag{1.7.14}$$

As a result,

$$\lim_{n\to\infty} \mathbb{E}[(1+S_n')^{r-1}] = \mathbb{E}[(1+S)^{r-1}]. \tag{1.7.15}$$

Since the limit does not depend on i, we obtain that

$$\lim_{n\to\infty} \mathbb{E}[S_n^r] = \mathbb{E}[(1+S)^{r-1}] \lim_{n\to\infty} \sum_{i=1}^n \sum_{1\leq a<b\leq d_i} \mathbb{P}(I_{ab,i}=1)$$
$$\mathbb{E}[(1+S)^{r-1}] \lim_{n\to\infty} \sum_{i=1}^n \frac{d_i(d_i-1)}{2}$$
$$= \frac{\nu}{2}\mathbb{E}[(1+S)^{r-1}] = \mathbb{E}[S^r]. \tag{1.7.16}$$

This advances the induction hypothesis, and completes the proof of (0.7.8).

To prove (0.7.9), we perform a similar induction scheme. Now we prove that, for all $r\geq 0$, $\mathbb{E}[S_n^r \widetilde{M}_n^s]$ converges to $\mathbb{E}[S^r M^s]$ by induction on s. The claim for $s=0$ follows from (0.7.8), which initializes the induction hypothesis, so we are left to advance the induction hypothesis. We follow the argument for S_n above. It is not hard to see that it suffices to prove that, for every ij,

$$\lim_{n\to\infty} \mathbb{E}[S_n^r \widetilde{M}_n^{s-1} | I_{s_1t_1,s_2t_2,ij}=1] = \mathbb{E}[S^r(1+M)^{s-1}]. \tag{1.7.17}$$

Note that when $I_{s_1t_1,s_2t_2,ij}=1$, then we know that two edges are paired together to form a multiple edge. Removing these two edges leaves us with a graph which is very close to the configuration model with degrees $(d_i')_{i\in[n]}$, where $d_i'=d_i-2$, and $d_j'=d_j-2$ and $d_t'=d_t$ for all $t\neq i,j$. The only difference is that when a stub connected to i is attached to a stub connected to j, then this creates an additional number of multiple edges. Ignoring this effect creates the lower bound

$$\mathbb{E}[S_n^r \widetilde{M}_n^{s-1} | I_{s_1t_1,s_2t_2,ij} = 1] \geq \mathbb{E}[S_n^r (\widetilde{M}_n + 1)^{s-1}], \tag{1.7.18}$$

which, by the induction hypothesis, converges to $\mathbb{E}[S^r(1+M)^{s-1},]$ as required.

Let $I'_{s_1t_1,s_2t_2,ij}$ denote the indicator that stub s_1 is connected to t_1, s_2 to t_2 and *no other stub* of vertex i is connected to a stub of vertex j. Then,

$$\frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} I'_{s_1t_1,s_2t_2,ij} \leq \widetilde{M}_n \leq \frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} I_{s_1t_1,s_2t_2,ij}. \tag{1.7.19}$$

Hence,

$$\mathbb{E}[S_n^r \widetilde{M}_n^s] \leq \frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} \mathbb{P}(I_{s_1t_1,s_2t_2,ij} = 1) \mathbb{E}\big[S_n^r \widetilde{M}_n^{s-1} | I_{s_1t_1,s_2t_2,ij} = 1\big], \tag{1.7.20}$$

and

$$\mathbb{E}[S_n^r \widetilde{M}_n^s] \leq \frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} \mathbb{P}(I'_{s_1t_1,s_2t_2,ij} = 1) \mathbb{E}\big[S_n^r \widetilde{M}_n^{s-1} | I'_{s_1t_1,s_2t_2,ij} = 1\big]. \tag{1.7.21}$$

Now, by the above, $\mathbb{E}\big[S_n^r \widetilde{M}_n^{s-1} | I_{s_1t_1,s_2t_2,ij} = 1\big]$ and $\mathbb{E}\big[S_n^r \widetilde{M}_n^{s-1} | I'_{s_1t_1,s_2t_2,ij} = 1\big]$ converge to $\mathbb{E}\big[S^r(M+1)^{s-1}\big]$, independently of s_1t_1, s_2t_2, ij. Further,

$$\frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} \mathbb{P}(I'_{s_1t_1,s_2t_2,ij} = 1) \to \nu^2/2, \tag{1.7.22}$$

and also

$$\frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} \mathbb{P}(I_{s_1t_1,s_2t_2,ij} = 1) \to \nu^2/2. \tag{1.7.23}$$

