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R.M. Goodwin

"A Growth Cycle"

Originally published in C.H. Feinstein (ed.), *Capitalism and Economic Growth*, Cambridge UP, Cambridge, 1967, pp. 54-8. Specially revised and enlarged for E.K. Hunt & Jesse G. Schwartz (eds.), *A Critique of Economic Theory*, Penguin, Harmondsworth, 1972, pp. 442-9.

Volterra-Lotka equations

As an introduction to the Goodwin model, let us explore the prey-predator Volterra-Lotka equations.[1]

Assume a two-species population, say, fish (prey) and sharks (predator). Fish [$x(t)$ is the fish population at time *t*] can reproduce without sharks at an exponential rate *a* so that $\dot{x} = ax(t)$ but with sharks [$y(t)$] the situation is different. They reproduce at a rate

$$\dot{x} = [a - by(t)]x(t) = ax(t) - by(t)x(t) \qquad \text{(VL Eq.) (1)}$$

The meaning of this equation is that the exponential growth of fish is dampened by the population of sharks.

Sharks feed on fish. If there are no fish they starve. Their population dynamics is given by

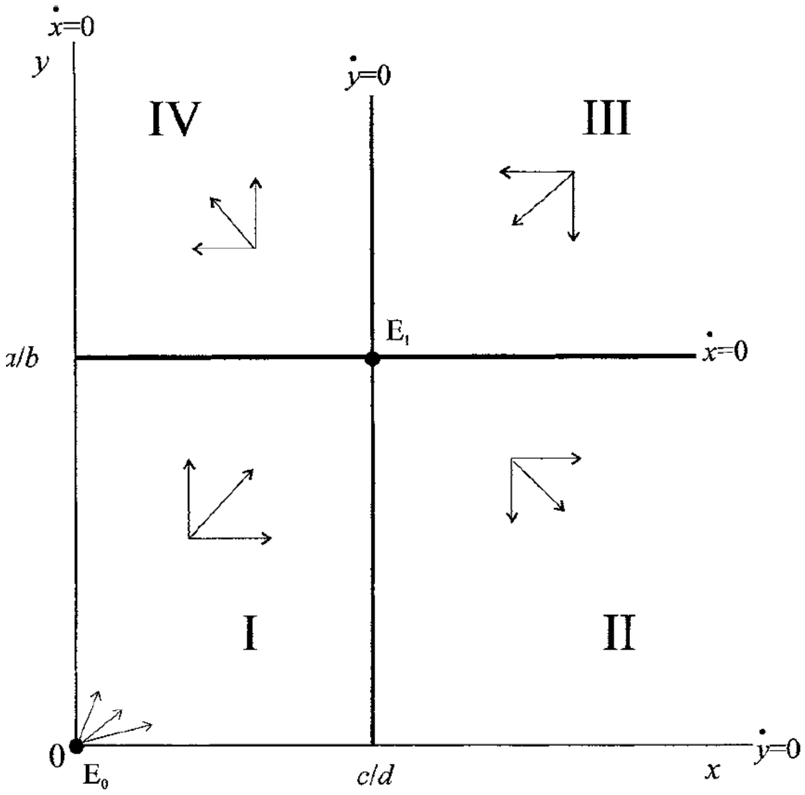
$$\dot{y} = [-c + dx(t)]y(t) = -cy(t) + dx(t)y(t) \qquad \text{(VL Eq.) (2)}$$

Their autonomous growth rate is negative and their rate of growth depends on the number of available fish.

Assume that all parameters *a*, *b*, *c* and *d* are positive. It is easy to compute the equilibrium solution for $\dot{x} = \dot{y} = 0$. So either $(x^*, y^*) = (0,0)$ or $(x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\right)$.

More interesting however is to observe what happens to the trajectories out of equilibrium. Take the non-zero equilibrium values and see what are the signs of $\dot{x}$ and $\dot{y}$ in the four quadrants that are formed in the *x*-*y* phase plane that are created by the intersection of *x** and *y**.

[1] Vito Volterra, *Leçons sur la théorie mathématique de la lutte pour la vie*, Paris: Gauthier-Villars, 1935. Alfred James Lotka (1880 – 1949), *Elements of Physical Biology*, Williams and Wilkins, Baltimore,1925. [Reprinted as *Elements of Mathematical Biology*, Inc., New York: Dover Publications, 1956]. Volterra has also published a relevant paper in 1926. This description is based on Richard Shone, *Economic Dynamics*, Cambridge UP, Cambridge, 1997, sec. 12.4.2.



