605: Exercices IV

1. Let $A \subseteq \mathbb{R}$. Show that the following are equivalent:

1. A is measurable.
2. For every $\varepsilon > 0$ there exists a closed set $F \subseteq \mathbb{R}$ with $F \subseteq A$ and $\lambda^*(A \setminus F) < \varepsilon$.
3. There exists an F_σ-set C such that $C \subseteq A$ and $\lambda^*\big(A \setminus C\big) = 0$.

2. (α) Let $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Show that

$$\lambda^*(A) = \lambda^*(A + t)$$

(outer measure is invariant under translation).

(β) If additionally A is measurable, then $A + t$ is measurable.

3. (α) Let A be a bounded subset of $\mathbb{R}$. Show that $\lambda^*(A) < +\infty$.

(β) Suppose that $A \subseteq \mathbb{R}$ has at least one interior point. Show that $\lambda^*(A) > 0$.

4. (α) If $A, B \subseteq \mathbb{R}$ and $\lambda^*(B) = 0$, show that $\lambda^*(A \cup B) = \lambda^*(A)$.

(β) If $A, B \subseteq \mathbb{R}$ και $\lambda^*(A \triangle B) = 0$, show that $\lambda^*(A) = \lambda^*(B)$ (the symbol $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of A and B).

5. (α) Let $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Write tA for the set $tA = \{tx \mid x \in A\}$. Show that $\lambda^*(tA) = |t|\, \lambda^*(A)$.

(β) Let $B \subseteq \mathbb{R}$ and let $f : B \to \mathbb{R}$ be a Lipschitz function with constant C, that is, satisfying $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in B$. Show that

$$\lambda^*(f(A)) \leq C\lambda^*(A)$$

for all $A \subseteq B$.

(γ) Let $A \subseteq \mathbb{R}$ with $\lambda(A) = 0$. Show that the set $A' = \{x^2 \mid x \in A\}$ also has measure $\lambda(A') = 0$.
Hint: First consider the case $A \subseteq [-M, M]$ for some $M > 0$.

6. Let $E \subseteq \mathbb{R}$ with $0 < \lambda^*(E) < +\infty$ and let $0 < \alpha < 1$. Show that *there exists* an open interval I with the property

$$\lambda^*(E \cap I) > \alpha\, \ell(I).$$

Hint: Assume the opposite and, for an arbitrary $\varepsilon > 0$, consider a sequence of intervals I_k such that $E \subseteq \bigcup_{k=1}^\infty I_k$ and $\sum_{k=1}^\infty \ell(I_k) < \lambda^*(E) + \varepsilon$.

7. Let A be a measurable set and let $\delta > 0$ be such that $\lambda(A \cap I) \geq \delta\, \ell(I)$ *for every* open interval I. Show that $\lambda(A^c) = 0$.