This implies that

$$\mathbb{E}[S_n^r \widetilde{M}_n^{s-1} | I_{s_1t_1,s_2t_2,ij} = 1] = \mathbb{E}[S_{n-1}^r \widetilde{M}_{n-1}^{s-1}] + o(1). \tag{1.7.24}$$

The remainder of the proof is identical to the one leading to (0.7.16). □

Solution to Exercise 7.14. To obtain a triangle we need to three connected stubs say $(s_1,t_1),(s_2,t_2),(s_3,t_3)$ where s_1 and t_3 belong to some vertex i with degree d_i, s_2 and t_1 to vertex j with degree d_j and s_3, t_2 to some vertex k with degree d_k. Obviously we have

$$\begin{aligned} 1 \leq\ & s_1 \leq d_i, \\ 1 \leq\ & t_1 \leq d_j, \\ 1 \leq\ & s_2 \leq d_j, \\ 1 \leq\ & t_2 \leq d_k, \\ 1 \leq\ & s_3 \leq d_k, \\ 1 \leq\ & t_3 \leq d_i. \end{aligned}$$

The probability of connecting s_1 to t_1 is $1/(\ell_n - 1)$. Furthermore, connecting s_2 to t_2 appears with probability $1/(\ell_n - 3)$ and s_3 to t_3 with probability $1/(\ell_n - 5)$. Of course we can pick all stubs of i to be s_1, and we have $d_i - 1$ vertices left from which we may choose t_3. Hence, for the amount of triangles,

$$\sum_{i<j<k} \frac{d_i d_j}{\ell_n - 1} \cdot \frac{(d_j-1)d_k}{\ell_n - 3} \cdot \frac{(d_k-1)(d_i-1)}{\ell_n - 5} = \sum_{i<j<k} \frac{d_i(d_i-1)}{\ell_n - 1} \cdot \frac{d_j(d_j-1)}{\ell_n - 3} \cdot \frac{d_k(d_k-1)}{\ell_n - 5} \tag{1.7.25}$$

$$\sim \frac{1}{6}\Big(\sum_{i=1}^{n} \frac{d_i(d_i-1)}{\ell_n}\Big)^3.$$

We will show that

$$\sum_{i<j<k} \frac{d_i(d_i-1)}{\ell_n - 1} \cdot \frac{d_j(d_j-1)}{\ell_n - 3} \cdot \frac{d_k(d_k-1)}{\ell_n - 5} \sim \frac{1}{6}\Big(\sum_{i=1}^{n} \frac{d_i(d_i-1)}{\ell_n}\Big)^3$$

by expanding the righthand-side. We define

$$S := \Big(\sum_{i=1}^{n} \frac{d_i(d_i-1)}{\ell_n}\Big)^3. \tag{1.7.26}$$

Then, we have

$$S = \sum_{i=1}^{n} \Big(\frac{d_i(d_i-1)}{\ell_n}\Big)^3 + 3\sum_{i=1}^{\infty} \sum_{j=1, j\neq i}^{\infty} \Big(\frac{d_i(d_i-1)}{\ell_n}\Big)^2 \Big(\frac{d_j(d_j-1)}{\ell_n}\Big) \tag{1.7.27}$$

$$+ \quad \sum_{i\neq j\neq k} \frac{d_i(d_i-1)}{\ell_n} \cdot \frac{d_j(d_j-1)}{\ell_n} \cdot \frac{d_k(d_k-1)}{\ell_n}, \tag{1.7.28}$$

where the first part contains n terms, the second $n(n-1)$ and the third $n(n-1)(n-2)$. So for large n we can say that

$$S \sim \sum_{i\neq j\neq k} \frac{d_i(d_i-1)}{\ell_n} \cdot \frac{d_j(d_j-1)}{\ell_n} \cdot \frac{d_k(d_k-1)}{\ell_n}. \tag{1.7.29}$$

