AN EXERCISE

Remark 1 *If $f \in L^1(\mathbb{R})$ then $|f|$ cannot of course be bounded below on an unbounded interval (such as $[a, \infty)$). But it does not follow necessarily that $\lim_{|x|\to\infty} f(x) = 0$, even when f is infinitely differentiable.*

Example [Greg] Let $\phi : [-1, 1] \to [0, 1]$ be a continuous function, with $\phi(0) = 1$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \phi(n^2(x - n))$ when $x \in [n - \frac{1}{n^2}, n + \frac{1}{n^2}]$ for some $n \in \mathbb{N}$, $n \geq 2$ and $f(x) = 0$ otherwise. Notice that f is supported on the union of the intervals $[n - \frac{1}{n^2}, n + \frac{1}{n^2}]$, $n \in \mathbb{N}$ which are disjoint. Hence f is continuous and $\|f\|_1 \leq \sum_n \frac{1}{n^2} \|\phi\|_1 < \infty$ so $f \in L^1(\mathbb{R})$. But $f(n) = 1$ for all $n \in \mathbb{N}$.
Notice that we can even choose ϕ to be infinitely differentiable: take $\phi(x) = \exp(\frac{-1}{1-x^2})$ when $|x| < 1$ and $\phi(x) = 0$ otherwise. Then f will also be infinitely differentiable.
However since the integral $\int_{I_n} |f'(x)|dx$ over the interval $I_n = [n - \frac{1}{n^2}, n + \frac{1}{n^2}]$ is a positive constant, independent of n, it follows that $\|f'\|_1 = \sum_n \int_{I_n} |f'(x)|dx = +\infty$, so f' is not in L^1.

Remark 2 *If f is differentiable and $f' \in L^1(\mathbb{R})$ it does not follow that $f \in L^1(\mathbb{R})$.*
For example take f to be the indefinite integral $f(x) = \int_{-\infty}^{x} \phi(t)dt$ of the function ϕ in the previous example. Then f is non-negative and increasing, so it cannot be in L^1 (if $a \geq 1$ then $f(a) = f(1) = \int_{\infty}^{1} \phi(t)dt$ so $\|f\|_1 \geq \int_a^\infty f(t)dt \geq f(1)m([a, \infty)) = +\infty$).

But if both f and f' are in L^1 then:

Exercise *Suppose $f \in L^1(\mathbb{R})$ is an everywhere differentiable function such that $f' \in L^1(\mathbb{R})$. Then*

$$\lim_{|x|\to\infty} f(x) = 0.$$

Proof For every $[a, b] \subseteq \mathbb{R}$ we have $f' \in L^1([a, b])$ and so (Koum-Negr. 7.19)

$$f(b) - f(a) = \int_a^b f'(t)dt. \qquad (*)$$

But since $\int_{-\infty}^{+\infty} |f'(t)|dt < \infty$ given $\epsilon > 0$ there exists $a_0 > 0$ so that

$$\int_{a_0}^{+\infty} |f'(t)|dt < \epsilon.$$

This shows that $\lim_{x\to\infty} f(x)$ exists. Then the fact that $\int_x^\infty |f(t)|dt \to 0$ as $x \to \infty$ forces $\lim_{x\to\infty} f(x) = 0$. A similar argument yields $\lim_{x\to-\infty} f(x) = 0$. In detail:

It follows from $(*)$ that for all $n, m \geq a_0$,

$$|f(n) - f(m)| \leq \left| \int_n^m f'(t)dt \right| \leq \left| \int_n^m |f'(t)|dt \right| \leq \int_{a_0}^{+\infty} |f'(t)|dt < \epsilon.$$

Thus $y := \lim_n f(n)$ exists. Take $n_0 \geq a_0$ with $|y - f(n_0)| < \epsilon$ and consider any $x > n_0$. Then

$$|f(x) - y| = \left| \int_{n_0}^x f'(t)dt + f(n_0) - y \right| \leq \int_{n_0}^x |f'(t)|dt + |f(n_0) - y| < 2\epsilon.$$

It follows that $\lim_{x\to\infty} f(x) = y$. If $a > n_0 > a_0$ then $|f(x) - y| < 2\epsilon$ when $x \geq a$ and so, for all $b > a$,

$$\left| \int_a^b f(x)dx - y(b-a) \right| \leq \int_a^b |f(x) - y|dx < 2\epsilon(b-a).$$

therefore

$$|y| < 2\epsilon + \frac{1}{b-a} \int_a^b |f(x)|dx$$

But since $\int_{-\infty}^\infty |f(x)|dx < \infty$, we may choose a so that $\int_a^\infty |f(x)|dx < \epsilon$. Then $|y| < 2\epsilon + \frac{\epsilon}{b-a}$ for all $b > a$ and hence $y = 0$.