605: Exercises III

1. If p and q are trigonometric polynomials, show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p(t-s)q(s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)q(t-x)dx := (p*q)(t)$$

for all t. Show that $p*q$ is a trigonometric polynomial and find $\widehat{p*q}(k)$ for each $k \in \mathbb{Z}$.

2. If q is a trigonometric polynomial and $f : \mathbb{T} \to \mathbb{C}$ is integrable, show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)q(s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)q(t-x)dx := (f*q)(t)$$

for all t. Show that $f*q$ is a trigonometric polynomial and find $\widehat{f*q}(k)$ for each $k \in \mathbb{Z}$.

3. If $f : \mathbb{T} \to \mathbb{C}$ is integrable, show that, for each $m \in \mathbb{N}$,

$$\sigma_m(f) = \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) \hat{f}(k)e_k\,.$$

4. If $f, g : \mathbb{T} \to \mathbb{C}$ are continuous, show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(t-x)dx := (f*g)(t)$$

for all t. Show that $f*g$ is continuous and find $\widehat{f*g}(k)$ for each $k \in \mathbb{Z}$.

5. Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function which is integrable over $[-\pi, \pi]$. Suppose that for some $x \in \mathbb{R}$ the limits

$$f(x^-) := \lim_{t\to x^-} f(t) \quad \text{and} \quad f(x^+) := \lim_{t\to x^+} f(t)$$

exist. Show that the Fourier series $S[f]$ of f is Abel summable at x: more precisely, show that

$$\lim_{r\to 1^-} f_r(x) = \frac{f(x^-)+f(x^+)}{2}.$$

You may use the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{0} P_r(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} P_r(x)\,dx.$$

(Reminder: $f_r(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)P_r(t-s)ds$.)