Now there are six possible orderings of i, j, k, hence

$$\frac{1}{6}S \sim \sum_{i<j<k} \frac{d_i(d_i-1)}{\ell_n} \cdot \frac{d_j(d_j-1)}{\ell_n} \cdot \frac{d_k(d_k-1)}{\ell_n} \sim \sum_{i<j<k} \frac{d_i(d_i-1)}{\ell_n - 1} \cdot \frac{d_j(d_j-1)}{\ell_n - 3} \cdot \frac{d_k(d_k-1)}{\ell_n - 5}. \tag{1.7.30}$$

□

Solution to Exercise 7.18. In this case we have $d_i = r$ for all $i \in [n]$. This gives us

$$\mathbb{E}[D] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{d_i(d_i-1)}{\ell_n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{r(r-1)}{nr} = r - 1. \tag{1.7.31}$$

Furthermore,

$$\prod_{i=1}^{n} d_i! = \prod_{i=1}^{n} r! = (r!)^n. \tag{1.7.32}$$

Finally we have for the total number of stubs $\ell_n = rn$. Substituting these variables in (7.5.1) gives us for the number of simple graphs with constant degree sequence $d_i = r$

$$e^{-\frac{(r-1)}{2}-\frac{(r-1)^2}{4}} \frac{(rn-1)!!}{(r!)^n}(1+o(1)). \tag{1.7.33}$$

□

1.8 Solutions to the exercises of Chapter 8.

Solution to Exercise 8.1. At time t, we add a vertex v_t, and connect it with each vertex v_i, $1 \leq i < t$ with probability p. In the previous chapters, we had the relation $p = \frac{\lambda}{n}$, but since n is increasing over time, using this expression for p will not result in an Erdős-Rényi random graph. We could off course wish to obtain a graph of size N, thus stopping the algorithm at time $t = N$, and using $p = \frac{\lambda}{N}$.

□

Solution to Exercise 8.2. We will use an induction argument over t. For $t = 1$ we have a single vertex v_1 with a self-loop, hence $d_1(1) = 2 \geq 1$.

Now suppose at time t we have $d_i(t) \geq 1 \, \forall_i$.
At time $t+1$ we add a vertex v_{t+1}. We do not remove any edges, so we only have to check whether the newly added vertex has a non-zero degree. Now the algorithm adds the vertex having a single edge, to be connected to itself, in which case $d_{t+1}(t+1) = 2$, or to be connected to another already existing vertex, in which case it's degree is 1. In the latter case, one is added to the degree of the vertex to which v_{t+1} is connected, thus that degree is still greater than zero. Hence we can say that $d_i(t+1) \geq 1 \, \forall_i$
We can now conclude that $d_i(t) \geq 1$ for all i and t. The statement $d_i(t) + \delta \geq 0$ for all $\delta \geq -1$ follows directly. □

Solution to Exercise 8.3. The statement

$$\frac{1+\delta}{t(2+\delta)+(1+\delta)} + \sum_{i=1}^{t} \frac{d_i(t)+\delta}{t(2+\delta)+(1+\delta)} = 1 \tag{1.8.1}$$

will follow directly if the following equation holds:

$$(1+\delta) + \sum_{i=1}^{t}(d_i(t)+\delta) = t(2+\delta) + (1+\delta). \tag{1.8.2}$$

Which is in its turn true if

$$\sum_{i=1}^{t}(d_i(t)+\delta)=t(2+\delta). \tag{1.8.3}$$

But since $\sum_{i=1}^{t} d_i(t) = 2t$ by construction, the latter equation holds. Hence, the upper statement holds and the probabilities do sum up to one.