We can also compute the trajectories. Since $y = f(x)$ we try to find dy/dx. By the chain rule this is $dy/dx = \dfrac{dy/dt}{dx/dt} = \dfrac{(-c+dx)y}{(a-by)x} \Rightarrow \left(\dfrac{a}{y} - b\right)dy = \left(-\dfrac{c}{x} + d\right)dx$.

Integrating both sides we get $a\ln y - by = -c\ln x + dx + k_1$. ($k_1$ is the constant of integration). Rearranging and raising to *e* we get $y^a x^b = ke^{by+dx}$ $\left(k = e^{k_1}\right)$. Hence, $k = \dfrac{y^a x^b}{e^{by+dx}}$. For each value of the constant *k* we get the trajectory in the phase plane. (See diagram). So we see that for out of equilibrium values, there is no tendency of the system to reach equilibrium, but it moves on a closed trajectory depending on the initial conditions. If we plot the two populations against time we see that there is a pattern of lagged oscillations.

[For a graphic portrayal of prey-predator models see the Wator: A Predator-Prey simulation available from https://beltoforion.de/en/wator/. See also https://en.wikipedia.org/wiki/Wa-Tor]

The Goodwin model envisages capitalism as a prey predator model and constructs a phase plane with the share of labor income and the proportion of the labor force employed in the place of prey and predator. So the antagonistic nature of capitalism together with its homeostatic property is mathematically modeled.

The Goodwin model

Assumptions.

1. Steady technical progress
2. Steady growth of the labor force.
3. Only two factors of production, labor and capital
4. All quantities real and net
5. All wages consumed, all profits saved (simplification for constant proportional savings)
6. Constant capital-output ratio
7. Real wage rate rises in the neighborhood of full employment

Symbols

q output, k capital, l employment , w wage rate, $a = a_0 e^{\alpha t} = \frac{q}{l}$ labor productivity, [(Greek letter) α constant]. $\sigma = \frac{k}{q}$ capital-output ratio, the inverse of capital productivity, $u = w/a$ the (employed) workers' share of output. Note that $\frac{w}{a} = \frac{wl}{q}$.

Since all output is divided between the two classes, the capitalists' share is $1 - \frac{w}{a}$.

The surplus=profits=investment=savings=$\left(1 - \frac{w}{a}\right) q = \dot{k}$. Since capital does not depreciate, the increase of capital (net investment) is the capitalists' profits and their (and the economy's) savings.

The profit rate is $\dot{k}/k$ and since we have a constant capital-output ratio this is equal to $\dot{q}/q$. Moreover the profit rate is equal to the capitalists' profits divided by their capital, so it is equal to $\frac{\left(1 - \frac{w}{a}\right) q}{k} = \frac{\left(1 - \frac{w}{a}\right)}{\sigma}$.

$n = n_0 e^{\beta t}$ is labor supply, with β constant. Labor supply is different from employment. When there is no unemployment $l = n$

The model

Now from $a = a_0 e^{\alpha t} = \frac{q}{l}$ we have:

$$\frac{d\left(\frac{q}{l}\right)\Big/dt}{\frac{q}{l}} = \frac{\dot{\overline{\frac{q}{l}}}}{\frac{q}{l}} = \frac{\frac{\dot{q}l - q\dot{l}}{l^2}}{\frac{q}{l}} = \frac{\dot{q}}{q} - \frac{\dot{l}}{l} \quad \text{and} \quad \frac{\frac{d\frac{q(t)}{l(t)}}{dt}}{\frac{q(t)}{l(t)}} = \frac{\frac{da(t)}{dt}}{a} = \frac{\frac{da_0e^{\alpha t}}{dt}}{a_0e^{\alpha t}} = \frac{\alpha a_0 e^{\alpha t}}{a_0 e^{\alpha t}} = \alpha$$

Since $\frac{\dot{q}}{q} = \frac{\left(1 - \frac{w}{a}\right)}{\sigma}$ is the profit rate we have

$$\frac{\dot{q}}{q} - \frac{\dot{l}}{l} = \frac{\left(1 - \frac{w}{a}\right)}{\sigma} - \frac{\dot{l}}{l} = \alpha \Rightarrow \frac{\dot{l}}{l} = \frac{\left(1 - \frac{w}{a}\right)}{\sigma} - \alpha = \frac{(1-u)}{\sigma} - \alpha \qquad \text{Eq. (1)}$$

So define $v = \frac{l}{n}$, the proportion of the labor force employed [unemployment is thus $1 - l/n$] and recall that $u = w/a$.

