605: Exercises V

1. (α) If $E \subseteq \mathbb{R}$ is measurable with $\lambda(E) < \infty$, show that for all $\epsilon > 0$ there exists a step function f vanishing outside a bounded interval so that $\|\chi_E - f\|_1 < \epsilon$.

   Hint Recall the first of the three principles of Littlewood.
   Note also that $f : \mathbb{R} \to \mathbb{R}$ is a step function vanishing outside a bounded interval if and only if there are $x_0, \dots x_n \in \mathbb{R},\ \ x_0 < x_1 < \cdots < x_n$ such that f is constant on each (x_{i-1}, x_i) and $f(t) = 0$ for all $t \notin [x_0, x_n]$.

   (β) If $I \subseteq \mathbb{R}$ is a bounded interval and $\epsilon > 0$, show that there is a continuous function g *with compact support* so that $\|\chi_I - g\|_1 < \epsilon$.

   (γ) using the above, show that the following linear spaces are dense in $L^1(\mathbb{R})$:

   (i) The space of simple integrable functions.

   (ii) The space of integrable step functions.

   (iii) The space $C_c(\mathbb{R})$ of continuous functions with compact support.

2. Suppose that $f, f_n : \mathbb{R} \to [0, +\infty]$, $n \in \mathbb{N}$, are nonnegative measurable functions and $f_n \searrow f$ a.e.. Show by example that one cannot conclude that $\int f d\lambda = \lim\limits_{n\to\infty} \int f_n d\lambda$.
   Assuming additionally that there is $k \in \mathbb{N}$ with $\int f_k d\lambda < \infty$, show that then $\int f d\lambda = \lim\limits_{n\to\infty} \int f_n d\lambda$.

3. Let $p \in [1, \infty)$ and $f \in \mathcal{L}^p(\mathbb{R})$. For all $t \in \mathbb{R}$, we define $f_t(s) = f(s - t)$. Show that $f_t \in \mathcal{L}^p(\mathbb{R})$ and that $\lim\limits_{t\to 0} \|f - f_t\|_p = 0$.

   Hint First consider the case $f \in C_c(\mathbb{R})$.

4. Let $f : \mathbb{R} \to [0, +\infty]$ be a measurable function. Suppose that $f > 0$ a.e.. If $\int_E f d\lambda = 0$ for some measurable set E, show that $\lambda(E) = 0$.

5. Let $f : \mathbb{R} \to [0, +\infty]$ be a nonnegative measurable function. Show that
$$\int_{\mathbb{R}} f d\lambda = \lim_{n\to\infty} \int_{[-n,n]} f d\lambda \quad \text{i.e.} \quad \lim_{n\to\infty} \int_{-n}^{n} f d\lambda = \int_{-\infty}^{\infty} f d\lambda\,.$$

6. Let $f : \mathbb{R} \to [0, +\infty]$ be a nonnegative measurable function. Show that
$$\int_{-\infty}^{\infty} f d\lambda = \lim_{n\to\infty} \int_{\{f \geq \frac{1}{n}\}} f d\lambda.$$

7. Let X be a measurable set and $f : X \to [-\infty, +\infty]$ a measurable function. Show that, for all $a \in \mathbb{R}$, the function $f_a : X \to [-\infty, +\infty]$ with
$$f_a(x) = \begin{cases} f(x) & \text{If } \ f(x) \leq a \\ a & \text{If } \ f(x) > a, \end{cases}$$
   is measurable.

8. Let $f : \mathbb{R} \to [0, +\infty]$ be a nonnegative integrable function. Show that
$$\int_{-\infty}^{\infty} f d\lambda = \lim_{n\to\infty} \int_{\{f \leq n\}} f d\lambda.$$

9. Let $f : \mathbb{R} \to [0, +\infty]$ be an integrable function. Is it true that $\lim_{x\to\pm\infty} f(x) = 0$?

10. (α) Show that for all $X \in \mathcal{M}$, $L^1(X) = \{fg : f, g \in L^2(X)\}$.

    (β) If $f \geq 0$, show that $f \in L^2([-\pi, \pi])$ if and only if $f^2 \in L^1([-\pi, \pi])$. Is the same true when $f([-\pi, \pi]) \subseteq \mathbb{R}$;