□

Solution to Exercise 8.6. We will again use an induction argument. At time $t = 1$ we have a single vertex v_1 with a self-loop, and the statement holds. At time $t = 2$ we add a vertex v_2 and connect it with v_1 with the given probability

$$\mathbb{P}\big(v_2 \to v_1 \big| \mathrm{PA}_{1,\delta}(1)\big) = \frac{2-1}{1} = 1. \tag{1.8.4}$$

Now suppose at time t we have a graph with one vertex v_1 containing a self-loop and $t-1$ other vertices having only one edge which connects it to v_1. In that case $d_1(t) = 2 + (t-1) = t+1$ and all other vertices have degree 1.
At time $t+1$ we add a vertex v_{t+1} having one edge which will be connected to v_1 with probability

$$\mathbb{P}\big(v_{t+1} \to v_1 \big| \mathrm{PA}_{1,\delta}(t)\big) = \frac{t+1-1}{t} = 1. \tag{1.8.5}$$

Hence, the claim follows by induction. □

Solution to Exercise 8.7. The proof is by induction on $t \geq 1$. For $t = 1$, the statement is correct, since, at time 2, both graphs consist of two vertices with two edges between them. This initializes the induction hypothesis.

To advance the induction hypothesis, we assume that the law of $\{\mathrm{PA}^{(b')}_{1,\alpha}(t)\}_{s=1}^{t}$ is equal to the one of $\{\mathrm{PA}^{(b)}_{1,\delta}(s)\}_{s=1}^{t}$, and, from this, prove that the law of $\{\mathrm{PA}^{(b')}_{1,\alpha}(s)\}_{s=1}^{t}$ is equal to the one of $\{\mathrm{PA}^{(b)}_{1,\delta}(s)\}_{s=1}^{t}$. The only difference between $\mathrm{PA}^{(b)}_{1,\delta}(t+1)$ and $\mathrm{PA}^{(b)}_{1,\delta}(t)$ and between $\mathrm{PA}^{(b')}_{1,\alpha}(t+1)$ and $\mathrm{PA}^{(b')}_{1,\alpha}(t)$ is to what vertex the $(t+1)^{\mathrm{st}}$ edge is attached. For $\{\mathrm{PA}^{(b)}_{1,\delta}(t)\}_{t=1}^{\infty}$ and conditionally on $\mathrm{PA}^{(b)}_{1,\delta}(t)$, this edge is attached to vertex i with probability

$$\frac{D_i(t)+\delta}{t(2+\delta)}, \tag{1.8.6}$$

while, for $\{\mathrm{PA}'_{1,\alpha}(t)\}_{t=1}^{\infty}$ and conditionally on $\mathrm{PA}'_{1,\alpha}(t)$, this edge is attached to vertex i with probability

$$\alpha\frac{1}{t} + (1-\alpha)\frac{D_i(t)}{2t}. \tag{1.8.7}$$

Bringing the terms in (0.8.7) onto a single denominator yields

$$\frac{D_i(t)+2\frac{\alpha}{1-\alpha}}{\frac{2}{1-\alpha}t}, \tag{1.8.8}$$

which agrees with (0.8.6) precisely when $2\frac{\alpha}{1-\alpha} = \delta$, so that

$$\alpha = \frac{\delta}{2+\delta}. \tag{1.8.9}$$

□

Solution to Exercise 8.9. We write

$$\Gamma(t+1) = \int_0^\infty x^t e^{-x} dx. \tag{1.8.10}$$

Using partial integration we obtain

$$\Gamma(t+1) = [-x^t e^{-x}]_{x=0}^\infty + \int_0^\infty t x^{t-1} e^{-x} dx = 0 + t \cdot \int_0^\infty x^{t-1} e^{-x} dx = t\Gamma(t).$$

In order to prove that $\Gamma(n) = (n-1)!$ for $n = 1, 2, \ldots$ we will again use an induction argument. For $n = 1$ we have

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx = 1 = (0)!.$$

Now the upper result gives us for $n = 2$

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1 = (2-1)!. \tag{1.8.11}$$

Suppose now that for some $n \in \mathbb{N}$ we have $\Gamma(n) = (n-1)!$. Again (8.3.2) gives us for $n+1$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!. \tag{1.8.12}$$