$v = \frac{l}{n} \Rightarrow \frac{\dot{v}}{v} = \frac{\dot{l}}{l} - \frac{\dot{n}}{n}$. From Eq. (1) and the equation of labor supply $n = n_0 e^{\beta t} \Rightarrow \frac{\dot{n}}{n} = \beta$ we have that

$$\frac{\dot{v}}{v} = \frac{\dot{l}}{l} - \frac{\dot{n}}{n} = \frac{(1-u)}{\sigma} - (\alpha + \beta) \qquad \text{Eq. (2)}$$

Assumption 7, namely, that real wage rate rises in the neighborhood of full employment is modeled with a specification that assumes a linear approximation

$$\frac{\dot{w}}{w} = -\gamma + \rho v \qquad \text{Eq. (3)}$$

Now, $u = \frac{w}{a} \Rightarrow \frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a}$. Substituting the first term of the right hand from Eq. (3) and noting that $a = a_0 e^{\alpha t} \Rightarrow \frac{\dot{a}}{a} = \alpha$ we have

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a} = -(\alpha + \gamma) + \rho v \qquad \text{Eq. (4)}$$

From Eqq (2) & (4) he have “a convenient statement” of the model.

$$\dot{v} = \left[\left\{ \frac{1}{\sigma} - (\alpha + \beta) \right\} - \frac{u}{\sigma} \right] v \qquad \text{Eq. (5)}$$

$$\dot{u} = \left\{ -(\alpha + \gamma) + \rho v \right\} u \qquad \text{Eq. (6)}$$

Note the similarity of Eqq (5) & (6) with VL Eqq. (1) & (2). As Goodwin observes this similarity is not entirely formal. Volterra’s problem – “the symbiosis of two populations – partly complementary, partly hostile – is helpful in the understanding of the dynamical contradictions of capitalism.”

We can eliminate time from Eqq (5) & (6) and write

$$du/dv = \frac{du/dt}{dv/dt} = \frac{\left\{ -(\alpha + \gamma) + \rho v \right\} u}{\left[\left\{ \frac{1}{\sigma} - (\alpha + \beta) \right\} - \frac{u}{\sigma} \right] v} \Rightarrow \left[\frac{\left\{ \frac{1}{\sigma} - (\alpha + \beta) \right\}}{u} - \frac{1}{\sigma} \right] du = \left\{ -\frac{(\alpha + \gamma)}{v} + \rho \right\} dv$$

Taking integrals from both sides we have

$$\left\{ \frac{1}{\sigma} - (\alpha + \beta) \right\} \ln u - \frac{1}{\sigma} u = -(\alpha + \gamma) \ln v + \rho v + ct \Rightarrow$$

$$\frac{1}{\sigma} u + \rho v - \left[\frac{1}{\sigma} - (\alpha + \beta) \right] \ln u - (\alpha + \gamma) \ln v = ct$$

[Mind the typo in the original model].

Goodwin simplifies this equation to

$$\theta_1 u + \theta_2 v - \eta_1 \ln u - \eta_2 \ln v = \ln H \quad ,$$

where

$\theta_1 = \frac{1}{\sigma}, \quad \theta_2 = \rho, \quad \eta_1 = \left[\frac{1}{\sigma} - (\alpha + \beta) \right], \quad \eta_2 = (\alpha + \gamma)$ and $\ln H$ the constant of integration.

Raising to *e* and expressing the equality in terms of functions of the variables of *u* and *v* we get:

$$\phi(u) = u^{\eta_1} e^{-\theta_1 u} = H v^{-\eta_2} e^{\theta_2 v} = H \psi(v)$$

This last equation gives as the trajectory of the economic system in the phase plane *v*-*u* for initial conditions described by *H*.