Induction yields $\Gamma(n) = (n-1)!$ for $n = 1, 2, \ldots$. □

Solution to Exercise 8.10. We rewrite (8.3.9) to be

$$\begin{aligned}
e^{-t} t^{t-\frac{1}{2}} \sqrt{2\pi} &\leq \Gamma(t+1) \leq e^{-t} t^t \sqrt{2\pi}\Big(1 + \frac{1}{12t}\Big), \\
(\frac{t}{e})^t \sqrt{\frac{2\pi}{t}} &\leq \Gamma(t+1) \leq (\frac{t}{e})^t \sqrt{2\pi}(1 + \frac{1}{12t}), \\
(\frac{t}{e})^t \sqrt{\frac{2\pi}{t}} &\leq t\Gamma(t) \leq (\frac{t}{e})^t \sqrt{2\pi}(1 + \frac{1}{12t}), \\
(\frac{t}{e})^t \sqrt{\frac{2\pi}{t}} \frac{1}{t} &\leq \Gamma(t) \leq (\frac{t}{e})^t \sqrt{\frac{2\pi}{t}} \sqrt{t}(1 + \frac{1}{12t}).
\end{aligned}$$

Using this inequality in the left-hand side of (8.3.8) we obtain

$$\begin{aligned}
\frac{(\frac{t}{e})^t \sqrt{\frac{2\pi}{t} \frac{1}{t}}}{(\frac{t-a}{e})^{t-a} \sqrt{\frac{2\pi}{t-a}} \sqrt{t-a}(1 + \frac{1}{12(t-a)})} &\leq \tfrac{\Gamma(t)}{\Gamma(t-a)} \leq \frac{(\frac{t}{e})^t \sqrt{\frac{2\pi}{t}} \sqrt{t}(1 + \frac{1}{12t})}{(\frac{t-a}{e})^{t-a} \sqrt{\frac{2\pi}{t-a} \frac{1}{t-a}}} \\
\frac{t^t}{(t-a)^{t-a}} \frac{e^{-a}}{t\sqrt{t}(1 + 12/(t-a))} &\leq \tfrac{\Gamma(t)}{\Gamma(t-a)} \leq \frac{t^t}{(t-a)^{t-a}} \frac{e^{-a}(1 + 1/12t)}{\sqrt{t-a}}.
\end{aligned}$$

We complete the proof by noting that $t - a = t(1 + O(1/t))$ and $1 + 1/12t = 1 + O(1/t)$. □

Solution to Exercise 8.11. This result is immediate from the collapsing of the vertices in the definition of $\mathrm{PA}_t(m,\delta)$, which implies that the degree of vertex $v_i^{(m)}$ in $\mathrm{PA}_t(m,\delta)$ is equal to the sum of the degrees of the vertices $v_{m(i-1)+1}^{(1)}, \ldots, v_{mi}^{(1)}$ in $\mathrm{PA}_{mt}(1,\delta/m)$. □

Solution to Exercise 8.16. We wish to prove

$$\mathbb{P}\Big(|P_{\geq k}(t) - \mathbb{E}[P_{\geq k}(t)]| \geq C\sqrt{t\log t}\Big) = o(t^{-1}). \tag{1.8.13}$$

First of all we have $P_{\geq k}(t) = 0$ for $k > mt$. We define, similarly to the proof of Proposition 8.3 the martingale

$$M_n = \mathbb{E}\big[P_{\geq k}(t)|\mathrm{PA}_{m,\delta}(n)\big]. \tag{1.8.14}$$

We have

$$\mathbb{E}[M_{n+1}|\mathrm{PA}_{m,\delta}(n)] = \mathbb{E}\Big[\mathbb{E}\big[P_{\geq k}(t)|\mathrm{PA}_{m,\delta}(n+1)\big]\Big|\mathrm{PA}_{m,\delta}(n)\Big] = \mathbb{E}\big[P_{\geq k}(t)|\mathrm{PA}_{m,\delta}(n)\big] = M_n. \tag{1.8.15}$$

Hence M_n is a martingale. Furthermore, M_n satisfies the moment condition, since

$$\mathbb{E}\big[M_n\big] = \mathbb{E}\big[P_{\geq k}(t)\big] \leq t < \infty. \tag{1.8.16}$$

Clearly, $\mathrm{PA}_{m,\delta}(0)$ is the empty graph, hence for M_0 we obtain

$$M_0 = \mathbb{E}\big[P_{\geq k}(t)|\mathrm{PA}_{m,\delta}(0)\big] = \mathbb{E}\big[P_{\geq k}(t)\big]. \tag{1.8.17}$$

We obtain for M_t

$$M_t = \mathbb{E}\big[P_{\geq k}(t)|\mathrm{PA}_{m,\delta}(t)\big] = \big[P_{\geq k}(t), \tag{1.8.18}$$

since $P_{\geq k}(t)$ can be determined when $\mathrm{PA}_{m,\delta}(t)$ is known. Therefore, we have

$$P_{\geq k}(t) - \mathbb{E}[P_{\geq k}(t)] = M_t - M_0. \tag{1.8.19}$$

To apply the Azuma-Hoeffding inequality, Theorem 2.25, we have to bound $|M_n - M_{n-1}|$. In step n, m edges are added to the graph. Now $P_{\geq k}$ only changes is an edge is added to a vertex with degree $k-1$. Now m edges have influence on the degree of at most $2m$ vertices, hence, the maximum amount of vertices of which de degree is increased to k is at most $2m$. So we have $|M_n - M_{n-1}| \leq 2m$. The Azuma-Hoeffding inequality now gives us

$$\mathbb{P}\Big(|P_{\geq k}(t) - \mathbb{E}[P_{\geq k}(t)]| \geq a\Big) \leq 2e^{-\frac{a^2}{8m^2t}}. \tag{1.8.20}$$

Taking $a = C\sqrt{t\log t}$, $C^2 \geq 8m$, we obtain

$$\mathbb{P}\Big(|P_{\geq k}(t) - \mathbb{E}[P_{\geq k}(t)]| \geq C\sqrt{t\log t}\Big) = o(t^{-1}). \tag{1.8.21}$$

□

Solution to Exercise 8.17. We have for $\kappa_k(t)$ and $\gamma_k(t)$ the following equation.

$$\begin{aligned}\kappa_k(t) &= \Big(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\Big)(k-1+\delta)p_{k-1} - \Big(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\Big)(k+\delta)p_k,\\ \gamma_k(t) &= -\mathbb{1}\{k=1\}\frac{1+\delta}{t(2+\delta)+(1+\delta)} + \mathbb{1}\{k=2\}\frac{1+\delta}{t(2+\delta)+(1+\delta)}.\end{aligned}$$

We start with C_γ. We have

$$|\gamma_k(t)| \le \frac{1+\delta}{t(2+\delta)+(1+\delta)} \le \frac{1}{t(\frac{2+\delta}{1+\delta})+1} \le \frac{1}{t+1}. \tag{1.8.22}$$

So indeed $C_\gamma = 1$ does the job. For $\kappa_k(t)$ we have

$$\kappa_k(t) = \Big(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\Big)\Big((k-1+\delta)p_{k-1} - (k+\delta)p_k\Big). \tag{1.8.23}$$

This gives us

$$\begin{aligned}|\kappa_k(t)| &\le \Big|\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\Big| \cdot \Big|(k-1+\delta)p_{k-1} - (k+\delta)p_k\Big|,\\ &\le \Big|\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\Big| \cdot \sup_{k\ge1}(k+\delta)p_k,\\ &= \Big|\frac{t(2+\delta)+(1+\delta)-(2+\delta)t}{t(2+\delta)^2+(1+\delta)(2+\delta)}\Big| \cdot \sup_{k\ge1}(k+\delta)p_k,\\ &= \Big|\frac{1+\delta}{t(2+\delta)^2+(1+\delta)(2+\delta)}\Big| \cdot \sup_{k\ge1}(k+\delta)p_k,\\ &= \Big|\frac{1}{2+\delta}\cdot\frac{1}{t(\frac{2+\delta}{1+\delta})+1}\Big| \cdot \sup_{k\ge1}(k+\delta)p_k,\\ &\le \Big|\frac{1}{t(\frac{2+\delta}{1+\delta})+1}\Big| \cdot \sup_{k\ge1}(k+\delta)p_k,\\ &\le \frac{1}{t+1} \cdot \sup_{k\ge1}(k+\delta)p_k.\end{aligned}$$

Hence, $C_\kappa = \sup_{k\ge1}(k+\delta)p_k$

□

Solution to Exercise 8.18. We note that

$$\sum_{i:D_i(t)\ge l} D_i(t) \ge l N_{\ge l}(t), \tag{1.8.24}$$

where we recall that $N_{\ge l}(t) = \#\{i \le t : D_i(t) \ge l\}$ is the number of vertices with degree at least l.