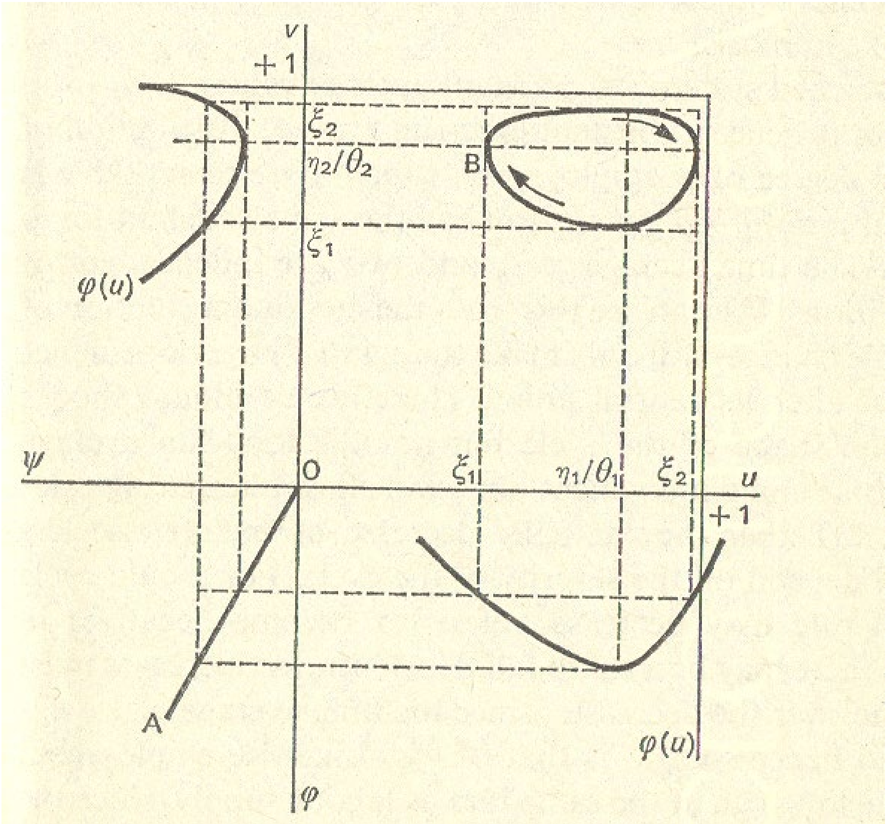
Goodwin takes the derivatives of the φ and ψ functions

$$\frac{d\phi}{du} = \left(-\theta_1 + \frac{\eta_1}{u}\right)\phi,$$
$$\frac{d\psi}{dv} = \left(\theta_2 - \frac{\eta_2}{v}\right)\psi$$
The extrema of these functions are at $u^* = \frac{\eta_1}{\theta_1}$, $v^* = \frac{\eta_2}{\theta_2}$

Since

$$\phi'' = -\frac{\eta_1}{u^2}\phi + \left(-\theta_1 + \frac{\eta_1}{u}\right)\phi'$$
$$\psi'' = \frac{\eta_2}{v^2}\psi + \left(\theta_2 - \frac{\eta_2}{v}\right)\psi'$$
at the extrema where $\phi' = \psi' = 0 \Rightarrow \phi'' < 0 \quad \psi'' > 0$

So Goodwin sketches the plots of the two functions for a part of the domain of u and v. [Remember that both variables must be positive and less than one]. Function φ has a maximum at $u^* = \frac{\eta_1}{\theta_1}$ and function ψ has a minimum at $v^* = \frac{\eta_2}{\theta_2}$. Since, $\phi = H\psi$ he devises a four quadrant positive diagram to derive the trajectory.



This is Diagram 3 in the original article. There are four quadrants connected in a single diagram. The bottom right quadrant plots in its abscissa axis the variable u and in its ordinate axis the function φ. The top left quadrant plots in its abscissa axis the variable v and in its ordinate axis the function ψ. [There is a typo in the article]. These two quadrants are connected through the bottom left quadrant that uses for its axes the ordinate axes of the preceding quadrants. This quadrant "solves" the $\phi = H\psi$ equation. Since H is a constant reflecting the initial conditions, the plot in this quadrant is a straight line through the origin with slope φ/ψ. Now, for every value of u read the corresponding value of φ (bottom right quadrant), transform it to a ψ value through the bottom left quadrant and from the top right quadrant read the corresponding value of v. This generates a locus of points in the top right quadrant that plots the trajectory of the system for the phase plane u-v. Rotating the line OA in the $\varphi\psi$ quadrant we get different trajectories. Note that there is a point at which u and v assume the values that maximize φ and minimize ψ respectively, this trajectory is a single point of equilibrium. But once this equilibrium is disturbed the system locks into a different trajectory. Goodwin notes that in the long run the time average of u and v are the undisturbed equilibrium values. Thus profits are constant in the long run, unemployment fluctuates around the undisturbed equilibrium value (1-v*) while employment grows at the rate of growth of the labour supply. Wages grow at the rate of productivity even if they may fluctuate widely depending on the initial conditions.

The constancy of the average rate of profit and the growth of wages according to productivity are the stylized facts that Goodwin attempted to explain in his model. A political corollary is "the fatuity and pusillanimity of social democracy".