By the proof of Proposition 8.3 (see also Exercise 8.16), there exists C_1 such that uniformly for all l,

$$\mathbb{P}\Big(|N_{\ge l}(t) - \mathbb{E}[N_{\ge l}(t)]| \ge C_1\sqrt{t\log t}\Big) = o(t^{-1}). \tag{1.8.25}$$

By Proposition 8.6, there exists a constant C_2 such that

$$\sup_{l\geq 1} |\mathbb{E}[P_l(t)] - tp_l| \leq C_2. \tag{1.8.26}$$

Therefore, we obtain that, with probability exceeding $1 - o(t^{-1})$,

$$\begin{aligned} N_{\geq l}(t) &\geq \mathbb{E}[N_{\geq l}(t)] - C_1\sqrt{t\log t} \geq \mathbb{E}[N_{\geq l}(t)] - \mathbb{E}[N_{\geq 2l}(t)] - C_1\sqrt{t\log t} \\ &\geq \sum_{l=l}^{2l-1}[tp_l - C_2] - C_1\sqrt{t\log t} \geq C_3 t l^{1-\tau} - C_2 l - C_1\sqrt{t\log t} \geq Btl^{2-\tau}, \end{aligned} \tag{1.8.27}$$

whenever l is such that

$$tl^{1-\tau} \gg l, \qquad \text{and} \qquad tl_t^{1-\tau} \gg \sqrt{t\log t}. \tag{1.8.28}$$

The first condition is equivalent to $l \ll t^{\frac{1}{\tau}}$, and the second to $l \ll t^{\frac{1}{2(\tau-1)}}(\log t)^{-\frac{1}{2(\tau-1)}}$. Note that $\frac{1}{\tau} \geq \frac{1}{2(\tau-1)}$ for all $\tau > 2$, so the second condition is the strongest, and follows when $tl^{2-\tau} \geq K\sqrt{t\log t}$ for some K sufficiently large.

Then, for l satisfying $tl^{2-\tau} \geq K\sqrt{t\log t}$, we have with probability exceeding $1 - o(t^{-1})$,

$$\sum_{i:D_i(t)\geq l} D_i(t) \geq Btl^{2-\tau}. \tag{1.8.29}$$

Also, with probability exceeding $1 - o(t^{-1})$, for all such l, $N_{\geq l}(t) \gg \sqrt{t}$. □

Solution to Exercise 8.19. We prove (8.7.3) by induction on $j \geq 1$. Clearly, for every $t \geq i$,

$$\mathbb{P}(D_i(t) = 1) = \prod_{s=i+1}^{t}\Big(1 - \frac{1+\delta}{(2+\delta)(s-1)+(1+\delta)}\Big) = \prod_{s=i+1}^{t}\Big(\frac{s-1}{s-1+\frac{1+\delta}{2+\delta}}\Big) = \frac{\Gamma(t)\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})\Gamma(i)}, \tag{1.8.30}$$

which initializes the induction hypothesis, since $C_1 = 1$.

To advance the induction, we let $s \leq t$ be the last time at which a vertex is added to i. Then we have that

$$\mathbb{P}(D_i(t) = j) = \sum_{s=i+j-1}^{t} \mathbb{P}\big(D_i(s-1) = j-1\big)\frac{j-1+\delta}{(2+\delta)(s-1)+1+\delta}\mathbb{P}\big(D_i(t) = j | D_i(s) = j\big). \tag{1.8.31}$$

By the induction hypothesis, we have that

$$\mathbb{P}\big(D_i(s-1) = j-1\big) \leq C_{j-1}\frac{\Gamma(s-1)\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(s-1+\frac{1+\delta}{2+\delta})\Gamma(i)}. \tag{1.8.32}$$

Moreover, analogously to (0.8.30), we have that

$$\mathbb{P}(D_i(t)=j|D_i(s)=j) = \prod_{q=s+1}^{t}\Big(1-\frac{j+\delta}{(2+\delta)(q-1)+(1+\delta)}\Big) \tag{1.8.33}$$
$$= \prod_{q=s+1}^{t}\Big(\frac{q-1-\frac{j-1}{2+\delta}}{q-1+\frac{1+\delta}{2+\delta}}\Big) = \frac{\Gamma(t-\frac{j-1}{2+\delta})\Gamma(s+\frac{1+\delta}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})\Gamma(s-\frac{j-1}{2+\delta})}.$$

Combining (0.8.32) and (0.8.33), we arrive at

$$\mathbb{P}(D_i(t)=j) \le \sum_{s=i+j-1}^{t}\Big(C_{j-1}\frac{\Gamma(s-1)\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(s-1+\frac{1+\delta}{2+\delta})\Gamma(i)}\Big)\Big(\frac{j-1+\delta}{(2+\delta)(s-1)+(1+\delta)}\Big)$$
$$\times\Big(\frac{\Gamma(t-\frac{j-1}{2+\delta})\Gamma(s+\frac{1+\delta}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})\Gamma(s-\frac{j-1}{2+\delta})}\Big). \tag{1.8.34}$$

We next use that

$$\Gamma(s-1+\frac{1+\delta}{2+\delta})((2+\delta)(s-1)+(1+\delta)) = (2+\delta)\Gamma(s+\frac{1+\delta}{2+\delta}), \tag{1.8.35}$$

to arrive at

$$\mathbb{P}(D_i(t)=j) \le C_{j-1}\frac{j-1+\delta}{2+\delta}\frac{\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(i)}\frac{\Gamma(t-\frac{j-1}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})}\sum_{s=i+j-1}^{t}\frac{\Gamma(s-1)}{\Gamma(s-\frac{j-1}{2+\delta})}. \tag{1.8.36}$$

We note that, whenever $l+b, l+1+a>0$ and $a-b+1>0$,

$$\sum_{s=l}^{t}\frac{\Gamma(s+a)}{\Gamma(s+b)} = \frac{1}{a-b+1}\Big[\frac{\Gamma(t+1+a)}{\Gamma(t+b)} - \frac{\Gamma(l+1+a)}{\Gamma(l+b)}\Big] \le \frac{1}{a-b+1}\frac{\Gamma(t+1+a)}{\Gamma(t+b)}. \tag{1.8.37}$$

Application of (0.8.37) for $a=-1, b=-\frac{j-1}{2+\delta}, l=i+j-1$, so that $a-b+1=\frac{j-1}{2+\delta}>0$ when $j>1$, leads to

$$\mathbb{P}(D_i(t)=j) \le C_{j-1}\frac{j-1+\delta}{2+\delta}\frac{\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(i)}\frac{\Gamma(t-\frac{j-1}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})}\frac{1}{\frac{j-1}{2+\delta}}\frac{\Gamma(t)}{\Gamma(t-\frac{j-1}{2+\delta})} \tag{1.8.38}$$
$$= C_{j-1}\frac{j-1+\delta}{j-1}\frac{\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(i)}\frac{\Gamma(t)}{\Gamma(t+\frac{1+\delta}{2+\delta})}.$$

Equation (0.8.38) advances the induction by (8.7.4). □

Solution to Exercise 8.24. Suppose $\alpha\delta_{\text{in}}+\gamma=0$, then, since all non-negative, we have $\gamma=0$ and either $\alpha=0$ or $\delta_{\text{in}}=0$.
Since $\gamma=0$, no new vertices are added with non zero in-degree.
In case of $\alpha=0$ we have $\beta=1$, and thus we only create edges in G_0. Hence, no vertices exist outside G_0 and thus there cannot exist vertices outside G_0 with in-degree

non zero.
In case of $\delta_{\text{in}} = 0$ (and $\gamma = 0$ still), vertices can be created outside G_0, but in in it's creation phase we will only give it an outgoïng edge. And this edge will be connected to a vertex inside G_0, since $\delta_{\text{in}} = 0$ and the possibility to is thus zero to create an ingoing edge to a vertex with $d_i(t) = 0$. Similarly, in case edges are created within the existing graphs, all ingoing edges will be in G_0 for the same reason. So, during all stages all vertices outside G_0 will have in-degree zero.

Now suppose $\gamma = 1$. Then the only edges being created during the process are those from inside the existing graph to the newly created vertex. So once a vertex is created and connected to the graph, it will only be able to gain out-going edges. Hence, the in-degree remains one for all vertices outside G_0 at all times.